On the family of $D(4)$-triples $\{k-2, k+2, 4k^3-4k\}$

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Abstract

In this paper we prove that if $k \geq 3$ and d are positive integers and the set $\{k-2, k+2, 4k^3-4k, d\}$ has the property that the product of any two of its distinct elements increased by 4 is a perfect square, then $d = 4k$ or $d = 4k^5 - 12k^3 + 8k$.

1 Introduction

Let $n \neq 0$ be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$, or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

Diophantus of Alexandria was the first to look for such sets and it was in the case $n = 1$. He found a set of four positive rational numbers with the above property $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$. However, Fermat found a first $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$. Euler was later able to add the fifth positive rational, $\frac{777480}{8288641}$, to Fermat's set. There is a folklore conjecture that there does not exist a $D(1)$-quintuple. Recently, Dujella [2] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples. Considering congruences modulo 8, it is easy to prove that a $D(4)$-m-tuple can contain at most two odd numbers. So Dujella's result implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples (see [4]). The second author has given several improvements to this result, proving that there does not exist

Received by the editors in May 2012 - In revised form in November 2012.
Communicated by M. Van den Bergh.
2010 *Mathematics Subject Classification* : Primary 11D09; Secondary 11J86.
Key words and phrases : Diophantine tuples, system of Diophantine equations.

a $D(4)$-sextuple, there are only finitely many $D(4)$-quintuples, and that an irregular $D(4)$-quadruple cannot be extended to a quintuple (see [5, 6, 7, 8]). In recent years there is a lot of work done on $D(n)$-m-tuples, specially in the cases $n = 1$, $n = -1$ and $n = 4$. To see all details the reader can visit the webpage http://web.math.pmf.unizg.hr/~duje/dtuples.html. Here, we will consider only the case $n = 4$.

For $n = 4$ it is conjectured that there does not exist a $D(4)$-quintuple. Actually, there is a stronger version of that conjecture (see [4, Conjecture 1]), that if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then $d = d_+ = a + b + c + \frac{1}{2}(abc + rst)$, where r, s and t are positive integers given by $ab + 4 = r^2$, $ac + 4 = s^2$ and $bc + 4 = t^2$. The $D(4)$-quadruple $\{a, b, c, d\}$, where $d > \max\{a, b, c\}$ is called a regular quadruple if $d = d_+$. We also define $d_- = a + b + c + \frac{1}{2}(abc - rst)$. The set $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple if $d_- \neq 0$, but $d_- < c$.

Mohanty and Ramasamy [17] were the first to study the non-extendibility of $D(4)$-m-tuples. They proved that the $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Kedlaya [15] later proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$. There are some generalizations of this result that support the conjecture. One was given by Dujella and Ramasamy [4] who proved conjecture for a parametric family of $D(4)$-quadruples. They showed that if k and d are positive integers and $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where F_k and L_k are Fibonacci and Lucas numbers. Same result for more parametric families with Fibonacci, Lucas and Pell numbers was given in [9]. Another generalization was given by Fujita [12], who proved that if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = 4k^3 - 4k$. Using linear forms in two logarithms the second author, He and Togbe [10] considered the extension of two-parametric family of $D(4)$-triples. In all those examples it was considered the extension of $D(4)$-triple with smallest possible c, that is when you fix a, b, then c is smallest such that $c > \max\{a, b\}$ and $\{a, b, c\}$ is a $D(4)$-triple.

In this paper we use the following (which can be proven in the same way as [14, Theorem 8]): if $\{k-2, k+2, c\}$ is a $D(4)$-triple, then $c = c_\nu$ for some $\nu \geq 1$, where

$$c_\nu := \frac{4}{k^2-4}\left\{\left(\frac{k+\sqrt{k^2-4}}{2}\right)^{2\nu+1} + \left(\frac{k-\sqrt{k^2-4}}{2}\right)^{2\nu+1} - k\right\}. \tag{1.1}$$

It is easy to see that $c_1 = 4k$, $c_2 = 4k^3 - 4k$ and $c_3 = 4k^5 - 12k^3 + 8k$.

