On Weierstrass' Monsters and lineability*

P. Jiménez-Rodríguez G. A. Muñoz-Fernández
J. B. Seoane-Sepúlveda

Abstract

Let E be a topological vector space and let us consider a property $\mathcal{P}$. We say that the subset M of E formed by the vectors in E which satisfy $\mathcal{P}$ is μ-lineable (for certain cardinal μ, finite or infinite) if $M \cup \{0\}$ contains an infinite dimensional linear space of dimension μ. In 1966 V. Gurariy provided a non-constructive proof of the $\aleph_0$-lineability of the set of *Weierstrass' Monsters* (continuous nowhere differentiable functions on $\mathbb{R}$). Here we provide the first constructive proof of the $\mathfrak{c}$-lineability of this set (where $\mathfrak{c}$ denotes the continuum). Of course, this result is the best possible in terms of dimension.

1 Preliminaries. Weierstrass' Monster and its variants

It came as a general *shock* when in 1872, and during a presentation before the Berlin Academy, K. Weierstrass provided the (nowadays) classical example of a function that was continuous everywhere but differentiable nowhere. The particular example was defined as

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x) \tag{1.1}$$

where $0 < a < 1$, b is any odd integer, and $ab > 1 + 3\pi/2$ (see Figure 1).

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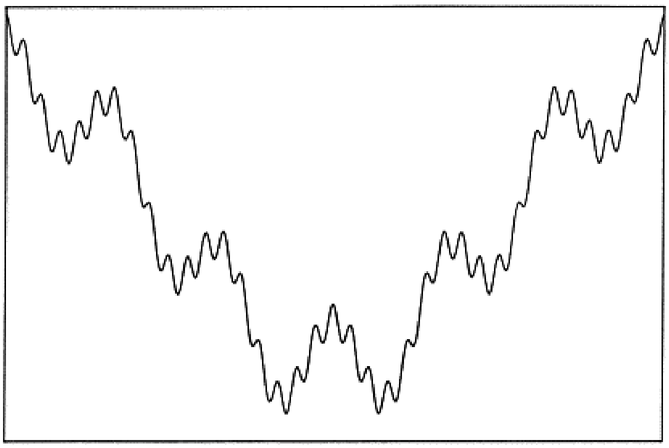

Figure 1: A sketch of Weierstrass' Monster

This apparent *shock* was a consequence of the general thought that most mathematicians shared, namely that a continuous function must have derivatives at a significant set of points (even A. M. Ampère attempted to give a theoretical justification for this). Although the first published example is certainly due to Weierstrass, already in 1830 the Czech mathematician B. Bolzano exhibited a continuous nowhere differentiable function (although it was not published until 1922). Let us make a brief account of the appearance throughout history of "Weierstrass' Monsters" (check, e.g. [19] for a thorough study of this topic): B. Bolzano (≈1830), M. Ch. Cellérier (≈1830), B. Riemann (1861), H. Hankel (1870), or K. Weierstrass (1872). After 1872 many other mathematicians also constructed similar functions, just to cite a partial list of these we have: H. A. Schwarz (1873), M. G. Darboux (1874), U. Dini (1877), K. Hertz (1879), G. Peano (1890), D. Hilbert (1891), T. Takagi (1903), H. von Koch (1904), W. Sierpiński (1912), G. H. Hardy (1916), A. S. Besicovitch (1924), B. van der Waerden (1930), S. Mazurkiewicz (1931), S. Banach (1931), S. Saks (1932), W. Orlicz (1947), or G. de Rham (1957), among others.

At the end of the twentieth century (and also nowadays) there have been many authors who have, as well, constructed other variants of the classical "*Weierstrass' Monster*" (1.1), as we shall mention in what follows.

Recall that, as it has become an standard notion in Mathematics, given a subset M of a topological vector space E, we say that M is μ-lineable (for certain cardinal μ, finite or infinite) if there exists a linear space $V \subset M \cup \{0\}$ of dimension μ.

