Approximate injectivity of dual Banach algebras

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Abstract

A new notion of injectivity is introduced. It is shown that approximate Connes-amenability and approximate injectivity are the same properties. As a consequence, approximate Connes-amenability of the direct sum of dual Banach algebras is discussed. A characterization is given for approximate Connes-amenability of dual Banach algebras in terms of the approximate splitting of certain short exact sequence.

1 Preliminaries

The notion of amenability for Banach algebras introduced by Johnson [12], has proved to be of enormous importance in Banach algebra theory (see [4]). Several modifications of this notion were introduced by Ghahramani and Loy in [8]. The reader may find more detail in [1, 2, 9, 10].

The concept of Connes-amenability, which is a natural generalization of amenability for dual Banach algebras, was introduced by Runde in [13], see also [3, 11]). For more information on this subject, see [14]. There is a characterization of Connes-amenability in terms of splitting of an admissible short exact sequence, a fact noted by Daws [5, Prop. 4.4]. The notion of injectivity for dual Banach algebras was introduced by Daws in [6]. A dual Banach algebra is injective if and only if it is Connes-amenable [6, Theorem 6.13] The concept of approximate

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Connes-amenability which is a generalization of Connes-amenability was introduced in [7].

The purpose of the present paper is to generalize [5, Prop. 4.4] and [6, Theorem 6.13] to the approximate case. The organization of the paper is as follows. In Sec. 2 we define and study the notion of approximate injectivity for dual Banach algebras. Then we show that approximate injectivity is equivalent to approximate Connes-amenability.

The results of Sec. 2 are applied in Sec. 3 to investigate the approximate Connes-amenability of the direct sum of dual Banach algebras. We prove that the direct sum of two approximately Connes-amenable dual Banach algebras, when at least one of them has an identity, is approximately Connes-amenable. For a Banach algebra $\mathcal{A}$, we study the relation between approximate Connes-amenability of $WAP(\mathcal{A}^*)^*$ and continuous representations of $\mathcal{A}$ on reflexive Banach spaces.

In Sec. 4 we give a characterization of approximate Connes-amenability of a dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ in terms of the approximate splitting of the short exact sequence

$$0 \longrightarrow \mathcal{A}_*^\sharp \overset{\Delta^*|_{\mathcal{A}_*^\sharp}}{\longrightarrow} \sigma WC(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* \longrightarrow \sigma WC(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* / \Delta^*(\mathcal{A}_*^\sharp) \longrightarrow 0\,.$$

Before proceeding further we recall some terminology. Throughout, if $\mathcal{A}$ is a Banach algebra we shall write $\mathcal{A}^\sharp$ for the *forced unitization* of $\mathcal{A}$. The adjoined identity element will usually be denoted by e.

Let E and F be Banach spaces. We write $\mathcal{L}(E, F)$ for the space of all bounded linear operators from E into F, and $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$. The closed unit ball of a Banach space E is denoted by *ball* E.

For a Banach algebra $\mathcal{A}$, a *Banach $\mathcal{A}$-bimodule E*, is a Banach space which is algebraically an $\mathcal{A}$-bimodule, and for which there is a constant $C \geq 0$ such that

$$||a\,.\,x|| \leq C||a||\,||x|| \text{ and } ||x\,.\,a|| \leq C||a||\,||x|| \quad (a \in \mathcal{A}, x \in E).$$

Let E be a Banach $\mathcal{A}$-bimodule and E_* be a closed submodule of E^* such that $E = (E_*)^*$. Then we say that E is a *dual Banach $\mathcal{A}$-bimodule* with *predual E_**.

A Banach algebra $\mathcal{A}$ is called a *dual Banach algebra* if it is a dual Banach $\mathcal{A}$-bimodule. For a dual Banach algebra $\mathcal{A}$, A dual Banach $\mathcal{A}$-bimodule E is *normal* if the maps

$$\mathcal{A} \longrightarrow E\ , \quad a \longmapsto \begin{cases} a\,.\,x \\ x\,.\,a \end{cases} ,$$

are w^*-w^*-continuous, for each $x \in E$.

A continuous *derivation* from a Banach algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule E is a bounded linear map $D : \mathcal{A} \longrightarrow E$, satisfying $D(ab) = a\,.\,Db + Da\,.\,b$. For $x \in E$, the derivation $ad_x := a \longmapsto a\,.\,x - x\,.\,a$, is called an *inner derivation*. A derivation $D : \mathcal{A} \longrightarrow E$ is *approximately inner* if there exists a net $(x_\alpha)_\alpha$ in E, such that $D = \lim_\alpha ad_{x_\alpha}$, the limit being in *strong operator topology*.

A Banach algebra $\mathcal{A}$ is *amenable* if for any Banach $\mathcal{A}$-bimodule E, every derivation from $\mathcal{A}$ to E^* is inner. We say that $\mathcal{A}$ is *approximately amenable* if for any Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \longrightarrow E^*$ is approximately inner.

Let $\mathcal{A}$ be a dual Banach algebra. Then $\mathcal{A}$ is *Connes-amenable* if every w^*-continuous derivation from $\mathcal{A}$ to a normal dual Banach $\mathcal{A}$-bimodule is inner. Similarly, a dual Banach algebra $\mathcal{A}$ is *approximately Connes-amenable* if for every normal, dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is approximately inner.

