On the Norm of a Self-Adjoint Operator and Applicationsto the Hilbert's Type Inequalities

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Abstract

The norm of a bounded self-adjoint operator $T : l^2 \to l^2$ is considered. As applications, a new bilinear inequality with a best constant factor and some Hilbert's type inequalities are built.

Let H be a real separable Hilbert space and $T : H \to H$ be a bounded self-adjoint semi-positive definite operator. Then (see [1],(17))

$$|(a, Tb)| \leq \frac{||T||}{\sqrt{2}}(||a||^2||b||^2 + (a, b)^2)^{\frac{1}{2}} \;\; (a, b \in H), \tag{1}$$

where (a, b) is the inner product of a and b, and $||a|| = \sqrt{(a, a)}$ is the norm of a.

Note 1. By Cauchy-Schwarz's inequality (see [2]), (1) can imply to

$$|(a, Tb)| \leq ||T||||a||||b|| \; (a, b \in H). \tag{2}$$

It is obvious that the constant factor $||T||$ in (2) is the best possible and then the constant factor $||T||/2$ in (1) is still the best possible since (1) is an improvement of (2).

In this paper, the norm of a bounded self-adjoint operator $T : l^2 \to l^2$ is considered. As applications, a new bilinear inequality with a best constant factor and some new Hilbert's type inequalities are built by using (1), (2) and the given norm.

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For this, we consider some firsthand corollaries of (1) as follows:

(a) Since we have (see [3])

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^{\lambda}}\right]^2 \le \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \sum_{n=1}^{\infty} n^{1-\lambda}a_n^2, \tag{3}$$

where the constant factor $\left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2$ $(0<\lambda\le 4)$ is the best possible and $B(u,v)$ is the Beta function. Replacing $n^{\frac{1-\lambda}{2}}a_n$ by a_n in (3), we have an equivalent form of (3) as

$$\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{(mn)^{(\lambda-1)/2}}{(m+n)^{\lambda}}a_m\right]^2 \le \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \sum_{n=1}^{\infty} a_n^2. \tag{4}$$

If we set a self-adjoint semi-positive definite operator $T: l^2 \to l^2$ as:

$$Ta := b = \left\{\sum_{m=1}^{\infty}\frac{(mn)^{(\lambda-1)/2}}{(m+n)^{\lambda}}a_m\right\}_{n=1}^{\infty}, \quad a=\{a_m\}_{m=1}^{\infty}\in l^2,$$

then Inequality (4) is equivalent to $||Ta|| \le B(\frac{\lambda}{2},\frac{\lambda}{2})||a||$. Since the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(0<\lambda\le 4)$ in (4) is the best possible, we can conclude that T is a bounded operator and $||T|| = B(\frac{\lambda}{2},\frac{\lambda}{2})$. Hence, if T is shown being of semi-positive definite, then by (1) and Note 1, one has: If $\{a_m\}_{m=1}^{\infty}$,$\{b_n\}_{n=1}^{\infty}\in l^2$, then for $0<\lambda\le 4$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}a_m b_n}{(m+n)^{\lambda}} \le \frac{1}{\sqrt{2}}B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2+\sum_{n=1}^{\infty}a_n b_n\right\}^{\frac{1}{2}}, \tag{5}$$

where the constant factor $\frac{1}{\sqrt{2}}B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible.

(b) Since we have (see [4])

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{m^{\lambda}+n^{\lambda}}\right]^2 \le (\frac{\pi}{\lambda})^2 \sum_{n=1}^{\infty} n^{1-\lambda}a_n^2, \tag{6}$$

where the constant factor $(\frac{\pi}{\lambda})^2$ $(0<\lambda\le 2)$ is the best possible. By the same way of (a), we have:

If $\{a_m\}_{m=1}^{\infty}$,$\{b_n\}_{n=1}^{\infty}\in l^2$, then for $0<\lambda\le 2$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(mn)^{\frac{\lambda-1}{2}}a_m b_n}{m^{\lambda}+n^{\lambda}} \le \frac{\pi}{\lambda\sqrt{2}}\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2+\sum_{n=1}^{\infty}a_n b_n\right\}^{\frac{1}{2}}, \tag{7}$$

where the constant factor $\frac{\pi}{\lambda\sqrt{2}}$ is the best possible.