In this paper we will prove the following result.

Theorem 1.1. *Let $k \geq 3$ and d be positive integers such that the set $\{k-2, k+2, 4k^3 - 4k, d\}$ is $D(4)$-quadruple. Then, $d = 4k$ or $d = 4k^5 - 12k^3 + 8k$.*

In the proof we will use already known methods in solving similar problems on the extension of Diophantine triples. Firstly, we will transform the problem of the extension of $D(4)$-triple to a quadruple to solving the system of simultaneous pellian equations. It furthermore leads to finding intersection of binary recurrence sequences which can be solved using the Baker's theory of linear forms in logarithms. That method can be used in general when we try to solve the problem of extendibility of $D(4)$-triple $\{a, b, c\}$, where $a < b < c$. But the problem with that is when we have parametric family of $D(4)$-triples, usually the upper bound for the parameter k that we get will be too large to reduce it using computer program. We will get the better bound if we use hypergeometric method, but it usually works only when b and c are not too close, which is not case here. So, one solution for that is to use some improvement on lower bound for linear forms in three logarithms as it was used by Bugeaud, Dujella and Mignotte in [1]. They proved that if $k \geq 2$ is an integer and $\{k-1, k+1, 16k^3-4k, d\}$ is a $D(1)$-quadruple, then $d = 4k$ or $d = 64k^5 - 48k^3 + 8k$. In particular, Theorem 1.1 for k even is direct consequence of the main result of [1]. But we will get even better bound for the parameter k using linear forms in two logarithms introduced in [13], which can be used here even the number c is not the smallest possible, similarly as it was done in [11]. After we get the satisfactory upper bound for the parameter k, we will solve the remaining cases using Baker-Davenport reduction.

2 Preliminaries

Let $\{a, b, c\}$ be a $D(4)$-triple such that $a < b < c$. Furthermore, let r, s and t be positive integers defined by $ab + 4 = r^2$, $ac + 4 = s^2$ and $bc + 4 = t^2$. In order to extend $\{a, b, c\}$ to a $D(4)$-quadruple $\{a, b, c, d\}$, we have to solve the system

$$ad + 4 = x^2, \qquad bd + 4 = y^2, \qquad cd + 4 = z^2. \tag{2.1}$$

with positive integers x, y, z. Eliminating d from (2.1), we get the following system of pellian equations

$$az^2 - cx^2 = 4(a - c), \tag{2.2}$$

$$bz^2 - cy^2 = 4(b - c). \tag{2.3}$$

By [6, Lemma 1], all solutions of (2.2) are given by $z = v_m^{(i)}$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = \frac{1}{2}\left(sz_0^{(i)} + cx_0^{(i)}\right), \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)}, \tag{2.4}$$

and $|z_0^{(i)}| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$. Similarly, all solutions of (2.3) are given by $z = w_n^{(j)}$, where

$$w_0^{(j)} = z_1^{(j)}, \quad w_1^{(j)} = \frac{1}{2}\left(tz_1^{(j)} + cy_1^{(j)}\right), \quad w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)} \tag{2.5}$$

and $|z_1^{(j)}| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}}$.

The initial terms $z_0^{(i)}$ and $z_1^{(j)}$ are almost completely determined in the following lemma (see [6, Lemma 9]).

Lemma 2.1. *(i) If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$*. Moreover,* $|z_0| = 2$ *or* $|z_0| = \frac{1}{2}(cr - st)$ *or* $|z_0| < 1.608a^{\frac{-5}{14}}c^{\frac{9}{14}}$.

(ii) If the equation $v_{2m+1} = w_{2n}$ *has a solution, then* $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0 z_1 < 0$.

(iii) If the equation $v_{2m} = w_{2n+1}$ *has a solution, then* $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0 z_1 < 0$.