Let $\mathcal{A}$ be a dual Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. We write $\sigma WC(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \longrightarrow E \ , \quad a \longmapsto \begin{cases} a \, . \, x \\ x \, . \, a \end{cases} ,$$

are w^*-weak continuous. It is clear that $\sigma WC(E)$ is a closed submodule of E.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the standard way. Define $\Delta_{\mathcal{A}} : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\Delta_{\mathcal{A}}(a \otimes b) = ab$. Then $\Delta_{\mathcal{A}}$ is an $\mathcal{A}$-bimodule homomorphism. In the sequel, simply we write Δ for both $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{A}^\sharp}$.

For a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$, it is shown in [15, Cor. 4.6] that $\Delta^*(\mathcal{A}_*) \subseteq \sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$, so that Δ^* maps $\mathcal{A}_*$ into $\sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$. Consequently, taking the adjoint of $\Delta^*|_{\mathcal{A}_*}$, we can extend Δ to an $\mathcal{A}$-bimodule homomorphism $\Delta_{\sigma WC} : \sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \longrightarrow \mathcal{A}$.

For a Banach algebra $\mathcal{A}$, recall that $(\mathcal{A}\hat{\otimes}\mathcal{A})^* = \mathcal{L}(\mathcal{A}, \mathcal{A}^*)$ with duality defined by

$$\langle a \otimes b, T \rangle = \langle a, Tb \rangle \quad (T \in \mathcal{L}(\mathcal{A}, \mathcal{A}^*), \ a, b \in \mathcal{A}) \, .$$

According to [5, 6], it is useful to identify $\sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$ with $\sigma WC(\mathcal{L}(\mathcal{A}, \mathcal{A}^*))$.

Connes-amenability of a dual Banach algebra $\mathcal{A}$ is equivalent to existence of a *σWC-virtual diagonal*, which is an element $M \in \sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a \, . \, M = M \, . \, a \ \text{ and } \ a\Delta_{\sigma WC}(M) = a \quad (a \in \mathcal{A}) \, ,$$

see [15, Theorem 4.8].

2 Approximate Injectivity

Our first proposition is a characterization of approximately Connes-amenable dual Banach algebras which improves [7, Theorem 3.3].

Proposition 2.1. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then the following are equivalent:

(i) $\mathcal{A}$ is approximately Connes-amenable.

(ii) There is a net $(M_\alpha)_\alpha \subseteq \sigma WC((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ such that

$$a \, . \, M_\alpha - M_\alpha \, . \, a \longrightarrow 0 \ \text{ and } \ \Delta_{\sigma WC} M_\alpha \longrightarrow e \quad (a \in \mathcal{A}^\sharp) \, .$$

(iii) There is a net $(M'_\alpha)_\alpha \subseteq \sigma WC((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ such that

$$a \, . \, M'_\alpha - M'_\alpha \, . \, a \longrightarrow 0 \ \text{ and } \ \Delta_{\sigma WC} M'_\alpha = e \quad (a \in \mathcal{A}^\sharp) \, .$$

Proof. Using [7, Prop. 2.3], the equivalences of (i) and (ii) is just [7, Theorem 3.3]. $(i) \implies (iii)$ also follows from the proof in [7]. The implication $(iii) \implies (ii)$ is obvious. ∎

Let $\mathcal{F}$ be a subset of an algebra $\mathcal{H}$. The *commutant* of $\mathcal{F}$ is

$$\mathcal{F}^c = \{T \in \mathcal{H} \,:\, TS = ST,\, S \in \mathcal{F}\} .$$

It is obvious that $\mathcal{F}^c$ is a closed subalgebra of $\mathcal{H}$.

Let E be a Banach space, and let $\mathcal{F} \subseteq \mathcal{L}(E)$ be a subalgebra. A *quasi-expectation* for $\mathcal{F}$ is a *projection* $Q : \mathcal{L}(E) \longrightarrow \mathcal{F}^c$ such that $Q(STU) = SQ(T)U$ for $S, U \in \mathcal{F}^c$ and $T \in \mathcal{L}(E)$.

When E is a reflexive Banach space, $E^* \hat{\otimes} E$ is the canonical predual for $\mathcal{L}(E)$, see Example 3 in the introduction in [13], so that it induces a w^*-topology on $\mathcal{L}(E)$.

From [6], we recall that a unital dual Banach algebra $\mathcal{A}$ is *injective* if whenever $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous, unital representation on a reflexive Banach space E, there is a quasi-expectation $Q : \mathcal{L}(E) \longrightarrow \pi(\mathcal{A})^c$ for $\pi(\mathcal{A})$.

Let E be a Banach space and let $\mathcal{F} \subseteq \mathcal{L}(E)$ be a subalgebra. It is easy to see that a bounded linear map $Q : \mathcal{L}(E) \longrightarrow \mathcal{F}^c$ is a quasi-expectation for $\mathcal{F}$ if and only if

(1) The map Q is the identity on $\mathcal{F}^c$,

(2) $SQ(T) - Q(T)S = 0, \ \ (S \in \mathcal{F},\, T \in \mathcal{L}(E))$, and

(3) $Q(STU) = SQ(T)U \ \ (S, U \in \mathcal{F}^c,\, T \in \mathcal{L}(E))$.

