Convergence of Bieberbach polynomials inside domains of the complex plane

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Abstract

Let $G \subset C$ be a finite Jordan domain, $z_0 \in G$; $B \Subset G$ be an arbitrary closed disk with $z_0 \in B$, and $w = \varphi(z, z_0)$ be the conformal mapping of G onto a disk $\{w : |w| < r\}$ normalized by $\varphi(z_0, z_0) = 0$, $\varphi'(z_0, z_0) = 1$. It is well known that the Bieberbach polynomials $\{\pi_n(z, z_0)\}$ for the pair (G, z_0) converge uniformly to $\varphi(z, z_0)$ on compact subsets of the Jordan domain G. In this paper we study the speed of $\|\varphi - \pi_n\|_{C(B)} \to 0$, $n \to \infty$, in domains of the complex plane with a complicated boundary structure.

1 Introduction

Let $G \subset C$ be a finite domain bounded by a Jordan curve L ; $z_0 \in G$ and let $w = \varphi(z, z_0)$ denotes the conformal mapping of G onto $\{w : |w| < r\}$ normalized by $\varphi(z_0, z_0) = 0$, $\varphi^{'}(z_0, z_0) = 1$. Let $\wp_n$ be the class of all algebraic polynomials P_n of degree at most n,with complex coefficients and satisfying the conditions $P_n(z_0, z_0) = 0$, $P_n^{'}(z_0, z_0) = 1$. The Bieberbach polynomials $\pi_n(z, z_o)$ for the pair (G, z_0) are defined as the polynomials that minimize the norm

$$\|P_n'\|_{L_2(G)} := \left(\iint_G |P_n'(z)|^2 \, d\sigma_z\right)^{\frac{1}{2}} \tag{1.1}$$

in the class $\wp_n$. It is easy to check that π_n also minimizes the norm $\left\|\varphi^{'} - P_n'\right\|_{L_2(G)}$ in that class $\wp_n$.

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Let $B \Subset G$ be a arbitrary closed disk such that $z_0 \in B$. It is well known that if G is a Caratheodory domain, then the Bieberbach polynomials π_n converge uniformly to φ on compact subsets of G. Thus, for all $z,\ z_0 \in B \Subset G$

$$\omega_n(B) := \sup_{z,z_0 \in B, B \Subset G} |\varphi(z, z_0) - \pi_n(z, z_o)| \to 0,\ \ n \to \infty. \tag{1.2}$$

The fact of the uniform convergence of Bieberbach polynomials π_n to φ on the closure of domain G was first observed by Keldysh [17], for the domains bounded smooth Jordan curve with bounded curvature. In [17] he also constructed an example of domain, bounded by a piecewise analytic curve with one singular points where Bieberbach polynomials diverge on the boundary singular point. Therefore, the uniform convergence in $\overline{G}$ of the Bieberbach polynomials for given pair (G, z_0) depends on the geometric properties of domain G. This problem has been studied by some authors, see, for example, [2], [5], [8], [12], [15], [16] (for more references see [15]).

It is well-known in the approximation theory that, generally, the rate of approximations of a given function in the domain G is better than the rate of approximation in $\overline{G}$.For which domains is this property valid with respect to the approximations by Bieberbach polynomials? Firstly, Suetin [23] studied this problem for domains G with $\partial G \in C(p+1, \alpha)$, $p \geq 0, 0 < \alpha < 1$, and obtained following estimation for (1.2):

$$\omega_n(B) \leq const\, [dist\,(B, L)]^{-2p-6}\, n^{-2p-2\alpha}. \tag{1.3}$$

Comparing this estimation from [23, Th.'s 5.2-5.4] we see that the above property respect the rate of the convergence of Bieberbach polynomials in G and in $\overline{G}$ holds for domains $C(p, \alpha)$ in case of $p = 2$ and does not hold in $p = 1$.

In 1997 D. Gaier [13, Res. Prob. 97-1] during solving a problem about of analytic continuity of the function φ on $\overline{G}$, he asked the question: "How fast is the convergence of the π_n to φ on $B \Subset G$?"

One of the authors [6] investigated this problem in various domains of the complex plane.

In this paper, we continue to study the estimation

$$\omega_n(B) \leq const\ \delta^{-q}(B)\ \eta_n\ ,\ \delta(B) := dist(B, L), \tag{1.4}$$

where $q > 0$, and $\eta_n \to 0$, $n \to \infty$, in domains of the complex plane with a more general boundary structure, in particular for domains having exterior zero angles.

2 Main definition and results

Let G be a finite domain in the complex plane bounded by a Jordan curve $L := \partial G$, $\Omega := C\overline{G}$; $w = \Phi(z)$ be a conformal mapping of Ω onto $\Omega' := \{w : |w| > 1\}$ normalized by $\Phi'(\infty) > 0$, and $\Psi = \Phi^{-1}$.

Let us begin with some definitions. Throughout this paper, we denote by $c, c_1, c_2, ...$ positive constants, and by $\varepsilon, \varepsilon_1, \varepsilon_2, ...$ sufficiently small positive constants in general different at different occurrences, but only depending on the geometry of G.

Definition 1. *[18, p.97] The Jordan arc or curve L is called a K-quasiconformal ($K \geq 1$), if there exists a K-quasiconformal mapping f of a domain $D \supset L$ such that $f(L)$ is a line segment or circle.*

Let $F(L)$ denote the set of all sense-preserving plane homeomorphisms f of domains $D \supset L$ such that $f(L)$ is a line segment or circle and let

$$K_L := \inf\{K(f) : f \in F(L)\}$$

where $K(f)$ is the maximal dilatation of a such mapping f. Then L is quasiconformal if and only if $K_L < +\infty$. If L is K -quasiconformal, then $K_L \leq K$.