(iv) If the equation $v_{2m+1} = w_{2n+1}$ *has a solution, then* $|z_0| = t$, $|z_1| = s$, $z_0 z_1 > 0$.

We have omitted the superscripts (i) and (j) here, and we will continue to do so. Also, we will assume that $k > 10^6$, which will not be a problem because at the end we will check what is happening for smaller k. In our case, we have $a = k-2$, $b = k+2$, $c = 4k^3 - 4k$, $r = k$, $s = 2k^2 - 2k - 2$ and $t = 2k^2 + 2k - 2$. Therefore, $\frac{1}{2}(cr - st) = 4k^2 - 2$, and we can easily check that $|z_0| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} < k^2\sqrt{\frac{8\sqrt{k}}{\sqrt{k-2}}} < 4k^2 - 2$ and $|z_1| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}} < k^2\sqrt{\frac{8\sqrt{k}}{\sqrt{k+2}}} < 4k^2 - 2$. From the proof of [6, Lemma 9] , the third possibility in (i) appears only if there is a positive integer $d_0 = (z_0^2 - 4)/c$, $d_0 < c$ such that $c \geq 0.036 d_0^{3.5} a^{2.5}$. Then by [6, Proposition 1] $\{a, b, d_0, c\}$ is an irregular $D(4)$-quadruple. But this is not possible in our case, since $d_0 = c_1 = 4k$ and the quadruple $\{k-2, k+2, 4k, 4k^3 - 4k\}$ is regular.

Hence, the only possibilities which may occur in our case are (i) with $|z_0| = 2$, when$x_0 = y_1 = 2$ and (iv) in which case $x_0 = y_1 = r = k$.

Now the equation $v_m = w_n$ can be in standard way transformed into a logarithmic inequality.

Let us denote

$$\alpha_1 := \frac{s + \sqrt{ac}}{2}, \qquad \alpha_2 := \frac{t + \sqrt{bc}}{2},$$
$$\alpha_3 := \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}, \qquad \alpha_4 := \frac{\sqrt{b}(k\sqrt{c} \pm t\sqrt{a})}{\sqrt{a}(k\sqrt{c} \pm s\sqrt{b})}.$$

Lemma 2.2. *Let* $k \geq 3$ *be an integer.*

(i) If $v_m = w_n$ *has a solution with* $m \equiv n \equiv 0 \pmod 2$, $m \geq 2$ *and* $z_0 = z_1 = \pm 2$, *then we have*

$$0 < m \log \alpha_1 - n \log \alpha_2 + \log \alpha_3 < 4.05\alpha_1^{-2m} < 1.1k^2\alpha_1^{-2m}. \tag{2.6}$$

(ii) If $v_m = w_n$ *has a solution with* $m \equiv n \equiv 1 \pmod 2$, $m \geq 1$ *and* $z_0 = \pm t, z_1 = \pm s$ $(z_0 z_1 > 0)$, *then we have*

$$0 < m \log \alpha_1 - n \log \alpha_2 + \log \alpha_4 < 1.1k^2\alpha_1^{-2m}. \tag{2.7}$$

Proof.

(i) By (2.4), (2.5) and $z_0 = z_1 = \pm 2$, we have

$$v_m = \frac{1}{\sqrt{k-2}}\Bigg\{(\sqrt{c} \pm \sqrt{k-2})\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^m - (\sqrt{c} \mp \sqrt{k-2})\left(\frac{s-\sqrt{(k-2)c}}{2}\right)^m\Bigg\},$$

$$w_n = \frac{1}{\sqrt{k+2}}\Bigg\{(\sqrt{c} \pm \sqrt{k+2})\left(\frac{t+\sqrt{(k+2)c}}{2}\right)^n - (\sqrt{c} \mp \sqrt{k+2})\left(\frac{t-\sqrt{(k+2)c}}{2}\right)^n\Bigg\}.$$

