Maps with dense orbits: Ansari's theorem revisited and the infinite torus

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Abstract

Let B be a Banach space and T a bounded linear operator on B. A celebrated theorem of Ansari says that whenever T is hypercyclic so is any power T^n. We provide a very natural proof of this theorem by building on an approach by Bourdon. We also explore an extension to a non linear setting of a theorem of León-Saavedra and Müller which says that for $\lambda \in \mathbb{C}$ and $|\lambda| = 1$ the operator λT is hypercyclic whenever T is.

1 Introduction

A continuous linear operator T in a Fréchet space F is hypercyclic if there exists $x \in F$ such that $\mathrm{Orb}(T, x) := \{T^k x : k = 0, 1, 2, \ldots\}$ is dense in F. The properties of this type of operators have been intensively studied in the last decade. The Oberwolfach report [16] gives a good sample of results recently obtained in the area. It illustrates also the speed with which some of the main problems have been resolved and new ones are investigated. At the writing of the report one of the main problems was whether every hypercyclic operator satisfies the Hypercyclicity Criterion; soon after a paper by De la Rosa and Read [8] answered it in the negative. However, the importance of the Hypercyclicity Criterion is that if an operator satisfies it then it is hypercyclic. The first version of it was given by Kitai in [13] and by Gethner and Shapiro in [9].

In [1] Ansari solved a question posed in [13] by proving her beautiful theorem: If an operator T is hypercyclic so is every power T^n for any $n \in \mathbb{N}$. Bourdon [5]

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was inspired by this result and in trying to find a less mysterious way of proving it, he was able to prove it in a more transparent way when $n = 2$. In the next section we show how to push his method for the general case $n \in \mathbb{N}$. The proof is valid for Fréchet spaces.

Once we have a hypercyclic operator, Ansari's theorem provides us with a plethora of them. Also, as was shown in [20], when combined with other arguments it might give different classes of hypercyclic operators. For instance, the spectral mapping theorem might show that T^n and T^m are not similar to each other whenever $n \neq m$.

Let $\mathbb{T} = \{\alpha : \alpha \in \mathbb{C},\ |\alpha| = 1\}$. León-Saavedra and Müller proved in [14] that if T is a hypercyclic Banach space operator, then the operator λT is also hypercyclic whenever λ is in the unit circle $\mathbb{T}$. This motivates the question in Section 3. This last section shows some partial answer to the question; our setting is the infinite torus.

2 Iterates of a map with a dense orbit

Let X be a topological space and let $T : X \longrightarrow X$ be continuous. For an integer $n \geq 1$, we denote by T^n the nth iterate of T: $T \circ T \circ \cdots \circ T$, n times, and take T^0 to be the identity map. A subset A of X is called invariant under T if $T(A) \subseteq A$.

The following theorem is the key for our proof of Theorem 2.2.

Theorem 2.1 (Separation Theorem for prime n). *Let $T : X \longrightarrow X$ be a continuous map. Assume that there exists $x \in X$ such that $\mathrm{Orb}(T, x)$ is dense in X. Suppose also that*

$$D := \{y \in X : \mathrm{Orb}(T, y) \text{ is dense in } X\}$$

is invariant under T. If $\mathrm{Orb}(T^n, x)$ is not dense in X for some prime $n \geq 2$, then D is separated by n pairwise disjoint open sets G_i, that is,

$$D = \bigcup_{i=1}^{n} (D \cap G_i),$$

and $T(D \cap G_i) \subseteq D \cap G_{i+1}$ for $1 \leq i \leq n-1$, $T(D \cap G_n) \subseteq D \cap G_1$. Consequently, each $D \cap G_i$ is invariant under T^n, $i = 1, 2, \ldots, n$.