$D = \mathbb{C}$ gives the *global* definition of a K-quasiconformal arc or curve consequently. This definition is common in the literature.

At the same time, we can consider the domain $D \supset L$ as the neighborhood of the curve L. In this case, Definition 1 will be called *local definition* of a quasiconformal arc or curve. Through this work we consider the local definition. This local definition has an advantage in determining the coefficients of quasiconformality for some simple arcs and curves.

Theorem 1. *Let L be a $K-$ quasiconformal curve. Then, for every $n \geq 2$*

$$\omega_n(B) \leq c\delta^{-3}(B)n^{-\gamma}, \tag{2.1}$$

where $0 < \gamma < \frac{1}{K^4}$ is arbitrary.

Definition 2. *We say that $G \in PQ(K, \alpha, \beta)$, $K \geq 1$, $\alpha > 0$, $\beta > 0$, if $L := \partial G$ is expressed as the union of a finite number of K_j-quasiconformal arcs, $K = \max\limits_{1 \leq j \leq m} \{K_j\}$, connecting at $z_1, ..., z_m$ points, so that L is locally K-quasiconformal at z_1, and if in (x, y) local co-ordinate system with origin at z_j, $2 \leq j \leq m$, the following conditions hold:*

a) for $j = \overline{2, p}$,

$$\begin{aligned} \{z &= x + iy : a_1 x^{1+\alpha} \leq y \leq a_2 x^{1+\alpha}, 0 \leq x \leq \varepsilon_1\} \subset C\overline{G}, \\ \{z &= x + iy : |y| \geq \varepsilon_2 x,\ 0 \leq x \leq \varepsilon_1\} \subset \overline{G}. \end{aligned}$$

b) for $\overline{p+1, m}$

$$\begin{aligned} \{z &= x + iy : a_3 x^{1+\beta} \leq y \leq a_4 x^{1+\beta}, 0 \leq x \leq \varepsilon_3\} \subset \overline{G}, \\ \{z &= x + iy : |y| \geq \varepsilon_4 x, 0 \leq x \leq \varepsilon_3\} \subset C\overline{G}. \end{aligned}$$

for some certain constants $-\infty < a_1 < a_2 < \infty$, $-\infty < a_3 < a_4 < \infty$, $\varepsilon_i > 0$, $i = 1, 2$.

It is clear from Definition 2 that each domain $G \in PQ(K, \alpha, \beta)$ may have $p - 1$ exterior and $m - p$ interior zero angles. If a domain G does not have exterior zero angles ($p = 1$) (interior zero angles ($m = p$)), then we write $G \in PQ(K, 0, \beta)$ ($G \in PQ(K, \alpha, 0)$).

Theorem 2. *Let $G \in PQ(K, \alpha, \beta)$, $\alpha < 1$, $\beta \geq 0$. Then, for every $n \geq 3$, we have*

$$\omega_n(B) \leq c\delta^{-3}(B) \ln \ln n (\ln n)^{\frac{\alpha-1}{\alpha}}. \tag{2.2}$$

Theorem 3. *Let $G \in PQ(K, 0, \beta)$. Then, for every $n \geq 2$, we have*

$$\omega_n(B) \leq c\delta^{-3}(B) n^{-\gamma}, \tag{2.3}$$

where

$$0 < \gamma < \begin{cases} \frac{1}{K^4}, & if \quad \beta < \frac{K^2-1}{K^2+1}, \\ \frac{1-\beta}{(1+\beta)K^2}, & if \quad \frac{K^2-1}{K^2+1} \leq \beta < 1 \end{cases}$$

is arbitrary.

Comparing Theorem's 1, 3 with [5, Th.2.3, Th.2.4] and Theorem 2 with [8, Th.2] we see that the degree of convergence π_n to φ in G is much better than in $\overline{G}$. We also note that the degree of the $\delta(B)$ in Theorem 3 is reduced from 6 to 3 compared with [6, Theorem 2.6].

Definition 3. *We say that $G \in Q^\alpha$, $0 < \alpha \leq 1$, if*

a) $L := \partial G$ *is a quasicircle,*

b) $\Psi \in Lip\alpha$, $w \in \overline{\Omega}'$.

Theorem 4. *Let $G \in Q^\alpha$, $0 < \alpha \leq 1$. Then, for every $n \geq 2$, we have*

$$\omega_n(B) \leq c\delta^{-3}(B) n^{-\gamma}, \tag{2.4}$$

where $0 < \gamma < \frac{\alpha}{2(2-\alpha)}$ is arbitrary.

Remark 1. *1.*

2. *If G is convex, then $\Psi \in Lip1$ [21], hence $\gamma < \frac{1}{2}$.*

b) *If L is a smooth curve having continuous tangent line (the class of these curves we denote by C_θ, and write $G \in C_\theta \Leftrightarrow L \in C_\theta$), then $G \in Q^\alpha$, for all $0 < \alpha < 1$, and hence $\gamma < \frac{1}{2}$.*

c) *If L is quasi-smooth, that is, for every pair $z_1, z_2 \in L$, if $s(\ z_1, z_2)$ represents the smaller of the length of the arcs joining z_1 to z_2 on L , there exists a constant $c > 1$such that $s(z_1, z_2) \leq c\,|z_1 - z_2|$, then $\Psi \in Lip\frac{c}{(1+c)^2}$ [24], and it is an easy calculation to find γ associated with these values.*

d) *If L is "c-quasiconformal" (see, for example[19]), then $\Psi \in Lip\alpha$ for $\alpha = \frac{2(\arcsin\frac{1}{c})^2}{\pi^2 - \pi\arcsin\frac{1}{c}}$. Also, if L is an asymptotic conformal curve, then $\Psi \in Lip\alpha$ for $\alpha < 1$ [19]*

Definition 4. *We say that* $G \in Q(\nu), 0 < \nu < 1$, *if*

i) $L := \partial G$ *is quasicircle.,*

ii) For $\forall z \in L$, *there exists a* $r > 0$ *and* $0 < \nu < 1$ *such that a closed circular sector*

$$S(z; r, \nu) := \left\{\zeta : \zeta = z + re^{i\theta}, 0 \leq \theta_0 < \theta < \theta_0 + \nu\right\}$$

of radius r *and opening* $\nu\pi$ *lies in* $\overline{\Omega}$ *with vertex at* z.