This notion of lineability was originally coined by Gurariy and it first appeared in [1, 18]. Since this concept appeared, a trend has started in which many authors became interested in the study of subsets of $\mathcal{C}[0,1]$ enjoying certain special properties. Prior to the publication of [1, 18], some authors, when working with infinite dimensional spaces, already found large linear structures enjoying these type of special properties (even though they did not explicitly used the term *lineability*). Probably, the first result in this line this was due to Levine and Milman (1940, [15]):

Theorem 1.1 (Levine and Milman, 1940)**.** *There are no infinite dimensional closed subspaces of $\mathcal{C}[0,1]$ composed by just functions of bounded variation.*

Authors such as R. Aron, L. Bernal-González, G. Botelho, P. Enflo, G. Godefroy, V. Fonf, V. Gurariy, V. Kadets, and D. Pellegrino (among many others) have been working on this topic in the last decade and on many different frameworks (see, e.g. [2–8, 10, 14, 16] for a wider range of examples).

Coming back to the set of continuous nowhere differentiable functions, let us recall that the lineability of this type of function has been thoroughly studied in the last years (although the very first result in this direction was due to V. I. Gurariy in 1966, [11, 12], who showed that the set of continuous nowhere differentiable functions on $[0, 1]$ is $\aleph_0$-lineable.) The lineability of this class of functions has been studied in depth, as we summarize next. V. Fonf, V. Gurariy and V. Kadeč [9], in 1999, showed that the set of continuous nowhere differentiable functions on $[0, 1]$ is spaceable (that is, there exists an infinite dimensional and closed subspace of $\mathcal{C}[0, 1]$ every non-zero element of which is continuous and nowhere differentiable). Much more is true, L. Rodríguez-Piazza showed that the X in [9] can be chosen to be isometrically isomorphic to any separable Banach space [17]. Also, some years ago, S. Hencl [13] showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0, 1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder.

In this note we contribute to the above results by providing a constructive proof of the $\mathfrak{c}$-lineability of the set of Weierstrass' Monsters. In [11, 12] V. Gurariy provided a non-constructive proof of the $\aleph_0$-lineability of this set and later (as we mentioned above) the existence of infinite dimensional Banach spaces of such functions was also obtained (which, in particular, gives $\mathfrak{c}$-lineability). Here, we give the first constructive proof of the $\mathfrak{c}$-lineability of this set. Of course, since $\mathfrak{c}$ denotes the continuum, all the previous results are the best possible in terms of dimension.

2 Our construction

2.1 Some remarks on the "original" Monster

For the sake of completeness of this paper (and since it shall be needed in what follows), we shall now give a simplified proof of the fact that Weierstrass' Monster is, actually, continuous and nowhere differentiable. The steps followed in the forthcoming proof shall be needed in our construction in Subsection 2.2.

Theorem 2.1 (Weierstrass, 1872). *The function $W : \mathbb{R} \longrightarrow \mathbb{R}$ given by*

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$ and b is an odd natural number greater than 1, is a continuous nowhere differentiable function on $\mathbb{R}$

Proof. Since $0 < a < 1$ we have that $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty$. Hence, together with $\sup_{x\in\mathbb{R}} |a^k \cos(b^k \pi x)| \leq a^k$ for all $k \in \mathbb{N}$ and *Weierstrass Convergence Criterion*, we can conclude that $\sum_{k=0}^{n} a^k \cos(b^k \pi x)$ converges uniformly to $W(x)$ on $\mathbb{R}$. The continuity of W follows then from the continuity of $\sum_{k=0}^{n} a^k \cos(b^k \pi x)$.