Proof. Observe that $\mathrm{Orb}(T, x) \subseteq D$ since $T(D) \subseteq D$ and $x \in D$. Thus D is dense in X. For $i = 1, 2, \ldots, n$, let C_i be the closure of the set $\mathrm{Orb}(T^n, T^{i-1}x)$. Thus $C_1 \neq X$ but

$$X = \overline{\mathrm{Orb}(T, x)} = \overline{\bigcup_{i=1}^{n} \mathrm{Orb}(T^n, T^{i-1}x)} = \bigcup_{i=1}^{n} C_i.$$

It would be tempting to define G_i as the interior of C_i. A modification of this idea will work.

Step 1. The continuity of T implies

$$T(C_i) \subseteq C_{i+1} \quad \text{for } i = 1, 2, \ldots, n-1 \quad \text{and} \quad T(C_n) \subseteq C_1. \tag{1}$$

By applying (1) appropriately $1, 2, \ldots, n-1, n$ times we have, respectively,

$$T(C_n) \subseteq C_1, T^2(C_{n-1}) \subseteq C_1, \ldots, T^{n-1}(C_2) \subseteq C_1 \quad \text{and} \quad T^n(C_1) \subseteq C_1.$$

That is,

$$T^r(C_{n-r+1}) \subseteq C_1, \quad r = 1, 2, \ldots, n. \tag{2}$$

Using the same arguments we obtain

$$T^n(C_i) \subseteq C_i, \quad i = 1, 2 \ldots, n.$$

Hence it is clear that for $m = 0, 1, 2, \ldots$,

$$T^{mn}(C_i) \subseteq C_i \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{3}$$

Step 2. We will show that

$$D \cap C_1 \cap C_2 \cdots \cap C_n = \emptyset. \tag{4}$$

Assume that $z \in D \cap C_1 \cap C_2 \cdots \cap C_n$. Let $k = mn + r$ be a nonnegative integer with $0 \leq r < n$. Since $z \in C_{n-r+1}$ for $r > 0$ and $T^0 z = z \in C_1$, by (2) $T^r z \in C_1$, and this together with (3) implies that $T^k z = T^{mn}(T^r z) \in C_1$ for $k = 0, 1, 2, \ldots$. Thus we obtain $X = \overline{\mathrm{Orb}(T, z)} \subseteq C_1$, which is a contradiction.

Step 3. There exist an integer $r \in [1, n-1]$ and $A \subset \{1, 2, \ldots n\}$ with Card(A)= r such that

$$D \cap (\bigcap_{i \in A} C_i) \neq \emptyset \quad \text{but} \quad D \cap (\bigcap_{i \in B} C_i) = \emptyset \tag{5}$$

whenever $B \subset \{1, 2, \ldots, n\}$ and Card(B)$> r$.

The set

$$M := \{s : s \leq n \text{ and } D \cap (\bigcap_{i \in B} C_i) = \emptyset \text{ whenever } \mathrm{Card}(B) = s\}$$

satisfies that $n \in M$ because of (4), and $1 \notin M$ since $x \in C_1 \cap D$. Thus, we have obtained

$$r = \min M - 1.$$

According to equation (5) let $z \in D \cap (\bigcap_{i \in A_1} C_i)$ with $A_1 \subset \{1, 2, \ldots, n\}$ and Card $(A_1) = r$. In the next step we will need the following.

Fact: For $i = 1, 2, \ldots, n$ the sets $A_i := \{s + i - 1 \ (\mathrm{mod}\ n) : s \in A_1\}$ are all different.

Since card(A_1)=$r < n$, the sets A_1 and A_2 are different.

Define the maps $L^j : \{A_1, A_2, \ldots, A_n\} \longrightarrow \{A_1, A_2, \ldots, A_n\}$ by $L^j(A_k) := A_{k+j\ (\mathrm{mod}\ n)}$. The cyclic group $\{L^j : j = 0, 1, 2, \ldots, n-1\}$, where $L^{j_1} * L^{j_2} = L^{j_1 + j_2\ (\mathrm{mod}\ n)}$, has more than one element since $L^1(A_1) = A_2 \neq L^0(A_1) = A_1$.