Let us define

$$P := \frac{2(\sqrt{c} \pm \sqrt{k-2})}{\sqrt{k-2}}\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^m, \quad Q := \frac{2(\sqrt{c} \pm \sqrt{k+2})}{\sqrt{k+2}}\left(\frac{t+\sqrt{(k+2)c}}{2}\right)^n.$$

It follows from $v_m = w_n$ that

$$P - \frac{4(c-k+2)}{k-2}P^{-1} = Q - \frac{4(c-k-2)}{k+2}Q^{-1}. \tag{2.8}$$

Since $P > 0$, $Q > 0$ and

$$\begin{aligned} P - Q &= \frac{4(c-k+2)}{k-2}P^{-1} - \frac{4(c-k-2)}{k+2}Q^{-1} \\ &> \frac{4(c-k+2)}{k-2}(P^{-1} - Q^{-1}) = \frac{4(c-k+2)}{k-2}(Q-P)P^{-1}Q^{-1}, \end{aligned}$$

we have $P > Q$.
Furthermore, since $m \geq 2$, we have

$$P \geq \frac{2(\sqrt{c} - \sqrt{k-2})}{\sqrt{k-2}}\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^2 > 2c.$$

Then, we conclude

$$\begin{aligned}\frac{P-Q}{P} &= \frac{4(c-k+2)}{k-2}P^{-2} - \frac{4(c-k-2)}{k+2}P^{-1}Q^{-1} < \frac{4(c-k+2)}{k-2}P^{-2} \\ &< \frac{4(c-k+2)}{k-2} \cdot \frac{1}{4c^2} < 0.011.\end{aligned}$$

Hence, we have

$$\begin{aligned}0 &< \log\frac{P}{Q} = \log\left(\frac{Q}{P}\right)^{-1} = -\log\frac{Q}{P} = -\log\left(1-\frac{P-Q}{P}\right) \\ &< \frac{4(c-k+2)}{k-2}P^{-2} + \left(\frac{4(c-k+2)}{k-2}\right)^2 P^{-4} \\ &\qquad = \left(1+\frac{4(c-k+2)}{k-2}P^{-2}\right)\frac{4(c-k+2)}{k-2}P^{-2} \\ &\le \left(1+\frac{4(c-k+2)}{k-2}P^{-2}\right)\frac{4(c-k+2)}{k-2} \\ &\qquad \left(\frac{\sqrt{k-2}}{\sqrt{c}-\sqrt{k-2}}\right)^2\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^{-2m} \\ &= \left(1+\frac{4(c-k+2)}{k-2}P^{-2}\right)\frac{4(c-k+2)}{(\sqrt{c}-\sqrt{k-2})^2}\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^{-2m} \\ &< 1.011\cdot 4\cdot\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^{-2m} < 4.05\left(\frac{s+\sqrt{(k-2)c}}{2}\right)^{-2m}.\end{aligned}$$

The statement (ii) can be proved in the same way. ∎

Lemma 2.3. *Let $k > 10^6$ be an integer. Assume that $v_m = w_n$ with $n \geq 2$. Then*

$$0 < m-n < \frac{1.01n}{k\log k}.$$

Proof. Firstly, it is easy to see that if $n \geq 2$, then $m > n$. It follows from $v_2 < w_2$ in all cases and the sequence (w_n) grows more quickly. Then from Lemma 2.2 and $\alpha_3, \alpha_4 > 1$, we can conclude

$$m\log\alpha_1 - n\log\alpha_2 < 1.1k^2\alpha_1^{-6} < 1.1k^{-10}.$$

Hence, we have

$$\begin{aligned}\frac{m-n}{n} &< \frac{\log\alpha_2-\log\alpha_1}{\log\alpha_1} + \frac{1.1}{nk^{10}\log\alpha_1} = \frac{\log\left(\frac{\alpha_2}{\alpha_1}\right)}{\log\alpha_1} + \frac{1.1}{nk^{10}\log\alpha_1} \\ &< \frac{\log\left(\frac{2t}{2s-1}\right)}{\log\alpha_1} + \frac{0.28}{k^{10}\log k} < \frac{2t-2s+1}{(2s-1)\cdot 2\log k} + \frac{0.28}{k^{10}\log k} \\ &< \frac{8k+1}{(8k^2-8k-10)\log k} + \frac{0.28}{k^{10}\log k} < \frac{1.01}{k\log k}\end{aligned}$$

since $k > 10^6$. ∎

So if we put $\Delta = m - n$, we have just proved the following lemma.