Let us now prove the non-differentiability of W at any $x_0 \in \mathbb{R}$. For every $m \in \mathbb{N}$, choose $a_m \in \mathbb{Z}$ such that $b^m x_0 - a_m \in (-\frac{1}{2}, \frac{1}{2}]$ and call $x_{m+1} = b^m x_0 - a_m$. Also, define

$$y_m = \frac{a_m - 1}{b^m} \quad \text{and} \quad z_m = \frac{a_m + 1}{b^m}. \tag{2.1}$$

Then, we have

$$y_m - x_0 = -\frac{1 + x_{m+1}}{b^m} < 0 < \frac{1 - x_{m+1}}{b^m} = z_m - x_0,$$

where $x_{m+1} \in (-\frac{1}{2}, \frac{1}{2}]$, and therefore $y_m < x_0 < z_m$. Hence, $0 < x_0 - y_m, z_m - x_0 < z_m - y_m = \frac{2}{b^m}$ and then $y_m \longrightarrow x_0^-$ and $z_m \longrightarrow x_0^+$ as $m \to \infty$.

Now, consider the quotient

$$\begin{aligned}\frac{W(y_m) - W(x_0)}{y_m - x_0} &= \sum_{k=0}^{m-1} \left((ab)^k \frac{\cos(b^k \pi y_m) - \cos(b^k \pi x_0)}{b^k (y_m - x_0)} \right) \\ &+ \sum_{k=0}^{\infty} \left(a^{m+k} \frac{\cos(b^{m+k} \pi y_m) - \cos(b^{m+k} \pi x_0)}{y_m - x_0} \right) =: S_1 + S_2.\end{aligned}$$

It can be seen, after performing some simple calculations (and by means of the mean value theorem), that

$$|S_1| \leq \frac{\pi (ab)^m}{ab - 1}. \tag{2.2}$$

Focusing now on S_2, and since $b > 1$ is odd and $a_m \in \mathbb{Z}$, we have

$$\cos(b^{m+k} \pi y_m) = -(-1)^{a_m}$$

and

$$\cos(b^{m+k} \pi x_0) = (-1)^{a_m} \cos(b^k \pi x_{m+1}).$$

Therefore, we have

$$\begin{aligned}S_2 = \sum_{k=0}^{\infty} a^{m+k} \frac{-(-1)^{a_m} - (-1)^{a_m} \cos(b^k \pi x_{m+1})}{-\frac{1+x_{m+1}}{b^m}} &= \\ (ab)^m (-1)^{a_m} \sum_{k=0}^{\infty} a^k \frac{1 + \cos(b^k \pi x_{m+1})}{1 + x_{m+1}}&.\end{aligned}$$

Since $x_{m+1} \in (-\frac{1}{2}, \frac{1}{2}]$ and $\cos(b^k \pi x_{m+1}) \geq 0$ we also have

$$\sum_{k=0}^{\infty} a^k \frac{1 + \cos(b^k \pi x_{m+1})}{1 + x_{m+1}} \geq \frac{1 + \cos(\pi x_{m+1})}{1 + x_{m+1}} \geq \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}. \tag{2.3}$$

Next, inequality (2.2) ensures the existence of $\epsilon_{1,m} \in [-1, 1]$ with

$$S_1 = (-1)^{a_m} \frac{\pi (ab)^m \epsilon_{1,m}}{ab - 1},$$

and inequality (2.3) guarantees the existence of $\eta_m \geq 1$ such that

$$\sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k \pi x_{m+1})}{1+x_{m+1}} = \eta_m \frac{2}{3}.$$

Thus, we have

$$\frac{W(y_m)-W(x_0)}{y_m-x_0} = (-1)^{a_m}(ab)^m \left(\frac{2}{3}\eta_m + \epsilon_{1,m} \cdot \frac{\pi}{ab-1}\right). \tag{2.4}$$