Since $(L^1)^n = L^0$ is the identity of the group, the order of L^1 divides n and therefore is n since n is prime. On the other hand, the map L^1 is one-to-one and so is onto, and consequently L^j is bijective for all j. Thus we can consider $\{L^j : j = 0, 1, 2, \ldots, n-1\}$ as a subgroup of the group of permutations S_m, where

m =Card$\{A_1, A_2, \dots A_n\}$. Consequently, by Lagrange's theorem, $n = \text{order}(\mathrm{L}^1)$ divides $m!$, and therefore $m = n$.

This means that the sets A_i are all different for $i = 1, 2, \dots, n$.

Step 4. With A_i as above, let us define

$$K_i := \bigcap_{j \in A_i} C_j$$

with $i = 1, 2, \dots, n$. Then by (1) we have

$$T(K_i) \subseteq K_{i+1} \quad \text{for } i = 1, 2, \dots, n-1 \quad \text{and} \quad T(K_n) \subseteq K_1, \tag{6}$$

and recall that $z \in D \cap K_1$.

Therefore the invariance of D under T and (6) imply that $T^{i-1}z \in D \cap K_i$ for $i = 1, 2, \dots, n$. Since (3) implies that $T^{mn} K_i \subset K_i$ for $m = 0, 1, 2, \dots$, we deduce that $\text{Orb}(T, z) \subseteq \bigcup_{i=1}^n K_i$. Then since each K_i is closed

$$X = \bigcup_{i=1}^n K_i \tag{7}$$

and thus $D = \bigcup_{i=1}^n (D \cap K_i)$.

Furthermore, since $\text{Card}(A_i \cup A_j) > r$ for $i \neq j$, (5) implies that

$$D \cap K_i \cap K_j = \emptyset. \tag{8}$$

So we have obtained a partition of D in n non-empty relatively closed sets.

Step 5. We are finally ready to obtain the promised open sets G_i. Set

$$G_i := X \setminus \bigcup_{j \neq i} K_j$$

for $i = 1, 2, \dots, n$. Due to (7),

$$G_i = K_i \setminus \bigcup_{j \neq i} K_j, \quad i = 1, 2, \dots, n. \tag{9}$$

Then G_i is open and for $i \neq j$

$$G_i \cap G_j = \emptyset.$$

For $i = 1, 2, \dots, n$ we have by (8) and (9),

$$D \cap G_i = D \cap K_i. \tag{10}$$

Thus

$$D = \bigcup_{i=1}^n (D \cap K_i) = \bigcup_{i=1}^n (D \cap G_i).$$

Finally, equations (6) and (10) imply

$$T(D \cap G_1) \subseteq D \cap G_2, T(D \cap G_2) \subseteq D \cap G_3, \dots, T(D \cap G_n) \subseteq D \cap G_1.$$

It follows that every set $D \cap G_i$ is nonempty and invariant under T^n. ∎

Note that if X is T_1 and without isolated points, then D is invariant under T.

Remark. We remark that Theorem 2.1 can also be obtained as a consequence of a result of Banks, Theorem 2.3 in [2]; however, the derivation of Theorem 2.1 in this way requires basically the same effort as obtaining it directly.

It is intriguing to speculate why Bourdon's approach in [5] was not pursued by other people. A possible explanation is that he and Feldman proved in [6] another beautiful and more encompassing theorem which also implies results previously obtained by Costakis [7] and by Peris [18]. Moreover, in [23] Wengenroth extended the results of [6] to non-locally convex topological vector spaces. We wrote the preceding speculations in the original version of this paper, but the referee informs us that Bourdon's approach has very recently been pursued by Grosse-Erdmann, León-Saavedra and Piqueras-Lerena [10]