The above observation is the motivation of the basic definition for the present paper.

Definition 2.2. Let $\mathcal{F}$ be a subalgebra of $\mathcal{L}(E)$ for some Banach space E. An *approximate quasi-expectation* for $\mathcal{F}$ is a net of bounded linear maps $Q_\alpha : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$, such that

(i) Each Q_α is the identity map on $\mathcal{F}^c$,

(ii) $SQ_\alpha(T) - Q_\alpha(T)S \longrightarrow 0\,,\ (S \in \mathcal{F},$ uniformly for all $T \in ball\mathcal{L}(E))$, and

(iii) $Q_\alpha(STU) = SQ_\alpha(T)U \ \ (S, U \in \mathcal{F}^c,\, T \in \mathcal{L}(E),$ and for all $\alpha)$.

We remark that (ii) of Definition 2.2 is exactly the condition

$$\sup_{T \in ball\mathcal{L}(E)} ||SQ_\alpha(T) - Q_\alpha(T)S|| \longrightarrow 0\,,\ \ (S \in \mathcal{F})\,.$$

Definition 2.3. A (unital) dual Banach algebra $\mathcal{A}$ is *approximately injective* if whenever $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a (unital) w^*-continuous representation on a reflexive Banach space E, there is an approximate quasi-expectation for $\pi(\mathcal{A})$.

In the above definition, we wish to stress that the representation π is assumed to be unital, when $\mathcal{A}$ is unital.

It is known that if $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra, then its unitization $\mathcal{A}^\sharp = \mathcal{A} \oplus^1 \mathbb{C}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus^\infty \mathbb{C}$, where $\oplus^1$ and $\oplus^\infty$ indicate ℓ^1 and ℓ^∞ direct sums, respectively. The duality pairing between $\mathcal{A}^\sharp$ and its predual is given by

$$\langle(\phi, \alpha), (a, \lambda)\rangle = \langle\phi, a\rangle + \lambda\alpha \ \ (a \in \mathcal{A},\, \phi \in \mathcal{A}_*,\, \alpha, \lambda \in \mathbb{C})\,.$$

We write I_E for the identity map on a Banach space E.

Proposition 2.4. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is approximately injective if and only if $\mathcal{A}^\sharp$ is approximately injective.

Proof. Let $\mathcal{A}$ be approximately injective and let $\pi : \mathcal{A}^\sharp \longrightarrow \mathcal{L}(E)$ be a unital w^*-continuous representation where E is a reflexive Banach space. Clearly $\pi|_{\mathcal{A}}$ is a w^*-continuous representation for $\mathcal{A}$. Approximate injectivity of $\mathcal{A}$ implies that there is an approximate quasi-expectation $Q_\alpha : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ for $\pi(\mathcal{A})$. It is easy to check that $\pi(\mathcal{A}^\sharp)^c = \pi(\mathcal{A})^c$. For $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$ and $T \in \mathcal{L}(E)$, we observe that

$$\pi(a,\lambda)Q_\alpha(T) - Q_\alpha(T)\pi(a,\lambda) = \pi(a)Q_\alpha(T) - Q_\alpha(T)\pi(a) \; .$$

Therefore (Q_α) is an approximate quasi-expectation for $\pi(\mathcal{A}^\sharp)$, as required.

Conversely, suppose that $\mathcal{A}^\sharp$ is approximately injective, and that $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on a reflexive Banach space E. We extend π to $\tilde{\pi}$ from $\mathcal{A}$ into $\mathcal{A}^\sharp$ by setting $\tilde{\pi}(a,\lambda) = \pi(a) + \lambda I_E$, for $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It is readily seen that $\tilde{\pi}$ is a w^*-continuous representation. By the assumption, there is an approximate quasi-expectation $(Q_\alpha)_\alpha$ for $\tilde{\pi}(\mathcal{A}^\sharp)$. Because $\tilde{\pi}(\mathcal{A}^\sharp)^c = \pi(\mathcal{A})^c$ and $\pi(a) = \tilde{\pi}(a)$ for every $a \in \mathcal{A}$, we conclude that $\mathcal{A}$ is approximately injective. ∎

The following is a part of [7, Prop. 6.1].

Proposition 2.5. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras and that $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a w^*-continuous homomorphism. Suppose that $\mathcal{A}$ is approximately Connes-amenable. Then there is a net $(Q_i) \subseteq \mathcal{L}(B)$ such that each Q_i is the identity map on $\theta(\mathcal{A})^c$,

$$\theta(a)\; Q_i(b) - Q_i(b)\; \theta(a) \longrightarrow 0 \quad (a \in \mathcal{A},\ \text{uniformly for all } \ b \in \text{ball}\, \mathcal{B}) \; ,$$

and

$$Q_i(z_1 b z_2) = z_1 Q_i(b) z_2 \quad (z_1, z_2 \in \theta(\mathcal{A})^c,\ b \in \mathcal{B}) \; .$$