It is well known that each quasicircle satisfies the condition ii). Nevertheless, this condition imposed on L gives a new geometric characterization of the curve or region. For example, if the region G^* is defined by

$$G^* := \left\{z : z = re^{i\theta}, 0 < r < 1, \frac{\pi}{2} < \theta < 2\pi\right\},$$

then the coefficient of quasiconformality K of the G^* does not obtain so easily, whereas $G^* \subset Q(\frac{1}{2})$.

Theorem 5. *Let* $G \in Q(\nu), 0 < \nu < 1,$. *Then, for every* $n \geq 2$

$$\omega_n(B) \leq c\delta^{-3}(B)n^{-\gamma}, \tag{2.5}$$

where $0 < \gamma < \frac{\nu}{2(2-\nu)}$ *is arbitrary.*

If, in addition we impose some conditions of smoothness of boundary curve $L = \partial G$, then on the right part of (2.5) their will be better degree.

Definition 5. *We say that* $G \in C_\theta(\lambda)$, *if* L *consist of the union of finite* $C_\theta-$*arc such that they have exterior angles* $\lambda_j\pi$ *at the corners where two arcs meet,* $0 < \lambda_j < 2$, $\min\limits_j \lambda_j = \lambda$.

Theorem 6. *Let* $G \in C_\theta(\lambda), 0 < \lambda < 2$. *Then, for every* $n \geq 2$

$$\omega_n(B) \leq c\delta^{-\frac{5-2\lambda}{2-\lambda}}(B)n^{-\gamma}, \tag{2.6}$$

where $0 < \gamma < \min\{1; \frac{2\lambda}{2-\lambda}\}$ *is arbitrary.*

We see that the estimation (2.6) is better than (2.5) for $0 < \lambda < 1$.

Comparing Theorem 6 with [6, Th.2.12] we see that the degree of convergence π_n to φ in G is much better than in $\overline{G}$ and the degree of the $\delta(B)$ is reduced.

3 Some auxiliary facts

We will use the notations "$a \prec b$" for $a \leq cb$ and "$a \asymp b$" if simultaneously $a \prec b$ and $b \prec a$.

For an arbitrary $z_0 \in B \Subset G$, let $w = g(z, z_0)$ be the conformal mapping of G onto the unit disk normalized by $g(z_0, z_0) = 0$, $g'(z_0, z_0) > 0$. Whenever we write $w = g(z)$, it will be understood that $w = g(z, z_0)$ for a fixed z_0.

For $t>0$, let $L_t := \{z : |g(z)| = t, \text{ if } t<1,\ |\Phi(z)| = t, \text{ if } t>1\}, L_1 \equiv L$; $G_t := intL_t$; $\Omega_t = extL_t$.

Let L be a $K-$quasiconformal curve and $D \subset C$. Then the region D can be chosen to be the region $G_{R_0} \backslash G_{r_0}$, for a certain number $1 < R_0 \leq 2$ depending on g, Φ, f and $r_0 = R_0^{-1}$[1, p.28]. In this case, it is known that the function $\alpha(.) = f^{-1}\{[\overline{f(.)}]^{-1}\}$ is a K^2-quasiconformal reflection across L as shown in [7, p.75], that is, $\alpha(.)$ is a K^2-antiquasiconformal mapping leaving points on L fixed and satisfying the conditions $\alpha(G_{\widetilde{R}} \backslash \overline{G}) \subset G \backslash \overline{G}_{r_0}$, $\alpha(G \backslash \overline{G}_{\widetilde{r}}) \subset G_{R_0} \backslash \overline{G}$ for some $1 < \widetilde{R} < R_0$, $r_0 < \widetilde{r} < 1$. By using the facts in [18, p.98], [7, p.76] we can find a $C(K)$ - quasiconformal reflection $\alpha^*(.)$ across L such that it satisfies the following:

$$|z_1 - \alpha^*(z)| \asymp |z_1 - z|\,,\ z_1 \in L,\ z \in D. \tag{3.1}$$

Lemma 1. *Let $G \in Q^\alpha$, $0 < \alpha \leq 1$; $z_0 \in B \Subset G$. Then for all u , $0 < u < R_0 - 1$, we have*

$$mes\ g[\alpha^*(G_{1+u} \backslash G), z_0] \prec \delta^{-1}(B)\delta^{\frac{1}{2(2-\alpha)}}(\zeta), \tag{3.2}$$

where $\zeta = g^{-1}(\tau, z_0) : |\tau| = \inf\{|w| : w \in g[\alpha^*(L_{1+u}), z_0]\}$.

Proof. It is obvious that

$$mes\ g[\alpha^*(G_{1+u} \backslash G), z_0)] \prec (1 - |\tau|). \tag{3.3}$$

We present the proof under several headings.