Next, since $ab > 1 + \frac{3}{2}\pi$ is equivalent to $\frac{2}{3} > \frac{\pi}{ab-1}$, we have

$$\frac{2}{3}\eta_m + \epsilon_{1,m} \cdot \frac{\pi}{ab-1} \geq \frac{2}{3} - \frac{\pi}{ab-1} > 0 \tag{2.5}$$

and, thus, the quantity $\frac{2}{3}\eta_m + \epsilon_{1,m} \cdot \frac{\pi}{ab-1}$ is bounded below by a strictly positive constant. This, along with $ab > 1$ allows us to conclude

$$\left| \frac{W(y_m)-W(x_0)}{y_m-x_0} \right| \xrightarrow[m\longrightarrow\infty]{} \infty$$

and then W is not differentiable in x_0. ∎

The following remark shall also be useful in our construction.

Remark 2.2. *Although the non-differentiability of W has already been proved, it shall be useful to study what happens with the sequence $\{z_m\}$ (see equation* (2.1)*) in order to show that the set of all continuous nowhere differentiable functions is lineable. As earlier, we have*

$$\frac{W(z_m)-W(x_0)}{z_m-x_0} =: S_1' + S_2', \textit{ with}$$

$$|S_1'| \leq \frac{\pi(ab)^m}{ab-1}, \cos(b^{m+k}\pi z_m) = -(-1)^{a_m}$$

and

$$S_2' = -(ab)^m(-1)^{a_m} \sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k \pi x_{m+1})}{1-x_{m+1}}$$

and, when it comes to consider the same constants as the ones appearing earlier, we see that

$$S_2' = -\eta_m \frac{2}{3},$$

since y_m and z_m do not appear in the expressions.
Hence, we would have

$$\frac{W(z_m)-W(x_0)}{z_m-x_0} = -(-1)^{a_m}(ab)^m \left(\frac{2}{3}\eta_m + \epsilon_{2,m} \cdot \frac{\pi}{ab-1}\right), \tag{2.6}$$

where $\epsilon_{2,m} \in [-1,1]$.

2.2 Our construction ($\mathfrak{c}$-lineability of ***Weierstrass' Monsters***)

In this section we give our construction of a $\mathfrak{c}$-dimensional linear space every non-zero element of which is a Weierstrass' Monster. Of course, this will not give a closed linear space (we will just simply consider the linear, and non-closed, span of a basis that we will provide). As we mentioned earlier, the existence of infinite dimensional closed linear spaces of *"monsters"* has already been proved in the past (see, e.g., [9,11,12]).

Theorem 2.3. *The set of all continuous nowhere differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is $\mathfrak{c}$-lineable.*

Proof. Consider, for $\frac{7}{9} < a < 1$,

$$W_a(x) = \sum_{k=0}^{\infty} a^k \cos(9^k \pi x).$$

These W_a's will be the basis of our linear space. We see that each W_a is a *Weierstrass function*, since $9a > 7 > 1 + \frac{3}{2}\pi$. Let now $\frac{7}{9} < a_1 < a_2 < \ldots < a_l < 1$, $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$, define $g(x) = \sum_{i=1}^{l} \lambda_i W_{a_i}(x)$ and assume $g = 0$. We shall prove by induction that

$$\sum_{i=1}^{l} \lambda_i a_i^n = 0 \quad \text{and} \quad \sum_{i=1}^{l} \lambda_i \frac{a_i^{n+1}}{1-a_i} = 0,$$

for all $n \in \mathbb{N}$, which would prove $\lambda_i = 0$ for all $1 \le i \le l$ since we would have a Vandermonde-like determinant.