We recall some preliminaries from [6] that are needed to prove the main theorem. Suppose that $\mathcal{A}$ is a dual Banach algebra and that $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on some reflexive Banach space E. We turn $\mathcal{L}(E)\hat{\otimes}E\hat{\otimes}E^*$ into a Banach $\mathcal{A}$-bimodule by setting

$$a\, .\, (T \otimes x \otimes \phi) := T \otimes \pi(a)(x) \otimes \phi \; , \text{ and } \; (T \otimes x \otimes \phi)\, .\, a := T \otimes x \otimes \pi(a)^*(\phi) \; ,$$

for all $a \in \mathcal{A}$, $x \in E$, $\phi \in E^*$ and $T \in \mathcal{L}(E)$. Hence the dual space $\mathcal{L}(\mathcal{L}(E))$, with predual $\mathcal{L}(E)\hat{\otimes}(E\hat{\otimes}E^*)$, has naturally a Banach $\mathcal{A}$-bimodule structure. One can check that $\mathcal{L}(E)$ becomes a Banach $\mathcal{A}$-bimodule through

$$a\, .\, T := \pi(a)T \; , \text{ and } \; T\, .\, a := T\pi(a) \quad (a \in \mathcal{A},\ T \in \mathcal{L}(E)) \; ,$$

and a Banach $\pi(\mathcal{A})^c$-bimodule in the obvious way. We write $\mathcal{L}_{\mathcal{A}}(\mathcal{L}(E))$ for the collection of all $\pi(\mathcal{A})^c$*-bimodule homomorphisms*, that is, maps $Q \in \mathcal{L}(\mathcal{L}(E))$ such that

$$Q(ST) = SQ(T) \; , \text{ and } \; Q(TS) = Q(T)S$$

for all $S \in \pi(\mathcal{A})^c$ and $T \in \mathcal{L}(E)$. Then, the $\mathcal{A}$-bimodule action on $\mathcal{L}_{\mathcal{A}}(\mathcal{L}(E))$ is given by

$$(a \,.\, Q)(T) = \pi(a)Q(T)\ , \ \text{ and } \ (Q \,.\, a)(T) = Q(T)\pi(a) \qquad (a \in \mathcal{A},\ T \in \mathcal{L}(E),\ Q \in \mathcal{L}_{\mathcal{A}}(\mathcal{L}(E)))\ .$$

Let $X \subseteq \mathcal{L}(E)\hat{\otimes}E\hat{\otimes}E^*$ be the closure of the linear span of elements

$$cT \otimes x \otimes \mu - T \otimes x \otimes c^*(\mu), \quad \text{and} \quad Tc \otimes x \otimes \mu - T \otimes c(x) \otimes \mu$$

for all $c \in \pi(\mathcal{A})^c$, $T \in \mathcal{L}(E)$, $x \in E$ and $\mu \in E^*$. Define $\theta : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{L}_{\mathcal{A}}(\mathcal{L}(E))$ by

$$\theta(a \otimes b)(T) = aTb \quad (a, b \in \mathcal{A},\ T \in \mathcal{L}(E))\ ,$$

and then define $\psi : \mathcal{L}(E)\hat{\otimes}E\hat{\otimes}\dfrac{E^*}{X} \longrightarrow \mathcal{L}(\mathcal{A}, \mathcal{A}^*)$ by

$$\langle a \otimes b, \psi(T \otimes x \otimes \phi + X)\rangle = \langle \theta(a \otimes b)(T)(x), \phi\rangle = \langle aTb(x), \phi\rangle\ ,$$

for $a, b \in \mathcal{A}$, $x \in E$, $\phi \in E^*$, and $T \in \mathcal{L}(E)$. It is shown in [6, Theorem 6.11] that there is a Banach space E and an isometric, w^*-continuous representation $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$, such that ψ is a bijection and

$$\psi^* : \sigma WC(\mathcal{L}(\mathcal{A}, \mathcal{A}^*))^* \longrightarrow \mathcal{L}_{\mathcal{A}}(\mathcal{L}(E))$$

is an isomorphism.

Theorem 2.6. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is approximately Connes-amenable if and only if $\mathcal{A}$ is approximately injective.

Proof. Let $\mathcal{A}$ be approximately Connes-amenable, and let $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ be a w^*-continuous representation on some reflexive Banach space E. Applying Proposition 2.5, with $\mathcal{L}(E)$ and π in place of $\mathcal{B}$ and θ respectively, we see that $\mathcal{A}$ is approximately injective.

Conversely, suppose that $\mathcal{A}$ is approximately injective. By Proposition 2.4 and [7, Prop. 2.3(i)] without loss of generality, we may suppose that $\mathcal{A}$ is unital with the identity e. Suppose that $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is the isometric, w^*-continuous representation constructed in [6, Theorem 6.11], so that ψ^* is an isomorphism. Let $(Q_\alpha)_\alpha$ be an approximate quasi-expectation for $\pi(\mathcal{A})$. Note that $Q_\alpha \in \mathcal{L}_{\mathcal{A}}(\mathcal{L}(E))$, by (iii) of Definition 2.2. Define $M_\alpha := (\psi^*)^{-1}(Q_\alpha)$, for each α. For every $a \in \mathcal{A}$ and $T \in \mathcal{L}(E)$, we see that

$$(a \,.\, Q_\alpha - Q_\alpha \,.\, a)(T) = \pi(a)Q_\alpha(T) - Q_\alpha(T)\pi(a)\ ,$$

so that $a \,.\, Q_\alpha - Q_\alpha \,.\, a \longrightarrow 0$ in $\mathcal{L}(\mathcal{L}(E))$. Then, because ψ^* is an $\mathcal{A}$-bimodule isomorphism, we conclude that $a \,.\, M_\alpha - M_\alpha \,.\, a \longrightarrow 0$ in $\sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, for each $a \in \mathcal{A}$.