1) Let $D \cap B = \emptyset$. Since $\Psi \in Lip\ \alpha$, then $g \in Lip\ \frac{1}{2-\alpha}$ by [19], and

$$1 - |\tau| \prec d^{\frac{1}{2-\alpha}}(\zeta, L). \tag{3.4}$$

2) Let us suppose $D \cap B \neq \emptyset$. Let $d(B, L) = |z - t|\,, z \in L, t \in B$. There are two cases to be considered:

2.1) $\alpha^*(B) \cap \overline{G}_{1+u} \neq \emptyset$. In this case, [1, Cor.1.3] and (3.1) imply

$$1 - |\tau| \prec 1 \prec \left|\frac{z - \zeta}{z - t}\right|. \tag{3.5}$$

2.2) $\alpha^*(B) \cap \overline{G}_{1+u} = \emptyset$. Let $\Gamma := \Gamma(z, \zeta; B, G)$ be a family of locally rectifiable curves separating in G z, ζ from B and $\Gamma' := g(\Gamma)$; and we also set

$$z^* = \frac{1}{z - z_1}; w^* = \frac{1}{w} \tag{3.6}$$

where $z_1 \in G$ is some fixed point, such that $d(z_1, L) \geq \varepsilon$, $|z_1 - z_0| > \varepsilon$. After that the domain G is transforming in some domains G^*, $\infty \in G^*$ with a quasiconformal boundary $L^* = \partial G^*$; $z \to z^*$, $\zeta \to \zeta^*$, $t \to t^*$, $\tau \to \tau^*$; $\Gamma \to \Gamma^* := \Gamma^*(z^*, \zeta^*; z^*(B), G^*)$ and $\Gamma' \to \widetilde{\Gamma}'$.

According to [9, Th.4.2] we may write

$$m(\Gamma^*) \geq \frac{1}{2\pi} \ln c_1 \frac{|z^* - t^*|}{|z^* - \zeta^*|}, \tag{3.7}$$

where c_1 is independent of z^*, t^*, ζ^*.

On the other hand, since $g \in Lip\ \frac{1}{2-\alpha}$, then $z^* \circ g \circ w^* \in Lip\ \frac{1}{2-\alpha}$, and therefore, [10] yields

$$m(\widetilde{\Gamma}') \leq \frac{2-\alpha}{\pi} \ln \frac{c_2}{|\tau^*|-1}, \tag{3.8}$$

where c_1 is independent of τ^*. Considering the conformal invariants of the modulus from (3.6), (3.7) and (3.8), we obtain

$$1 - |\tau| \prec \left| \frac{z-\zeta}{z-t} \right|^{\frac{1}{2(2-\alpha)}}. \tag{3.9}$$

Now (3.3)- (3.5) and (3.9) provide (3.2). ■

Corollary 1. *Let $G \in Q^\alpha$, $0 < \alpha \leq 1$; $z_0 \in B \Subset G$. Then for all u, $0 < u < R_0 - 1$, we have*

$$mes\ g[\alpha^*(G_{1+u} \backslash G), z_0] \prec \delta^{-1}(B) u^{\frac{\alpha}{2(2-\alpha)}}.$$

This follows from (3.1) and [1, Cor. 1.3].

Now, we give some properties of the domains $G \in PQ(K, \alpha, \beta)$. Suppose that a domain $G \in PQ(K, \alpha, \beta)$ is given. For the sake of simplicity, but without missing the generality, we assume that $\alpha > 0$, $\beta > 0$; $p = 2$, $m = 3$, $z_2 = 1$, $z_3 = -1$; $(-1, 1) \subset G$, that the local coordinate axis in Definition 2 be parallel to OX and OY .Set $L^1 := \{z \in L : \mathrm{Im} z \geq 0\}$, $L^2 := \{z \in L : \mathrm{Im} z \leq 0\}$. Then z_1 is taken as an arbitrary point on L^2(or on L^1 subject to the chosen direction).

We recall that the domain $G \in PQ(K, \alpha, \beta)$ has interior and exterior zero angles at the nearest-neighborhood of each points $z_2 = 1$ and $z_3 = -1$ respectively. Therefore, following the arguments mentioned in [8], we can say that the function $w = g(z)$ and $w = \Phi(z)$ for the domain $G \in PQ(K, \alpha, \beta)$ satisfy the conditions described in [1, Lemma 1.1 and 1.2] at the nearest-neighborhood of the point ± 1. So, we can easily get from [1, Lemma 1.1 and 1.2], that

$$\begin{aligned} d(z, L) &\prec (1 - |g(z)|)^{K^{-2}}; \ |z-1| \prec |g(z) - g(1)|^{K^{-2}}, \\ \forall z &\in G : |z+1| > \varepsilon_1; \\ d(z, L) &\prec (|\Phi(z)| - 1)^{K^{-2}}; \ |z+1| \prec |\Phi(z) - \Phi(-1)|^{K^{-2}}, \\ \forall z &\in \Omega : |z-1| > \varepsilon_2. \end{aligned} \tag{3.10}$$

On the other hand, using the properties of the functions g and Φ at the nearest-neighborhood of the point $z_2 = 1$ and $z_3 = -1$ respectively (see [10]) we obtain

$$|z-1| \prec [-\ln|\Phi(z) - \Phi(1)|]^{-\alpha^{-1}}, \ \ |z+1| \prec [-\ln|g(z) - g(-1)|]^{-\beta^{-1}}. \tag{3.12}$$

The following two Lemma's one proves just like that of [6, Lemma's 3.7 and 3.8].

Lemma 2. *Let $G \in PQ(K, \alpha, \beta)$, $z_0 \in B \Subset G$ and $z \in G \backslash B$ be such that $|z - z_j| < \varepsilon_j$, $j = 1, 2$. Then*

$$|g(z, z_0) - g(z_j, z_0)| \prec \delta^{-\frac{1}{2}}(B) |z - z_j|^{\frac{1}{2}}.$$

Lemma 3. *Let $G \in PQ(K, \alpha, \beta)$, $z_0 \in B \Subset G$ and $\zeta \in \Omega$ be such that $|\zeta - z_j| < \varepsilon_j$, $j = 1, 2$. Then*

$$1 - \left| g(\alpha_j^*(\zeta), z_0) \right| \prec \delta^{-K^{-2}}(B) d^{K^{-2}}(\zeta, L).$$

4 Approximation by polynomials in the L_2- norm.

Suppose that a domain $G \in PQ(K,\alpha,\beta)$, $\alpha > 0$, $\beta > 0$ is given. For the sake of simplicity, but without missing the generality, we take the domain G as at in section 3.