First, start with $n = 0$. We have

$$W_a(\frac{1}{3}) = \sum_{k=0}^{\infty} a^k \cos(\frac{9^k}{3}\pi) = \cos(\frac{\pi}{3}) + \sum_{k=1}^{\infty} a^k \cos(3^{k-1}3^k\pi) = \cos(\frac{\pi}{3}) - \frac{a}{1-a}$$

for every a with $\frac{7}{9} < a < 1$.
Hence,

$$g(\frac{1}{3}) = \sum_{i=1}^{l} \lambda_i \left(\cos\frac{\pi}{3} - \frac{a_i}{1-a_i}\right) = \cos\frac{\pi}{3}\left(\sum_{i=1}^{l}\lambda_i\right) - \sum_{i=1}^{l}\frac{\lambda_i a_i}{1-a_i} = 0.$$

Similarly,

$$g(\frac{1}{9}) = \sum_{i=1}^{l} \lambda_i \left(\cos\frac{\pi}{9} - \frac{a_i}{1-a_i}\right) = \cos\frac{\pi}{9}\left(\sum_{i=1}^{l}\lambda_i\right) - \sum_{i=1}^{l}\frac{\lambda_i a_i}{1-a_i} = 0,$$

which implies

$$\left(\sum_{i=1}^{l}\lambda_i\right)\left(\cos\frac{\pi}{9} - \cos\frac{\pi}{3}\right) = 0$$

from which $\sum_{i=1}^{l} \lambda_i = 0$ and $\sum_{i=1}^{l} \frac{a_i \lambda_i}{1-a_i} = 0$. Assume now $\sum_{i=1}^{l} \lambda_i a_i^n = 0$ and $\sum_{i=1}^{l} \lambda_i \frac{a_i^{n+1}}{1-a_i} = 0$, for all $0 \leq n \leq m$. Then,

$$\begin{aligned}W_{a_i}(\frac{1}{9^{m+2}}) &= \sum_{k=0}^{\infty} a_i^k \cos \frac{9^k \pi}{9^{m+2}} \\ &= \cos \frac{\pi}{9^{m+2}} + a_i \cos \frac{\pi}{9^{m+1}} + \ldots + a_i^{m+1} \cos \frac{\pi}{9} - \frac{a_i^{m+2}}{1-a_i}.\end{aligned}$$

Hence, using the induction hypothesis $\sum_{i=1}^{l} \lambda_i a_i^n = 0$ for all $0 \leq n \leq m$,

$$\begin{aligned}g(\frac{1}{9^{m+2}}) &= \sum_{i=1}^{l} \lambda_i W_{a_i}(\frac{1}{9^{m+2}}) \\ &= \sum_{i=1}^{l} \lambda_i \left[\cos \frac{\pi}{9^{m+2}} + a_i \cos \frac{\pi}{9^{m+1}} + \ldots + a_i^{m+1} \cos \frac{\pi}{9} - \frac{a_i^{m+2}}{1-a_i}\right] \\ &= \left(\sum_{i=1}^{l} \lambda_i a_i^{m+1}\right) \cos \frac{\pi}{9} - \sum_{i=1}^{l} \lambda_i \frac{a_i^{m+2}}{1-a_i} = 0,\end{aligned} \tag{2.7}$$

which allows us to conclude, using the induction hypothesis $\sum_{i=1}^{l} \lambda_i \frac{a_i^{n+1}}{1-a_i} = 0$, that

$$\left(\sum_{i=1}^{l} \lambda_i a_i^{m+1}\right) \cos \frac{\pi}{9} - \sum_{i=1}^{l} \lambda_i \frac{a_i^{m+2}}{1-a_i} + \sum_{i=1}^{l} \lambda_i \frac{a_i^{m+1}}{1-a_i} = 0.$$

Then,

$$\left(\sum_{i=1}^{l} \lambda_i a_i^{m+1}\right) \cos \frac{\pi}{9} - \sum_{i=1}^{l} \lambda_i \frac{a_i^{m+1}(a_i - 1)}{1-a_i} = \left(\sum_{i=1}^{l} \lambda_i a_i^{m+1}\right) (\cos \frac{\pi}{9} + 1) = 0.$$