Since $\mathcal{A}\hat{\otimes}\mathcal{A}$ is w^*-dense in $\sigma WC((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, there exists a bounded net $(\tau_{\alpha,i})$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $M_\alpha = w^* - \lim_i \tau_{\alpha,i}$, for each α. Let $\tau_{\alpha,i} = \sum_{n=1}^{\infty} a_n^{\alpha,i} \otimes b_n^{\alpha,i}$. For $x \in E$ and $\phi \in E^*$, there exists $\lambda \in \mathcal{A}_*$, the predual of $\mathcal{A}$, such that

$\langle a, \lambda \rangle = \langle \phi, \pi(a)x \rangle$, for $a \in \mathcal{A}$. The same argument as in the proof of [6, Theorem 6.13] shows that $\langle ab, \lambda \rangle = \langle \phi, \pi(a) I_E \pi(b)(x) \rangle$, for all $a, b \in \mathcal{A}$, and

$$\langle \phi, Q_\alpha(I_E)(x) \rangle = \lim_i \sum_{n=1}^{\infty} \langle \phi, \pi(a_n^{\alpha,i} b_n^{\alpha,i})(x) \rangle .$$

Since Q_α is identity on $\pi(\mathcal{A})^c$, $Q_\alpha(I_E) = I_E$. Then, since x and ϕ are arbitrary, we must have $w^* - \lim_i \Sigma_{n=1}^{\infty} a_n^{\alpha,i} b_n^{\alpha,i} = e$. So that $\Delta_{\sigma WC}(M_\alpha) = e$, for each α. Therefore, by Proposition 2.1, $\mathcal{A}$ is approximately Connes-amenable. ∎

Let $\mathbb{Z}$ be the group of integers. It is known that $\ell^1(\mathbb{Z})$ is amenable and so it is not surprising that $\ell^1(\mathbb{Z})$ is approximately Connes-amenable. Here, we shall directly (although in the argument, we use the fact that $\mathbb{Z}$ is an amenable group) show that $\ell^1(\mathbb{Z})$ is approximately injective . Let E be a reflexive Banach space, and let $\pi : \ell^1(\mathbb{Z}) \longrightarrow \mathcal{L}(E)$ be a unital w^*-representation. For each $n \in \mathbb{N}$, we define $Q_n : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ by

$$Q_n(T) := \frac{1}{n} \sum_{k=1}^{n} \pi(\delta_{-k})\, T\, \pi(\delta_k) .$$

It is readily seen that each Q_n is identity on $\pi(\ell^1(\mathbb{Z}))^c$, and $Q_n(STU) = SQ_n(T)U$ for $S, U \in \pi(\ell^1(\mathbb{Z}))^c$ and $T \in \mathcal{L}(E)$. For $m \geq 0$ and $T \in$ *ball* $\mathcal{L}(E)$, we have

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} ||\pi(\delta_m)\, Q_n(T) - Q_n(T)\, \pi(\delta_m)|| \\
&\quad = \lim_{n \longrightarrow \infty} ||\frac{1}{n} \sum_{k=1}^{n} (\pi(\delta_{m-k})\, T\, \pi(\delta_k) - \pi(\delta_{-k})\, T\, \pi(\delta_{k+m}))|| \\
&\quad = \lim_{n \longrightarrow \infty} ||\frac{1}{n} \sum_{k=1}^{m} (\pi(\delta_{m-k})\, T\, \pi(\delta_k))|| \\
&\quad \leq \lim_{n \longrightarrow \infty} \frac{m}{n}\, ||T||\, ||\pi||^2 \\
&\quad \leq \lim_n \frac{m}{n}\, ||\pi||^2 ,
\end{aligned}$$

so that $\pi(\delta_m)\, Q_n(T) - Q_n(T)\, \pi(\delta_m) \longrightarrow 0$, for $m \geq 0$ and uniformly for all $T \in$ *ball* $\mathcal{L}(E)$.

A similar argument holds for $m < 0$. Therefore the sequence $(Q_n)_n$ is an approximate quasi-expectation for $\pi(\ell^1(\mathbb{Z}))$.

3 Application to direct sums and $WAP(\mathcal{A}^*)^*$

In view of Theorem 2.6, we give a new proof for [7, Proposition 2.3(ii)] concerning the approximate Connes-amenability of the direct sum of dual Banach algebras.

Proposition 3.1. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable dual Banach algebras and each has an identity. Then $\mathcal{A} \oplus^1 \mathcal{B}$ is approximately Connes-amenable.