As a consequence of the Separation Theorem we recover, in a more general setting but which is also contained in [6], Ansari's Theorem. For a continuous linear map T on a Fréchet space X, let

$$H_T := \{x : \overline{\mathrm{Orb}(T,x)} = X\}.$$

Theorem 2.2 (Ansari). *Let X be a Fréchet space and let T be a continuous linear map acting on X. If T is hypercyclic, then so is T^n for all $n \in \mathbb{N}$. Moreover, the sets of hypercyclic vectors for T and T^n coincide; i.e, $H_T = H_{T^n}$.*

Proof. For prime n, we argue as in Theorem 3.4 of [5]. Assume that the vector x is hypercyclic for T but not for T^n. We keep (until we get a contradiction) the notation for D and G_i of Theorem 2.1. Let $z \in D \cap G_1$. Since $z \in D$ the set

$$E = \{p(T)(z) : p \text{ is a polynomial}\} \setminus \{0\}$$

is also contained in D and is connected, see [4] for the complex case and [3] for the real case. On the other hand, $T^{i-1}z \in E \cap (G_i \cap D)$, $i = 1, 2, \ldots, n$. Thus the $G_i's$, with $i = 1, 2, \ldots, n$, separate not only D but also E, which is impossible. Thus we have proved that $D = H_T \subseteq H_{T^n}$. Since the other inclusion is immediate because $\mathrm{Orb}(T^n, x) \subset \mathrm{Orb}(T, x)$, we have equality $H_T = H_{T^n}$ for n prime.

The general case follows from Theorem 2.4 of [2]. We provide a proof for the sake of completeness. Assume by induction that $H_T = H_{T^l}$ whenever $1 \leq l < n$. Then we need to consider only the case in which n is not prime. Let $n = mp$ with p prime. Since $m < n$ we deduce that $H_T = H_{T^m}$, but since $T^n = (T^m)^p$ we conclude that $H_T = H_{T^m} = H_{(T^m)^p} = H_{T^n}$. ∎

The example in p. 1580 of [5] can be extended to the general case when p is prime by considering p different circles in the complex plane which are tangent in one point. Concretely, let $X = \bigcup_{k=1}^{p} C_k$ where $C_k = \{-k + 1 + kz : z \in \mathbb{T}\}$ for $k = 1, 2, \ldots, p$. (Recall that $\mathbb{T}$ is the unit circle.) Let $f : X \longrightarrow X$ be defined in C_k as $f(-k + 1 + kz) = -k + (k+1)z^2$ for $1 \leq k < p$ and $f(-p + 1 + pz) = z^2$. Calculations show that $f(C_k) = C_{k+1}$ for $1 \leq k < p$ and $f(C_p) = C_1$ and also $f^p(C_k) \subset C_k$ for $k = 1, 2, \ldots, p$.

Thus for each p prime there is a connected metric space X without isolated points and a function $f : X \longrightarrow X$ with a dense orbit but f^p doesn't have a dense orbit.

For a map $T : X \longrightarrow X$ there is another concept related to having dense orbit, which is (topological) transitivity. Actually for a compact metric space, transitivity is defined in p. 127 [22] as having a dense orbit. Another, more common definition, is the one given in [2]: given two nonempty open sets U, V there is $k > 0$ such that $T^k(U) \cap V \neq \emptyset$. We point out that Lemma 1.1 in [2] is not correct with this definition, but it becomes correct if k is allowed to be 0. Peris showed in [17] that if T is transitive on a Baire subspace X with at most one point of discontinuity, then T still has a dense orbit.

In [19] there is an example of a linear but unbounded operator T which is hypercyclic but neither T^2 nor $-T$ is hypercyclic, which shows that neither theorem mentioned in the abstract is valid for unbounded operators.

3 The infinite torus

The theorem of León-Saavedra and Müller [14] mentioned in the introduction is true also in the context of composition operators C_φ on $H(U) = \{$holomorphic functions on $U\}$, where φ is a holomorphic self map on the open unit disk U. Shapiro showed in [21] that C_φ is hypercyclic if and only if φ is univalent and doesn't have a fixed point in U. Moreover, he showed that these operators are chaotic, which means that they have in addition a dense set of periodic points. In [24] Yousefi and Rezaei studied hypercyclicity for weighted composition operators $M_f C_\varphi$ on $H(U)$, where $f \in H(U)$. In particular, they showed that when $\lambda \in \mathbb{T}$ and C_φ is hypercyclic so is λC_φ.