Each L^j, $j = 1,2$, is a K_j-quasiconformal arc. Let $\alpha_j^*(.)$ be the quasiconformal reflection across L^j. Let us also set

$$\begin{aligned}
\gamma_1^1 \ &: \ = \{z = x+iy : y = \frac{2a_1+a_2}{3}(1-x)^{1+\alpha}\},\\
\gamma_1^2 \ &: \ = \{z = x+iy : y = \frac{a_1+2a_2}{3}(1-x)^{1+\alpha}\},\\
\gamma_2^1 \ &: \ = \alpha_1^*\{z = x+iy : y = \frac{2a_1+a_2}{3}(x+1)^{1+\beta}\},\\
\gamma_2^2 \ &: \ = \alpha_2^*\{z = x+iy : y = \frac{a_1+2a_2}{3}(x+1)^{1+\beta}\},
\end{aligned}$$

where constants $a_j, j = 1,2$, are taken from the Definition 2.

According to [8, Lemma 5], for all $\zeta_1,\zeta_2 \in \gamma_j^i$, we get

$$mes\ \gamma_j^i(\zeta_1,\zeta_2) \prec |\zeta_1 - \zeta_2|.$$

For an $n > N(R_0)$ big enough and an arbitrary small $\varepsilon < 1$, let us choose $R = 1 + cn^{\varepsilon-1}$ such that $1 < R < R_0$. Let us choose points $z_j^i, i,j = 1,2$, such that they are intersections of L_R with γ_j^i, and either the first point is in $\widetilde{L}_R^1 := \{z : z \in L_R, \mathrm{Im}z \geq 0\}$, or $\widetilde{L}_R^1 := L_R \backslash \widetilde{L}_R^1$ (according to motion on L_R). These points divide L_R into four parts: $L_R^1 := L_R^1(z_1^1, z_2^1)$-an arc connecting points and z_1^1, z_2^1 , $L_R^2 := L_R^2(z_2^2, z_1^2)$, $L_R^3 := L_R^3(z_1^2, z_1^1)$, $L_R^4 := L_R^4(z_2^1, z_2^2)$, $L_R := \bigcup_{j=1}^4 L_R^j$; $\Gamma_R^j := \gamma_1^j \cup \gamma_2^j \cup L_R^j$; $U_J := int(\Gamma_R^j \cup L^j)$, $\gamma_i^j(R) = \Gamma_R^j \cap \gamma_i^j$, $i,j = 1,2$.

We extend the function $w = g(z,z_0)$ to $U_1 \cup U_2$ in the following way:

$$\widetilde{g}(z,z_0) := \begin{cases} g(z,z_0), & z \in \overline{G}, \\ \frac{1}{\overline{g(\alpha_j^*(z),z_0)}}, & z \in U_j, \quad j = 1,2 \end{cases} \cdot \tag{4.1}$$

Then using the above notations, from the Cauchy-Pompeii′s formula [18, p.148] we obtain

$$\begin{aligned}
g(z,z_0) \ &= \ \frac{1}{2\pi i}\int_{L_R} \frac{f(\zeta,z_0)}{\zeta - z} d\varsigma \\
&+ \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{2\pi i}\int_{\gamma_i^j(R)} \frac{\widetilde{g}(\zeta,z_0) - g((-1)^i,z_0)}{\zeta - z} d\varsigma \\
&- \frac{1}{\pi}\iint_{U_1\cup U_2} \frac{\widetilde{g}_{\overline{\zeta}}(\zeta,z_0)}{\zeta - z} d\sigma_\zeta,
\end{aligned} \tag{4.2}$$

where

$$f(\zeta,z_0) := \begin{cases} \widetilde{g}(\zeta,z_0), & \zeta \in L_R^1 \cup L_R^1, \\ g(1,z_0) & \zeta \in L_R^3, \\ g(-1,z_0) & \zeta \in L_R^4. \end{cases}$$

Lemma 4. *Let $G \in PQ(K, \alpha, \beta)$ for some $0 < \alpha < 1$, $\beta \geq 0$; $z_0 \in B \Subset G$. Then, for any $n \geq 3$, we have*

$$\left\| \varphi'(.,z_0) - \pi_n'(.,z_0) \right\|_{L_2(G)} \prec \delta^{-\frac{1}{2}}(B)\sqrt{\ln\ln n}(\ln n)^{\frac{\alpha-1}{2\alpha}}. \tag{4.3}$$

Proof. Lemma 4 is set up analogously to Lemma [6, Lemma 4.2] . The difference is that in Lemma [6, Lemma 4.2] the domain $G \in PQ(K, \beta)$ has interior zero angles only at the points $z_2 = 1$ and $z_3 = -1$. On the other hand we consider the domain $G \in PQ(K, \alpha, \beta)$ with an interior zero angle at z_3, but having the exterior zero angle at the point z_2. By this reason, following the scheme of [6, Lemma 4.2] proof, we give the estimations relatively to the point z_2 only.

There is a polynomial $P_n(z)$ of degree $\leq n$ [22, p.142], such that

$$\begin{aligned} & \left\| g'(.,z_0) - P_n'(.,z_0) \right\|_{L_2(G)} \\ \prec \; & \frac{1}{n} + \sum_{i=1}^{2}\sum_{j=1}^{2} \left\| \int_{\gamma_i^j(R)} \frac{\widetilde{g}(\zeta,z_0) - g((-1)^i,z_0)}{(\zeta-z)^2} d_\zeta \right\|_{L_2(G)} \\ & + \left\| \iint_{U_1\cup U_2} \frac{\widetilde{g}_{\overline{\zeta}}(\zeta,z_0)}{(\zeta-z)^2} d\sigma_\zeta \right\|_{L_2(G)} \\ = \; & : \frac{1}{n} + J_1(-1) + J_2(-1) + J_3(+1) + J_4(+1) + J_5. \end{aligned} \tag{4.4}$$

The estimate for the $J_k(-1), k = 1, 2$, is set up completely analogously to the $J_k, k = 1, 2$, in [6, (4.6), (4.8)].