Proof. Let E be a reflexive Banach space, and let $\pi : \mathcal{A} \oplus^1 \mathcal{B} \longrightarrow \mathcal{L}(E)$ be a w^*-continuous representation. We define homomorphisms $\phi : \mathcal{A} \longrightarrow \mathcal{A} \oplus^1 \mathcal{B}$ and $\psi : \mathcal{B} \longrightarrow \mathcal{A} \oplus^1 \mathcal{B}$ by setting $\phi(a) = (a, e_{\mathcal{B}})$ and $\psi(b) = (e_{\mathcal{A}}, b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ are the identities of $\mathcal{A}$ and $\mathcal{B}$, respectively. Clearly ϕ and ψ are w^*-continuous and therefore we may consider the w^*-continuous representations $\pi_{\mathcal{A}} := \pi\phi$ and $\pi_{\mathcal{B}} := \pi\psi$. Notice that $\pi_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{B})^c$, $\pi_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(\mathcal{A})^c$ and $\pi(\mathcal{A} \oplus^1 \mathcal{B})^c = \pi_{\mathcal{A}}(\mathcal{A})^c \cap \pi_{\mathcal{B}}(\mathcal{B})^c$. As $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable, there are approximate quasi-expectations $(P_i)_{i\in I}$ and $(Q_j)_{j\in J}$ for $\pi_{\mathcal{A}}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{B})$, respectively. Put $q_{(j,i)} := \dfrac{1}{2}(Q_j + P_i)$ for each $(j,i) \in J \times I$. It is readily seen that $q_{(j,i)}(S) = S$, for every $S \in \pi(\mathcal{A} \oplus^1 \mathcal{B})^c$.

Next, for each $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $T \in ball\mathcal{L}(E)$, using (iii) of Definition 2.2, we have

$$
\begin{aligned}
&\pi(a,b)q(T) - q_{(j,i)}(T)\pi(a,b) = \\
&\qquad \frac{1}{2}(\pi_{\mathcal{A}}(a)\ \pi_{\mathcal{B}}(b)Q_j(T) + \pi_{\mathcal{A}}(a)\ \pi_{\mathcal{B}}(b)P_i(T) - \\
&\qquad Q_j(T)\pi_{\mathcal{A}}(a)\ \pi_{\mathcal{B}}(b) - P_i(T)\pi_{\mathcal{A}}(a)\ \pi_{\mathcal{B}}(b)) = \\
&\frac{1}{2}\pi_{\mathcal{A}}(a)\ (\pi_{\mathcal{B}}(b)Q_j(T) - Q_j(T)\pi_{\mathcal{B}}(b)) + \frac{1}{2}\pi_{\mathcal{B}}(b)\ (\pi_{\mathcal{A}}(a)P_i(T) - P_i(T)\pi_{\mathcal{A}}(a)) + \\
&\qquad \frac{1}{2}(Q_j(\pi_{\mathcal{A}}(a)T)\ \pi_{\mathcal{B}}(b) - Q_j(T\pi_{\mathcal{A}}(a))\ \pi_{\mathcal{B}}(b)) + \\
&\qquad\qquad \frac{1}{2}(P_i(\pi_{\mathcal{B}}(b)T)\ \pi_{\mathcal{A}}(a) - P_i(T\pi_{\mathcal{B}}(b))\ \pi_{\mathcal{A}}(a))\ .
\end{aligned}
$$

Therefore $\pi(a,b)q_{(j,i)}(T) - q_{(j,i)}(T)\pi(a,b) \longrightarrow 0$ uniformly for all $T \in ball\mathcal{L}(E)$ and for $a \in \mathcal{A}$, and $b \in \mathcal{B}$. Finally, for $S, U \in \pi_{\mathcal{A}}(\mathcal{A})^c \cap \pi_{\mathcal{B}}(\mathcal{B})^c$ and for $T \in \mathcal{L}(E)$, we have $q_{(j,i)}(STU) = Sq_{(j,i)}(T)U$.

We conclude that $(q_{(j,i)})_{(j,i)\in J\times I}$ is an approximate quasi-expectation for $\pi(\mathcal{A} \oplus^1 \mathcal{B})$, as required. ∎

We can improve Proposition 3.1 as follows.

Theorem 3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable dual Banach algebras and that one of $\mathcal{A}$ or $\mathcal{B}$ has an identity. Then $\mathcal{A} \oplus^1 \mathcal{B}$ is approximately Connes-amenable.

Proof. Let E be a reflexive Banach space, and let $\pi : \mathcal{A} \oplus^1 \mathcal{B} \longrightarrow \mathcal{L}(E)$ be a w^*-continuous representation. We extend π to $\tilde{\pi}$ from $\mathcal{A}^\sharp \oplus^1 \mathcal{B}$ into $\mathcal{L}(E)$ by defining $\tilde{\pi}(e) = I_E - \pi(e_{\mathcal{B}})$, where e and $e_{\mathcal{B}}$ are the identities of $\mathcal{A}^\sharp$ and $\mathcal{B}$, respectively. It is readily seen that after this extension $\tilde{\pi}$ is still a w^*-continuous representation. Since $\mathcal{A}^\sharp \oplus^1 \mathcal{B}$ is approximately Connes-amenable by Proposition 3.1, there exists an approximate quasi-expectation $Q_i : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$ for $\tilde{\pi}(\mathcal{A}^\sharp \oplus^1 \mathcal{B})$. Clearly $\tilde{\pi}(\mathcal{A}^\sharp \oplus^1 \mathcal{B})^c = \pi(\mathcal{A} \oplus^1 \mathcal{B})^c$, and therefore we conclude that (Q_i) is an approximate quasi-expectation for $\pi(\mathcal{A} \oplus^1 \mathcal{B})$. ∎

Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is called *weakly almost periodic* if the maps

$$\mathcal{A} \longrightarrow E \ , \quad a \longmapsto \begin{cases} a \, . \, x \\ x \, . \, a \end{cases} ,$$

are weakly compact. The set of all weakly almost periodic elements in E is denoted by $WAP(E)$.