Question: Suppose that a topological space X admits a multiplication $M : \mathbb{T} \times X \longrightarrow X$, defined by $M(\lambda, x) = \lambda x$ such that $1x = x$ and $(\lambda\mu)x = \lambda(\mu x)$ which is a continuous mapping in both variables. Let $f : X \longrightarrow X$ be a map with a dense orbit. What are the conditions on X and f for which λf has also a dense orbit for all $\lambda \in \mathbb{T}$?

An example of such an X is when it is a subset of a topological vector space over the complex numbers and such that it is invariant under multiplying by λ for all $\lambda \in \mathbb{T}$.
Another example is the regular 2-torus since it can be seen as $\mathbb{T} \times \mathbb{T}$. (The unit circle $\mathbb{T}$ is the 1-torus.) More generally we can consider the n-torus and the infinite torus

$$\mathbb{T}^n = \underbrace{\mathbb{T} \times \cdots \times \mathbb{T}}_{n} \qquad \text{and} \qquad \mathbb{T}^\infty = \prod_{j \in J} X_j,$$

with J an infinite countable set, $X_j = \mathbb{T}$ for all $j \in J$, and $\mathbb{T}^\infty$ is equipped with the product topology. Thus $\mathbb{T}^\infty$ is an abelian topological group with multiplication coordinate-wise and with normalized Haar measure. In the sequel, until the last comments, we assume that the index set J is $\mathbb{N}$.

An easy example showing that the answer to the question is not always positive is the map $f : \mathbb{T} \longrightarrow \mathbb{T}$ defined by $f(z) = \alpha z$, where $\alpha = e^{2\pi t i}$ is such that $t \in \mathbb{R} \setminus \mathbb{Q}$. The fact that f has a dense orbit was already known to Dirichlet in 1845, p. 157 in [11]; but $\alpha^{-1} f = I$, which doesn't have a dense orbit. More generally, we have

Proposition 3.1. *For each $j \in \mathbb{N}$, let $\alpha_j = e^{2\pi t_j i}$ be such that the sets $\{1, t_1, \dots, t_n\}$ are linearly independent over $\mathbb{Q}$ for all $n \in \mathbb{N}$. Let $f : \mathbb{T}^\infty \longrightarrow \mathbb{T}^\infty$ be defined as $f(z_1, z_2, \dots) = (\alpha_1 z_1, \alpha_2 z_2, \dots)$. Then f has a dense orbit but $\alpha_j^{-1} f$ doesn't have a dense orbit for all $j \in \mathbb{N}$.*

Proof. We claim that the point $(1, 1, \dots)$ has a dense orbit under f. Let

$$(z_1, z_2, \dots, z_m, z_{m+1}, \dots)$$

be a point of $\mathbb{T}^\infty$. Let $\epsilon > 0$. Since $\mathbb{T}^\infty$ has the product topology, it is enough to find an n such that $|\alpha_j^n - z_j| < \epsilon$ for $j = 1, 2, \dots, m$. But this is what Kroenecker's theorem says, p. 158 in [11] or p. 381 in [12]. ∎

An affine map in $\mathbb{T}^\infty$ is of the form $L(z) = aB(z)$, where $a \in \mathbb{T}^\infty$ is fixed and B is a continuous endomorphism. If B is the identity, the map is called a rotation. The map in the proposition above is a rotation with $a = (\alpha_1, \alpha_2, \dots)$.