Since, for all $\zeta \in \gamma_2^i(R)$, $i = 1, 2$, we have

$$|\widetilde{g}(\zeta, z_0) - g((+1), z_0)| \prec \delta^{-\frac{1}{2}}(B) \, |\zeta - (+1)|^{\frac{1}{2}}$$

from (3.1) and Lemma 2, then, using [4, Lemma 5.2], we obtain

$$\begin{aligned} J_k(-1) &= \left\| \int_{\gamma_j^2(R)} \frac{\widetilde{g}(\zeta,z_0) - g(1,z_0)}{(\zeta-z)^2} d_\zeta \right\|_{L_2(G)} \prec \delta^{-\frac{1}{2}}(B) \, |\ln \ell_{j,2}| \, \ell_{j,2}^{1-\alpha}, \\ k &= 3, 4, \end{aligned} \tag{4.5}$$

where $\ell_{j,i} := mes \; \gamma_i^j(R)$, $i, j = 1, 2$. According to [1, Cor. 1.3], (3.10), (3.12) and [8, Lemma 5], we get

$$\ell_{j,2} \prec \left|1 - z_2^j\right| \prec (\ln n)^{-\alpha^{-1}}, j = 1, 2. \tag{4.6}$$

Thus, (4.5) implies

$$J_k(+1) \prec \delta^{-\frac{1}{2}}(B)\sqrt{\ln\ln n}(\ln n)^{\frac{\alpha-1}{2\alpha}}, k = 3, 4. \tag{4.7}$$

Since the Hilbert transformation

$$(Tf)(z) := -\frac{1}{\pi} \iint \frac{f(\zeta)}{(\zeta - z)^2} d\sigma_\zeta$$

is a bounded linear operator from $L_2 \to L_2$, (3.1) yields

$$J_5 \prec \left\| \widetilde{g}_{\overline{\zeta}} \right\|_{L_2(U_1\cup U_2)} \prec \left(\sum_{j=1}^{2} mes \; g(\alpha_j^*(U_j), z_0) \right)^{\frac{1}{2}}. \tag{4.8}$$

For a sufficiently large c and small $0<\varepsilon_0<\frac{1}{2}$, let us set

$$V_1^j:=\{\zeta:\zeta\in\alpha_j^*(U_j),|\zeta-1|\le c(\ln n)^{-\alpha^{-1}}\};V_2^j:=\alpha_j^*(U_j)\backslash V_1^j, j=1,2,\alpha>0;$$

$$U_{\varepsilon_0}:=\{\zeta:|\zeta+1|\le\varepsilon_0\},\widetilde{V}_j^i:=U_j\cap U_{\varepsilon_0},\ j=1,2,\alpha=0.$$

Then, by [6, Lemma 3.4] and 3, we obtain

$$\begin{array}{rcl} mes\ g(V_1^j) & \prec & \delta^{-1}(B)(\ln n)^{-\alpha^{-1}}, \\ mes\ g(V_2^j) & \prec & \delta^{-1}(B)n^{\frac{\varepsilon-1}{K^2}}, \end{array} \tag{4.9}$$

and

$$J_5^2\prec\delta^{-1}(B)(\ln n)^{-\alpha^{-1}}. \tag{4.10}$$

¿From (4.4), (4.7) and (4.10) we get

$$\left\|g'(.,z_0)-P_n'(.,z_0)\right\|_{L_2(G)}\prec\delta^{-\frac{1}{2}}(B)\sqrt{\ln\ln n}(\ln n)^{\frac{\alpha-1}{2\alpha}}. \tag{4.11}$$

Now, let $\widetilde{P}_n(z,z_0)$ is defined by

$$\widetilde{P}_n(z,z_0):=\begin{cases} P_n(z,z_0)-P_n(z_0,z_0)+(z-z_0)[g'(z_0,z_0)-P_n'(z_0,z_0)], & n>N(R_0),\\ (z-z_0)g'(z_0,z_0), & n\le N(R_0).\end{cases}$$

Then $\widetilde{P}_n(z_0,z_0)=0$, $\widetilde{P}_n'(z_0,z_0)=1$ and according to means value theorem, we get

$$\left\|g'(.,z_0)-\widetilde{P}_n'(.,z_0)\right\|_{L_2(G)}\prec(1+\delta^{-1}(z_0))\left\|g'(.,z_0)-P_n'(.,z_0)\right\|_{L_2(G)}. \tag{4.12}$$

Since $\varphi=rg$, where $r=[g'(z_0,z_0)]^{-1}\asymp\delta(z_0)$, we let $S_n:=r\widetilde{P}_n$. Then, (4.12) yields

$$\left\|\varphi'(.,z_0)-S_n'(.,z_0)\right\|_{L_2(G)}\prec\delta^{-\frac{1}{2}}(B)\sqrt{\ln\ln n}(\ln n)^{\frac{\alpha-1}{2\alpha}}.$$

Since $S_n(z_0,z_0)=0$, $S_n'(z_0,z_0)=1$, then taking into account the extremely property of $\pi_n(z,z_0)$ we complete the proof. ■

Lemma 5. *Let $G\in Q^\alpha$ for some $0<\alpha\le1$, $z_0\in B\Subset G$. Then, for any $n\ge2$, we have*

$$\left\|\varphi'(.,z_0)-\pi_n'(.,z_0)\right\|_{L_2(G)}\prec\delta^{-\frac{1}{2}}(B)n^{-\gamma},$$

where

$$0<\gamma<\frac{\alpha}{4(2-\alpha)}$$

is arbitrary.