Lemma 2.4. *Let $k > 10^6$ be an integer. If $v_m = w_n$ with $n \geq 2$, then there exist a positive integer $\Delta \geq 2$ such that*

$$m > 0.99 \cdot \Delta \cdot k \log k.$$

3 Linear forms in two logarithms

In this section we will apply the following result due to Laurent, Mignotte and Nesterenko to our linear form $\Lambda = m \log \alpha_1 - n \log \alpha_2 + \log \alpha_3'$, where $\alpha_3' = \alpha_3, \alpha_4$, assuming $k > 10^6$.

Lemma 3.1. ([16, Corollary 2]) *Let γ_1 and γ_2 be multiplicatively independent and positive algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ and*

$$\Lambda = b_1 \log \gamma_1 + b_2 \log \gamma_2.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$, for $i = 1, 2$ let

$$h_i \geq \max \left\{ h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\},$$

where $h(\gamma)$ is the absolute logarithmic height of γ, and

$$b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

If $\Lambda \neq 0$, then we have

$$\log |\Lambda| \geq -24.34 \cdot D^4 \left(\max \left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 h_1 h_2.$$

So, we will now transform our linear form to $\Lambda = m \log \left(\frac{\alpha_1}{\alpha_2} \right) + \log(\alpha_2^\Delta \alpha_3')$, where $\Delta = m - n$ as before. So in the notation of the previous lemma we have $D = 4$, $b_1 = m$, $b_2 = 1$, $\gamma_1 = \frac{\alpha_1}{\alpha_2}$ and $\gamma_2 = \alpha_2^\Delta \alpha_3'$. Moreover, γ_1 and γ_2 are multiplicatively independent since the relation

$$\left(\frac{s + \sqrt{ac}}{2} \right)^{i_1} \left(\frac{t + \sqrt{bc}}{2} \right)^{i_2} = \left(\frac{\sqrt{b}(x_0\sqrt{c} \pm z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} \pm z_1\sqrt{b})} \right)^{i_3}$$

implies

$$\left(\frac{s - \sqrt{ac}}{2} \right)^{i_1} \left(\frac{t - \sqrt{bc}}{2} \right)^{i_2} = \left(\frac{\sqrt{b}(x_0\sqrt{c} \mp z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} \mp z_1\sqrt{b})} \right)^{i_3}$$

and by multiplying these two relations we obtain $\left(\frac{b(c-a)}{a(c-b)}\right)^{i_3} = 1$ and $a = b$, a contradiction.

Now we have

$$h(\gamma_1) = h\left(\frac{\alpha_1}{\alpha_2}\right) \leq h(\alpha_1) + h(\alpha_2) = \frac{1}{2}\log\alpha_1 + \frac{1}{2}\log\alpha_2 < 2.06\log k.$$

So, we can take $h_1 = 2.06\log k$. Furthermore, in [5] the second author proved for general triple $\{a, b, c\}$ that $h(\alpha_3') < 3.425\log c < 3.425\log(4k^3) < 10.62\log k$. Then,

$$h(\gamma_2) = h(\alpha_2^{\Delta}\alpha_3') < \frac{\Delta}{2}\log\alpha_2 + 10.62\log k < (1.03\Delta + 10.62)\log k.$$