For a Banach algebra $\mathcal{A}$, there is a well-defined product on $WAP(\mathcal{A}^*)^*$ turning it into a dual Banach algebra with a universal property, [15, Theorem 4.10]. In [6, Prop. 6.15], Daws gives a characterization of Connes-amenability of $WAP(\mathcal{A}^*)^*$ in terms of continuous representations of $\mathcal{A}$. For the approximate case, however, we are only able to obtain a weaker result as follows.

Proposition 3.3. Suppose that $\mathcal{A}$ is a Banach algebra for which $WAP(\mathcal{A}^*)^*$ is approximately Connes-amenable. Then, for every continuous representation $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ on a reflexive Banach space E, there exists an approximate quasi-expectation for $\pi(\mathcal{A})$.

Proof. Suppose that $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a continuous representation on a reflexive Banach space E, and that $\hat{\pi} : WAP(\mathcal{A}^*)^* \longrightarrow \mathcal{L}(E)$ is the unique w^*-continuous representation extending π, [15, Theorem 4.10]. The same argument as in [6, Proposition 6.15], shows that $\pi(\mathcal{A})^c = \hat{\pi}(WAP(\mathcal{A}^*)^*)^c$. Now, Theorem 2.6 yields the existence of an approximate quasi-expectation for $\hat{\pi}(WAP(\mathcal{A}^*)^*)$ and in particular for $\pi(\mathcal{A})$. ∎

Corollary 3.4. Suppose that $\mathcal{A}$ is a dual Banach algebra. If $WAP(\mathcal{A}^*)^*$ is approximately Connes-amenable, so is $\mathcal{A}$.

Proof. Immediate from Proposition 3.3 and Theorem 2.6. ∎

We conclude this section by giving a direct proof for Corollary 3.4. Let E be a normal, dual Banach $\mathcal{A}$-bimodule, and $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous derivation. Let $\imath : \mathcal{A}_* \longrightarrow WAP(\mathcal{A}^*)$ be the canonical map. Then $\imath^*$ is an $\mathcal{A}$-bimodule homomorphism from $WAP(\mathcal{A}^*)^*$ onto $\mathcal{A}$. We turn E into a Banach $WAP(\mathcal{A}^*)^*$-bimodule by

$$x \, . \, \Lambda := x \, . \, \imath^*(\Lambda) \ , \ \text{ and } \ \Lambda \, . \, x := \imath^*(\Lambda) \, . \, x \quad (x \in E, \Lambda \in WAP(\mathcal{A}^*)^*) \, .$$

Note that E is normal as Banach $WAP(\mathcal{A}^*)^*$-bimodule. Hence $D\imath^* : WAP(\mathcal{A}^*)^* \longrightarrow E$ is a w^*-continuous derivation. Since $WAP(\mathcal{A}^*)^*$ is approximately Connes-amenable, there exists a net (x_i) in E such that

$$(D\imath^*)(\Lambda) = \lim_i \Lambda \, . \, x_i - x_i \, . \, \Lambda \quad (\Lambda \in WAP(\mathcal{A}^*)^*) \, .$$

Consequently, $Da = \lim_i a \, . \, x_i - x_i \, . \, a$, for all a in $\mathcal{A}$, as required.

4 Approximate Splitting

Let $\mathcal{A}$ be a Banach algebra and let X, Y and Z be left Banach $\mathcal{A}$-modules. We recall that a short exact sequence

$$\sum :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is *admissible*, if there exists a bounded linear map $\rho : Y \longrightarrow X$ such that $\rho f = I_X$. If, further, we may choose ρ to be a left $\mathcal{A}$-module homomorphism, then $\sum$ is said to *split*. We say that $\sum$ *approximately splits*, if there exists a net $\rho_i : Y \longrightarrow X$ of left inverse maps to f such that

$$a \,.\, \rho_i(y) - \rho_i(a \,.\, y) \longrightarrow 0 \quad (a \in \mathcal{A}, \text{ uniformly for all } y \in ball\ Y)\ .$$

Similar definitions hold for right modules and bimodules.

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and consider the short exact sequence of $\mathcal{A}$-bimodules

$$\sum(\mathcal{A}) :\ 0 \longrightarrow \mathcal{A}_* \xrightarrow{\Delta^*|_{\mathcal{A}_*}} \sigma WC(\mathcal{A}\hat{\otimes}\mathcal{A})^* \longrightarrow \sigma WC(\mathcal{A}\hat{\otimes}\mathcal{A})^*/\Delta^*(\mathcal{A}_*) \longrightarrow 0\ .$$

If $\mathcal{A}$ has an identity e, then $\Delta^*|_{\mathcal{A}_*}$ is an injective map, and the map

$$\sigma WC(\mathcal{A}\hat{\otimes}\mathcal{A})^* \longrightarrow \mathcal{A}_*\ ,\ \ T \longmapsto T(e)$$

is a bounded left inverse to $\Delta^*|_{\mathcal{A}_*}$, so that $\sum(\mathcal{A})$ is admissible.