Let $(X, \mathcal{M}, \mu)$ be a probability space and let V be a measure-preserving map on X, i.e., $\mu(V^{-1}(A)) = \mu(A)$ for every measurable set $A \in \mathcal{M}$. Recall that V is called *ergodic* if whenever $A \in \mathcal{M}$ is invariant under V then $\mu(A)$ is either 0 or 1. Theorem 1.11 of [22] says that if X is a connected, metric, compact abelian group, μ is its normalized Haar measure and V is an affine map, then V being ergodic is equivalent to having a point with dense orbit, in which case the set of points with dense orbit has measure 1. Thus the map in the proposition above is ergodic with respect to the Haar measure in $\mathbb{T}^\infty$.

It is well-known that the only continuous endomorphisms of the unit circle $\mathbb{T}$ are of the form $\varphi(z) = z^n$ with $n \in \mathbb{Z}$. In the sequel we only consider continuous endomorphisms of either $\mathbb{T}^n$ or $\mathbb{T}^\infty$. In p. 15 of [22] the endomorphisms of the n-torus $\mathbb{T}^n$ are characterized in terms of $n \times n$ matrices with integers entries. This comes from identifying $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$ by the map

$$I_n(e^{2\pi t_1}, \dots, e^{e\pi t_n}) = (t_1, \dots, t_n); \tag{11}$$

observe that I_n is an isomorphism. A homomorphism of $\mathbb{T}^n$ is an epimorphism if the determinant of the corresponding matrix is non zero. It is an automorphism if the determinant is either 1 or -1. The ergodic endomorphisms are the epimorphisms for which the associated matrices don't have unit roots as eigenvalues. In particular for $\mathbb{T}$, if $|n| > 1$ then $\varphi(z) = z^n$ is ergodic. So, in the case of $\mathbb{T}$ the automorphisms are not ergodic; however, starting with the regular torus $\mathbb{T}^2$, automorphisms can be ergodic. An example is the map $B(z_1, z_2) = (z_1^3 z_2^7, z_1^2 z_2^5)$, whose associated matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}.$$

Several properties of ergodic endomorphisms of $\mathbb{T}^\infty$ have been studied, see for instance Lind [15]. We now characterize the endomorphisms of the infinite torus in a similar way by identifying the associated infinite matrices with integer entries. For instance the endomorphism

$$B(z_1, z_2, z_3, \dots) = (z_1^2, z_1 z_3^4, z_1^{-2} z_4, \dots)$$

has the associated matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 4 & 0 & 0 & \cdots \\ -2 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We start out by identifying the characters of $\mathbb{T}^\infty$, i.e., the homomorphisms of $\mathbb{T}^\infty$ into the unit circle $\mathbb{T}$. Let φ be such a character and let

$$i_m(z_1, \ldots, z_m) = (z_1, \ldots, z_m, 1, 1, \ldots) \tag{12}$$

be the natural injection of $\mathbb{T}^m$ into $\mathbb{T}^\infty$. Then $\varphi \circ i_m$ is a character of the m-torus. In p. 14 of [22] these characters are identified, and therefore

$$\varphi(z_1, \ldots, z_m, 1, 1, \ldots) = \varphi \circ i_m(z_1, \ldots, z_m) = \prod_{j=1}^{m} z_j^{n_j}.$$

Since $(z_1, \ldots, z_m, 1, 1, \ldots)$ goes to $(z_1, \ldots, z_m, z_{m+1}, z_{m+2}, \ldots)$, the continuity of φ implies

$$\varphi(z_1, \ldots, z_m, z_{m+1}, z_{m+2}, \ldots) = \prod_{j=1}^{\infty} z_j^{n_j},$$

but the only way that this is possible is that there exists l such that $0 = n_j$ for $l < j$. Thus

$$\varphi(z_1, \ldots, z_m, z_{m+1}, z_{m+2}, \ldots) = \prod_{j=1}^{l} z_j^{n_j}. \tag{13}$$