Proof. In [3, p.234-236] we have shown that if L is a quasiconformal curve, then there exists a polynomials $P_n(z,z_0)$,of $\deg P_n=n$, satisfying $P_n(z_0,z_0)=0$ and $P_n'(z_0,z_0)=g(z_0,z_0)$, such that

$$\left\|g'(.,z_0)-P_n'(.,z_0)\right\|_{L_2(G)}\prec\frac{1}{n}+\delta^{-1}(z_0)[mes\ g(\alpha^*(G_{1+n^{\varepsilon-1}}\backslash G),z_0)]^{\frac{1}{2}} \tag{4.13}$$

for arbitrary small $\varepsilon > 0$.

Since $\varphi = rg$, where $r = [g'(z_0, z_0)]^{-1} \asymp \delta(z_0)$, we define the polynomials $S_n := rP_n$. Then, (4.13) implies

$$\left\|\varphi'(., z_0) - \pi_n'(., z_0)\right\|_{L_2(G)} \leq \left\|\varphi'(., z_0) - S_n'(., z_0)\right\|_{L_2(G)}$$

$$\begin{aligned} &= r\left\|g'(., z_0) - P_n'(., z_0)\right\|_{L_2(G)} \prec \frac{\delta(z_0)}{n} + [mes\ g(\alpha^*(G_R\backslash G), z_0)]^{\frac{1}{2}} \\ &\prec \delta^{-\frac{1}{2}}(B)n^{-\frac{\alpha}{4(2-\alpha)}}, \end{aligned}$$

where in the last inequality we used Corollary 1. ∎

Corollary 2. *Let $G \in Q(\lambda)$ for some $0 < \lambda < 1$, $z_0 \in B \Subset G$. Then, for any $n \geq 2$, we have*

$$\left\|\varphi'(., z_0) - \pi_n'(., z_0)\right\|_{L_2(G)} \prec \delta^{-\frac{1}{2}}(B)n^{-\gamma}, \tag{4.14}$$

where

$$0 < \lambda < \frac{\lambda}{4(2-\lambda)}$$

is arbitrary.

Proof. Since $G \in Q(\lambda)$, then G satisfies the "$\lambda-$ wedge" conditions. Therefore, by [20] $\Psi \in Lip\lambda$, and (4.14) follows from Lemma 5. ∎

Lemma 6. *Let $G \in C_\theta(\lambda)$ for some $0 < \lambda < 2$, $z_0 \in B \Subset G$. Then, for any $n \geq 2$*

$$\left\|\varphi'(., z_0) - \pi_n'(., z_0)\right\|_{L_2(G)} \prec \delta^{-\frac{1}{2(2-\lambda)}}(B)n^{-\gamma},$$

where

$$0 < \gamma < \min\{\frac{1}{2}; \frac{\lambda}{2-\lambda}\}$$

is arbitrary.

This Lemma in case of $L_p(G)-$ norm with $p > 1$ is proved in [3, Cor.3.3].

5 Proofs of main results

First of all we shall establish the necessary facts about tie of the orthogonal polynomials with conformal mappings and Bieberbach polynomials.

Let G be an arbitrary finite domain of the complex plane C, bounded by a Jordan curve L, $z_0 \in B \Subset G$; $\{K_n(z)\}$ deg $K_n = n$, $n = 0, 1, 2, ...$, be a system of orthogonal polynomials over the domain G . It is well known (see, for example, [14]) that the conformal mappings $\varphi(z, z_0)$ of the domain G can be represented with the help of polynomials $\{K_n(z)\}$ in the following way:

$$\varphi(z, z_o) = \sum_{i=0}^{\infty} \overline{K_i(z_0)} \int_{z_0}^{z} K_i(t)dt \Big/ \sum_{i=0}^{\infty} |K_i(z_0)|^2 . \tag{5.1}$$

On the order hand, the Bergman kernel function $K(z,\overline{z}_0)$ can be written [11] as

$$K(z,\overline{z}_0)=\frac{1}{\pi}\frac{\overline{g'(z_0,z_0)}g'(z,z_0)}{[1-\overline{g(z_0,z_0)}g(z,z_0)]^2}=\sum_{i=0}^{\infty}\overline{K_i(z_0)}K_i(z),\ z\in G, \tag{5.2}$$

where the series in (5.2) converge uniformly in G. Taking into account that $\varphi=rg$, we put

$$S_n:=\sum_{i=0}^{n}|K_i(z_0)|^2, \qquad S_\infty:=\sum_{i=0}^{\infty}|K_i(z_0)|^2,$$

and get from (5.2),

$$\varphi'(z,z_o)=\pi r^2\sum_{i=0}^{\infty}\overline{K_i(z_0)}K_i(z),\ z\in G. \tag{5.3}$$

Hence

$$\pi_n'(z,z_o)=\frac{1}{S_{n-1}}\sum_{i=0}^{n}\overline{K_i(z_0)}K_i(z),\ z\in G, \tag{5.4}$$

$$\pi r^2\cdot S_\infty=1. \tag{5.5}$$

Lemma 7. *We have*

$$\left\|\varphi'(.,z_0)-\pi_n'(.,z_0)\right\|_{L_2(G)}^2=\frac{1}{S_{n-1}}-\frac{1}{S_\infty}. \tag{5.6}$$

Proof. ¿From (5.3) and (5.4) we get

$$\varphi'(z,z_0)-\pi_n'(z,z_0)=\sum_{i=0}^{n-1}\left(\pi r^2-\frac{1}{S_{n-1}}\right)^2\overline{K_i(z_0)}K_i(z)+\pi r^2\sum_{i=n}^{\infty}\overline{K_i(z_0)}K_i(z).$$