It is shown in [5, Proposition 4.4] that $\mathcal{A}$ is Connes-amenable if and only if the short exact sequence $\sum(\mathcal{A})$ splits, whenever $\mathcal{A}$ is a unital dual Banach algebra. For the approximate version, we have the following.

Theorem 4.1. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is approximately Connes-amenable if and only if the admissible short exact sequence $\sum(\mathcal{A}^\sharp)$ approximately splits.

Proof. We write $\mathcal{A}^\sharp_*$ for the predual of $\mathcal{A}^\sharp$. Suppose that $\sum(\mathcal{A}^\sharp)$ approximately splits, so that there exists a net $\rho_i : \sigma WC(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^* \longrightarrow \mathcal{A}^\sharp_*$ of left inverse maps to $\Delta^*|_{\mathcal{A}^\sharp_*}$, such that

$$a \,.\, \rho_i(T) - \rho_i(a \,.\, T) \longrightarrow 0 \ \text{ and } \ \rho_i(T) \,.\, a - \rho_i(T \,.\, a) \longrightarrow 0\ ,$$

for each $a \in \mathcal{A}^\sharp$ and uniformly for all $T \in$ ball $\sigma WC(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*$.

Setting $M_i := \rho_i^*(e)$, it is readily seen that $\Delta_{\sigma WC}(M_i) = e$, for each i. For $a \in \mathcal{A}^\sharp$ and $T \in$ ball $\sigma WC(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*$, we have

$$\begin{aligned}
|\langle T, a \,.\, M_i - M_i \,.\, a\rangle| &= |\langle \rho_i(T \,.\, a) - \rho_i(a \,.\, T), e\rangle| \\
&\le |\langle \rho_i(T \,.\, a) - \rho_i(T) \,.\, a, e\rangle| + |\langle a \,.\, \rho_i(T) - \rho_i(a \,.\, T), e\rangle| \\
&\le ||\rho_i(T \,.\, a) - \rho_i(a \,.\, T)|| + ||a \,.\, \rho_i(T) - \rho_i(a \,.\, T)||,
\end{aligned}$$

so that $a \,.\, M_i - M_i \,.\, a \longrightarrow 0$. Therefore, by Proposition 2.1, $\mathcal{A}$ is approximately Connes-amenable.

Conversely, suppose that $\mathcal{A}$ is approximately Connes-amenable. Take the net $(M_i)_i \subseteq \sigma WC((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ given in Proposition 2.1 (iii), and define a net ρ_i : $\sigma WC(\mathcal{L}(\mathcal{A}^\sharp, \mathcal{A}^{\sharp^*})) \longrightarrow \mathcal{A}^{\sharp^*}$ by

$$\langle a, \rho_i(T) \rangle := \langle T \, . \, a, M_i \rangle \quad (a \in \mathcal{A}^\sharp).$$

Suppose that $(a_\alpha)_\alpha$ is a bounded net in $\mathcal{A}^\sharp$ which tends to $a \in \mathcal{A}^\sharp$ in the w^*-topology. Then $T \, . \, a_\alpha \longrightarrow T \, . \, a$ weakly, for each $T \in \sigma WC(\mathcal{L}(\mathcal{A}^\sharp, \mathcal{A}^{\sharp^*}))$, so that $\langle a_\alpha, \rho_i(T) \rangle \longrightarrow \langle a, \rho_i(T) \rangle$. This implies that ρ_i maps into $\mathcal{A}^\sharp_*$, for every i, as required.

For $\phi \in \mathcal{A}^\sharp_*$ and $a \in \mathcal{A}^\sharp$, we have

$$\langle a, (\rho_i \, \Delta^*|_{\mathcal{A}^\sharp_*})(\phi) \rangle = \langle \Delta^*(\phi \, . \, a), M_i \rangle = \langle \phi \, . \, a, \Delta_{\sigma wc}(M_i) \rangle = \langle \phi \, . \, a, e \rangle = \langle a, \phi \rangle \, ,$$

so that $\rho_i \, \Delta^*|_{\mathcal{A}^\sharp_*} = I_{\mathcal{A}^\sharp_*}$. Finally, for $a, b, c \in \mathcal{A}^\sharp$ and $T \in$ ball $\sigma WC(\mathcal{L}(\mathcal{A}^\sharp, \mathcal{A}^{\sharp^*}))$, we note that

$$\begin{aligned} \langle c, \rho_i(a \, . \, T . \, b) - a \, . \, \rho_i(T) \, . \, b \rangle &= \langle a \, . \, T \, . \, bc, M_i \rangle - \langle (T \, . \, bc) \, . \, a, M_i \rangle \\ &= \langle T \, . \, bc, M_i \, . \, a \rangle - \langle T \, . \, bc, a \, . \, M_i \rangle \\ &= \langle T \, . \, bc, M_i \, . \, a - a \, . \, M_i \rangle \, , \end{aligned}$$

so that $\sum(\mathcal{A}^\sharp)$ approximately splits. ■

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