So, for h_2 we can take $h_2 = (1.03\Delta + 10.62)\log k$. That implies

$$\frac{b_2}{Dh_1} = \frac{1}{8.24\log k} < 0.009$$

and

$$b' = \frac{m}{4(1.03\Delta + 10.62)\log k} + 0.009.$$

Next, from Lemma 2.4 we conclude

$$b' > \frac{0.99\Delta k}{4(1.03\Delta + 10.62)}$$

and

$$\log b' + 0.14 > \log\frac{0.99\Delta k}{4(1.03\Delta + 10.62)} + 0.14 > \frac{21}{D}$$

for $k > 10^6$. Now, from Lemma 2.2 we have

$$\frac{2m\log\alpha_1}{(1.03\Delta + 10.62)\log k} < 24.34 \cdot 4^4 \cdot (\log b' + 0.14)^2 \cdot 8.24\log k + \frac{\log(1.1k^2)}{(1.03\Delta + 10.62)\log k}$$

$$< 51344(\log b' + 0.14)^2\log k.$$

Furthermore, $\log\alpha_1 > 2.05\log k$ yields

$$\frac{2m}{(1.03\Delta + 10.62)\log k} < 25046(\log b' + 0.14)^2$$

and

$$8b' < 25046(\log b' + 0.14)^2 + 0.072,$$

which implies $b' < 560141$. Moreover,

$$\frac{m}{4(1.03\Delta + 10.62)\log k} < b' < 560141$$

together with Lemma 2.4 gives us

$$\frac{k \cdot 0.99\Delta}{4(1.03\Delta + 10.62)} < 560141$$

and using $\Delta \geq 2$, $k < 1.44 \cdot 10^7$.

Because we know what is happening for small indices ($n < 2$) in the equation $v_m = w_n$, we know that the only solutions are $v_0 = w_0 = \pm 2$ which gives us $d = 0$, $v_1 = w_1 = \frac{1}{2}(cr - st) = 4k^2 - 2$ which gives us $d = 4k$ and $v_1 = w_1 = \frac{1}{2}(cr + st) = 4k^4 - 8k^2 + 2$, which gives us $d = 4k^5 - 12k^3 + 8k$. So, we have just proved the Theorem 1.1 for $k \geq 1.44 \cdot 10^7$. Let us mention that using the same theorem with linear forms in three logarithm as Bugeaud, Dujella and Mignotte in [1] we would get an upper bound for k, $k < 3 \cdot 10^8$. We see that using linear forms in two logarithms will give us slightly better bound, but it can save us a lot of time when we do reduction using computer program.

4 The reduction method and the proof of Theorem 1.1

In the previous section we have proven Theorem 1.1 for large parameter k. Therefore, we are left to treat the cases of small $k < 1.44 \cdot 10^7$. To deal with the remaining cases we will use the Baker-Davenport reduction that will give us that in all of the remaining cases we do not have any new extension of the triple $\{k-2, k+2, 4k^3-4k\}$.

Also from [5] we know that $v_m = w_n$ implies $m < 4 \cdot 10^{21}$ in all cases. It enables us to use the following version of Baker-Davenport lemma.

Lemma 4.1. ([3, Lemma 5a]) *Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < EB^{-m}$$

in integers m and n with

$$\frac{\log(EQ/\eta)}{\log B} \leq m \leq M.$$

We apply Lemma 4.1 with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3'}{\log \alpha_2}, \quad E = \frac{1.1k^2}{\log \alpha_2}, \quad B = \alpha_1^2$$

and $M = 4 \cdot 10^{21}$, where $\alpha_3' = \alpha_3, \alpha_4$. After two steps of reduction in all cases we get $m < 2$, which gives us the already known extensions of our triple to a quadruple. Actually, for $k > 10^5$ we needed only one step of reduction. We have done this in Mathematica 7 and the running time was less than 40 hours. It finishes the proof of Theorem 1.1.

Acknowledgment. The second author was supported by the Ministry of Science, Education and Sports, Republic of Croatia, grant 037-0372781-2821.

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