Using this last result we have $\sum_{i=1}^{l} \lambda_i a_i^{m+1} = 0$, which, together with the conclusion in (2.7), yields $\sum_{i=1}^{l} \lambda_i \frac{a_i^{m+2}}{1-a_i} = 0$. This proves the linear independency of the W_a's. Assume now that $\lambda_1 \cdot \ldots \cdot \lambda_l \neq 0$, $\frac{7}{9} < a_l < a_{l-1} < \ldots < a_1 < 1$ and $g(x) := \sum_{i=1}^{l} \lambda_i W_{a_i}(x)$ is differentiable at $x_0 \in \mathbb{R}$. Then, following the proof of Theorem 2.1 and focusing on the equations (2.4) and (2.6), we have that for each $m \in \mathbb{N}$ and each $1 \leq i \leq l$ there exist $\epsilon_{1,m}^i$, $\epsilon_{1,m}^i \in [-1, 1]$ and an $\eta_m^i \geq 1$ such that

$$\begin{aligned}g'(x_0) &= \lim_{m \to \infty} \left[(-1)^{a_m} \sum_{i=1}^{l} \lambda_i (9a_i)^m \left(\frac{\epsilon_{1,m}^i \pi}{9a_i - 1} + \eta_m^i \frac{2}{3}\right)\right] \\ &= \lim_{m \to \infty} \left[-(-1)^{a_m} \sum_{i=1}^{l} \lambda_i (9a_i)^m \left(\frac{\epsilon_{2,m}^i \pi}{9a_i - 1} + \eta_m^i \frac{2}{3}\right)\right]\end{aligned}$$

and putting both limits together we obtain

$$\lim_{m \to \infty} \left[\sum_{i=1}^{l} \lambda_i (9a_i)^m \left(\frac{\pi}{9a_i - 1}(\epsilon_{1,m}^i + \epsilon_{2,m}^i) + \frac{4}{3}\eta_m^i\right)\right] = 0.$$

In other words,

$$\lim_{m\to\infty}\left[(9a_1)^m\sum_{i=1}^{l}\lambda_i\left(\frac{a_i}{a_1}\right)^m\left(\frac{\pi}{9a_i-1}(\epsilon^i_{1,m}+\epsilon^i_{2,m})+\frac{4}{3}\eta^i_m\right)\right]=0.$$

Now, if we recall the steps of the proof of Theorem 2.1 where η^i_m was to appear and we have in mind that $1+x_{m+1}\geq\frac{1}{2}$, then

$$\begin{aligned}\eta^i_m\frac{4}{3}&=2\sum_{k=0}^{\infty}a_i^k\frac{1+\cos(9^k\pi x_{m+1})}{1+x_{m+1}}\\&\leq 4\sum_{k=0}^{\infty}a_i^k[1+\cos(9^k\pi x_{m+1})]\leq 8\frac{1}{1-a_i}\leq\frac{8}{1-a_1}<\infty.\end{aligned}$$

Hence, by the conclusion in equation (2.1) and the above, we have that

$$\frac{\pi}{9a_i-1}(\epsilon^i_{1,m}+\epsilon^i_{2,m})+\frac{4}{3}\eta^i_m$$

is bounded above and below (by an strictly positive quantity), for all $1\leq i\leq l$. Also, since $0<a_i<a_1$ for all $2\leq i\leq l$ and $9a_1>1$, we obtain that

$$\lim_{m\to\infty}\left[(9a_1)^m\sum_{i=1}^{l}\lambda_i\left(\frac{a_i}{a_1}\right)^m\left(\frac{\pi}{9a_i-1}(\epsilon^i_{1,m}+\epsilon^i_{2,m})+\frac{4}{3}\eta^i_m\right)\right]=\operatorname{sign}(\lambda_1)\cdot\infty,$$

contradicting the conclusion that the upper limit is null. ∎

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Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040 (Spain).

email:pablo.jimenez.rod@gmail.com, gustavo_fernandez@mat.ucm.es,
jseoane@mat.ucm.es