For our next results we need the projection of $\mathbb{T}^\infty$ on the first n coordinates:

$$p_n(z_1, \ldots, z_n, z_{n+1}, \ldots) = (z_1, \ldots, z_n). \tag{14}$$

Proposition 3.2. *Let B be an endomorphism of $\mathbb{T}^\infty$ and let A be its associated matrix. Then*
(i) *Each row of A has only a finite number of non-zero integer entries. Moreover, each matrix with this property is the associated matrix of an endomorphism.*
(ii) *B is an epimorphism if and only if the set of rows of A is linearly independent.*

Proof. To see (i) consider the projection on the m-th coordinate $\pi_m : \mathbb{T}^\infty \longrightarrow \mathbb{T}$,

$$\pi_m(z_1, \ldots, \underbrace{z_m}_{m}, \ldots) = z_m.$$

Thus $\pi_m \circ B$ is a character of $\mathbb{T}^\infty$, and therefore by (13) there is $k(m) \in \mathbb{N}$,

$$\pi_m \circ B(z_1, \ldots, z_m, z_{m+1}, z_{m+2}, \ldots) = \prod_{j=1}^{k(m)} z_j^{a_{mj}},$$

but this means that the m row of A is $(a_{m1}, a_{m2}, \ldots, a_{mk(m)}, 0, 0, \ldots)$.

We now prove (ii). Assume that the set of rows of A is linearly independent. We want to show that given $W \in \mathbb{T}^\infty$ there is $Z \in \mathbb{T}^\infty$ such that $B(Z) = W$. Consider the first n rows of A and let $m = \max\{k(j) : j = 1, \ldots, n\}$ (where $a_{jk(j)} \neq 0$). Since the n rows are linearly independent, it follows that $n \leq m$ and that the matrix A_{nm} consisting of the first n rows and m columns of A is a surjective linear transformation from $\mathbb{R}^m$ onto $\mathbb{R}^n$. Let $I_n(p_n(W)) = Y_n$, where I_n is given by equation (11) and $p_n(W)$ is given by (14) and so it has the first n coordinates of W. Then there is a solution $A_{nm}(X_m) = Y_n$. Let $Z_m \in \mathbb{T}^m$ be such that $I_m(Z_m) = X_m$. Since the infinite torus is compact there is a sequence $\{m_q\}_{q=1}^\infty$ such that $i_{m_q}(Z_{m_q})$, given by equation (12), converges to a $Z \in \mathbb{T}^\infty$. On the other hand, $i_{n_q}(p_{n_q}(W))$ goes to W and so does $B(i_{m_q}(Z_{m_q}))$ since its first n_q coordinates are the same as the first n_q coordinates of $i_{n_q}(p_{n_q}W)$. By continuity $B(i_{m_q}(Z_{m_q}))$ converges to $B(Z) = W$.

For the converse, assume that among the first n rows there is one, say the i row, which is linearly dependent on the remaining $n-1$ rows. This means that there exists $W \in \mathbb{T}^\infty$, whose first n coordinates are 1 except the i coordinate, which cannot be in $B(\mathbb{T}^\infty)$. ∎

Proposition 3.3. *Let B be a homomorphism such that its associated matrix A has (as a partitioned matrix) the form*

$$\mathbf{A} = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & \cdots \\ A_{21} & A_{22} & 0 & 0 & 0 & \cdots \\ A_{31} & A_{32} & A_{33} & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where A_{ii} is a finite square matrix for each $i \in \mathbb{N}$. Then
(i) *B is an epimorphism if and only if each A_{ii} has a non zero determinant.*
(ii) *B is an automorphism if and only if each A_{ii} has a determinant which is either* 1 *or* -1.
(iii) *B is ergodic if and only if it is an epimorphism and the eigenvalues of each A_{ii} are not roots of unit.*