By Parseval equation,

$$\begin{aligned}&\left\|\varphi'(.,z_0)-\pi_n'(.,z_0)\right\|_{L_2(G)}^2\\ =&\sum_{i=0}^{n-1}\left(\pi r^2-\frac{1}{S_{n-1}}\right)^2|K_i(z_0)|^2+\left(\pi r^2\right)^2\sum_{i=n}^{\infty}|K_i(z_0)|^2\\ =&\left(\pi r^2-\frac{1}{S_{n-1}}\right)^2\cdot S_{n-1}+\left(\pi r^2\right)^2(S_\infty-S_{n-1})\\ =&\frac{1}{S_{n-1}}-\frac{1}{S_\infty}.\end{aligned}$$

∎

Corollary 3.

$$\sum_{i=n}^{\infty}|K_i(z_0)|^2=S_{n-1}\cdot S_\infty\cdot\left\|\varphi'(.,z_0)-\pi_n'(.,z_0)\right\|_{L_2(G)}^2. \tag{5.7}$$

Lemma 8. *Let* $\eta_n(B):=\sup\left\{\left\|\varphi'(.,z_0)-\pi_n'(.,z_0)\right\|_{L_2(G)}^2,\quad z_0\in B\Subset G\right\}$, *then*

$$\omega_n(B)\prec\delta^{-2}(z_0)\eta_n(B). \tag{5.8}$$

Proof. For all z, $z_0 \in B \Subset G$ we have

$$[\varphi(z,z_0)-\pi_n(z,z_0)]S_{n-1} = \varphi(z,z_0)S_{n-1} - \sum_{i=0}^{n-1}\overline{K_i(z_0)}\int_{z_0}^{z}K_i(t)dt$$

$$= \left[\varphi(z,z_0)S_\infty - \sum_{i=0}^{n-1}\overline{K_i(z_0)}\int_{z_0}^{z}K_i(t)dt\right] - \varphi(z,z_0)\left[S_\infty - S_{n-1}\right]$$

$$= \sum_{i=n}^{\infty}\overline{K_i(z_0)}\int_{z_0}^{z}K_i(t)dt - \varphi(z,z_0)\left[S_\infty - S_{n-1}\right]$$

$$=: \Delta_n(z,z_0) - \varphi(z,z_0)\Delta_n^{'}(z_0,z_0),$$

where

$$\Delta_n(z,z_0) = \sum_{i=n}^{\infty}\overline{K_i(z_0)}\int_{z_0}^{z}K_i(t)dt;\ \Delta_n^{'}(z_0,z_0) = \sum_{i=n}^{\infty}\left|K_i(z_0)\right|^2.$$

by (5.3) and (5.4). Then

$$|\varphi(z,z_0)-\pi_n(z,z_0)| \le \frac{|\Delta_n(z,z_0)|}{S_{n-1}} + |\varphi(z,z_0)|\frac{\left|\Delta_n^{'}(z_0,z_0)\right|}{S_{n-1}}. \tag{5.9}$$

For the estimation of the first term, we are applying Schwarz inequality, and accordingly to Corollary 3, we get

$$|\Delta_n(z,z_0)| \le \left\{\sum_{i=n}^{\infty}|K_i(z_0)|^2\right\}^{\frac{1}{2}}\left\{\sum_{i=n}^{\infty}\left|\int_{z_0}^{z}K_i(t)dt\right|^2\right\}^{\frac{1}{2}}$$

$$= \left\{S_{n-1}\cdot S_\infty\cdot \eta_n(B)\right\}^{\frac{1}{2}}\left\{mesl(z,z_0)\sum_{i=n}^{\infty}\int_{z_0}^{z}|K_i(t)|^2\,dt\right\}^{\frac{1}{2}}$$

$$= \left\{S_{n-1}\cdot S_\infty\cdot \eta_n(B)\right\}^{\frac{1}{2}}\left\{mesl(z,z_0)\int_{z_0}^{z}\sum_{i=n}^{\infty}|K_i(t)|^2\,dt\right\}^{\frac{1}{2}}$$

$$\le \left\{S_{n-1}\cdot S_\infty\cdot \eta_n(B)\right\}^{\frac{1}{2}}(mesl(z,z_0))\max_{t\in l(z,z_0)}\left\{\sum_{i=n}^{\infty}|K_i(t)|^2\,dt\right\}^{\frac{1}{2}}$$

$$= S_{n-1}\cdot S_\infty\cdot \eta_n(B)\,(mesl(z,z_0)),$$

where $l(z,z_0) \Subset B$ is the rectifiable arc joining z_0 and z.Then, using (5.5) we have

$$\frac{|\Delta_n(z,z_0)|}{S_{n-1}} \prec S_\infty\cdot\eta_n(B) \prec \frac{\eta_n(B)}{r^2}. \tag{5.10}$$

For the estimation the second term, firstly we observe that $|\varphi(z,z_0)| < r$ for all z, $z_0 \in B$, and therefore

$$|\varphi(z,z_0)|\frac{\left|\Delta_n^{'}(z_0,z_0)\right|}{S_{n-1}} \prec r\cdot S_\infty\cdot\eta_n(B) \prec \frac{\eta_n(B)}{r}. \tag{5.11}$$

¿From (5.9)-(5.11) we obtain

$$|\varphi(z,z_0)-\pi_n(z,z_0)| \prec \frac{\eta_n(B)}{r^2},\quad \forall\ z,z_0\in B\Subset G.$$

Since $r = \frac{1}{g'(z,z_0)} \asymp \delta(z_0)$, we complete the proof.

Now, the proof of Theorems 1-5 easily follows from Lemma 8, Lemmas 4, 6, [6, Lemma 4.2] and Corollary 2. ∎

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