Proof. By using Proposition 3.2 we can prove (i). But (i), (ii) and (iii) can be proved as a limit case of finite tori. We prove (iii) by using an argument similar to one used in [9]. Specifically, we will show that B has a dense orbit, which by Theorem 1.11 of [22] means that B is ergodic. Actually, we will show that the set of points with dense orbit is a G_δ dense set. Let $\{V_j : j = 1, 2, \ldots\}$ be a family of open sets which is a basis for the topology of $\mathbb{T}^\infty$ and for each j there is an L_j such that

$$V_j = p_{L_j}(V_j) \times \underbrace{\mathbb{T}}_{L_j+1} \times \mathbb{T} \times \cdots .$$

Set $W(j, N) := \bigcup_{n=N}^\infty \{x : B^n(x) \in V_j\}$. We will show that this set, which is open because B is continuous, is also dense. Moreover

$$\bigcap_{j,N=1}^\infty W(j, N) = \{x : \overline{\mathrm{Orb}(B, x)} = \mathbb{T}^\infty\}$$

is a G_δ dense set by Baire Category Theorem. Let $A_{i,i}$ be a $d_i \times d_i$ matrix and $L = d_1 + \cdots + d_q$. Let $B_L = p_L \circ B \circ i_L$, where p_L is given by (14) and i_L is given by (12). Thus B_L is the ergodic endomorphism of $\mathbb{T}^L = p_L(\mathbb{T}^\infty)$ whose associated matrix is

$$\mathbf{C} = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ A_{q1} & A_{q2} & A_{q3} & A_{q4} & A_{q5} & \cdots & A_{qq} \end{pmatrix},$$

(Observe that C is the northwest corner of A consisting of the first L rows and columns.) Then $p_L \circ B^n(x) = B_L^n \circ p_L(x)$ for every $x \in \mathbb{T}^\infty$.

Given $W(j, N)$ and V_k we want to find $x \in \mathbb{T}^\infty$ such that $x \in W(j, N) \cap V_k$. Let $L =$max$\{L_j, L_k\}$. The ergodicity of B_L implies that there exists $x \in \mathbb{T}^\infty$ such that $p_L(x) \in p_L(V_k)$ and $B_L^n(p_L(x)) \in p_L(V_j)$ for some $n > N$. This implies that $x \in V_k$ and $B^n(x) \in V_j$, that is $x \in W(j, N) \cap V_k$. ∎

The following result can be compared to Proposition 2.4 of [24].

Proposition 3.4. *Let B be an ergodic endomorphism of $\mathbb{T}^\infty$ and $a \in \mathbb{T}^\infty$. Assume further that the endomorphism C defined by $C(x) = x^{-1}B(x)$ is surjective. Then the affine map $L = aB$ is also ergodic and in particular λL is ergodic for all $\lambda \in \mathbb{T}$.*

Proof. Since the infinite torus is a connected, metric, compact and abelian group, we can use equivalence (ii) of Theorem 1.11 of [22]; (ii) (a) is satisfied as well as (ii) (b), the first because B is ergodic, and the second because C is an epimorphism. ∎

To conclude, we present some ergodic automorphisms of $\mathbb{T}^\infty$ whose associated matrices are not as in Proposition 3.3. Let C be such that its associated matrix is a permutation matrix with a number of finite or infinite cycles, each of which is infinite.

In what follows we consider the infinite torus $\mathbb{T}^\infty$ represented with the index set J equal to $\mathbb{Z}$. The right shift

$$B((\ldots, z_{-1}, \underbrace{z_0}_{0}, z_1, z_2, \ldots)) = (\ldots, z_{-2}, \underbrace{z_{-1}}_{0}, z_0, z_1, \ldots)$$

is ergodic. B is isomorphic to any C such that its associated matrix is a permutation matrix which has just one cycle. In the general case C is isomorphic to a right shift of finite or infinite multiplicity acting either on $\mathbb{T}^\infty = \underbrace{\mathbb{T}^\infty \times \cdots \times \mathbb{T}^\infty}_{n}$ or on $\mathbb{T}^\infty = \prod_{j=1}^\infty X_j$ with $X_j = \mathbb{T}^\infty$.

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