(c) Since we have (see [5])

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})^{\lambda-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^2 \le \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^2, \tag{8}$$

where the constant factor $\left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2$ $(0<\lambda\le 2)$ is the best possible. By the same way, we have:

If $\{a_m\}_{m=0}^{\infty}$,$\{b_n\}_{n=0}^{\infty} \in l^2$, then for $0 < \lambda \leq 2$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{[(m+\frac{1}{2})(n+\frac{1}{2})]^{\frac{\lambda-1}{2}}}{(m+n+1)^{\lambda}} a_m b_n \leq \frac{1}{\sqrt{2}} B(\frac{\lambda}{2},\frac{\lambda}{2}) \left\{\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 + \sum_{n=0}^{\infty} a_n b_n\right\}^{\frac{1}{2}}, \quad (9)$$

where the constant factor $\frac{1}{\sqrt{2}}B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible. In particular, for $\lambda = 1$, we have the following improved Hilbert's inequality (see [1])

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \leq \frac{\pi}{\sqrt{2}} \left\{\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 + \sum_{n=0}^{\infty} a_n b_n\right\}^{\frac{1}{2}}. \quad (10)$$

(d) Since we have (see [7])

$$\sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^{\lambda}, n^{\lambda}\}}\right]^2 \leq (\frac{4}{\lambda})^2 \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2, \quad (11)$$

where the constant factor $(\frac{4}{\lambda})^2$ $(0 < \lambda \leq 2)$ is the best possible. By the same way, we have:

If $\{a_m\}_{m=1}^{\infty}$,$\{b_n\}_{n=1}^{\infty} \in l^2$, then for $0 < \lambda \leq 2$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} a_m b_n}{\max\{m^{\lambda}, n^{\lambda}\}} \leq \frac{4}{\lambda\sqrt{2}} \left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 + \sum_{n=1}^{\infty} a_n b_n\right\}^{\frac{1}{2}}, \quad (12)$$

where the constant factor $\frac{4}{\lambda\sqrt{2}}$ is the best possible. In particular, for $\lambda = 1$, we have the following improved Hilbert's type inequality (see [6]) :

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} \leq \frac{4}{\sqrt{2}} \left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 + \sum_{n=1}^{\infty} a_n b_n\right\}^{\frac{1}{2}}. \quad (13)$$

Theorem 1. *Let $k(x,y)$ be continuous in $(0,\infty) \times (0,\infty)$, satisfying:*
(i) $k(x,y) = k(y,x)$ (> 0), for $x, y \in (0,\infty)$;
(ii) for $x > 0$ and $\varepsilon \geq 0, k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}}$ is decreasing in $y \in (0,\infty)$;
(iii) for $x > 0$ and $\varepsilon \in [0, \varepsilon_0)$ (ε_0 is small enough), the integral $\int_0^{\infty} k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}} dy$ is a constant only dependent on ε, but independent on x, such that

$$k(\varepsilon) := \int_0^{\infty} k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}} dy = k(0) + o(1) \quad (\varepsilon \to 0^+); \quad (14)$$

$$\sum_{m=1}^{\infty} m^{-(1+\varepsilon)} \int_0^1 k(m,y)(\frac{m}{y})^{\frac{1+\varepsilon}{2}} dy = O(1) \quad (\varepsilon \to 0^+). \quad (15)$$

If l^2 is a real space, define the operator $T : l^2 \to l^2$ with the kernel $k(m,n)$ as: for $n \in N$,

$$Ta := b = \left\{\sum_{m=1}^{\infty} k(m,n) a_m\right\}_{n=1}^{\infty}, \quad a = \{a_m\}_{m=1}^{\infty} \in l^2.$$

Then T is a bounded self-adjoint operator and

$$||T|| = k := k(0) = \int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{2}}dy < \infty. \tag{16}$$

Proof. By Cauchy's inequality with weight (see[8]), we have from (i), (ii) and (14) that

$$\begin{aligned}
&\left(\sum_{m=1}^\infty k(m,n)a_m\right)^2 = \left\{\sum_{m=1}^\infty k(m,n)[(\frac{n}{m})^{\frac{1}{4}}][(\frac{m}{n})^{\frac{1}{4}}a_m]\right\}^2 \\
\le\ & [\sum_{m=1}^\infty k(n,m)[(\frac{n}{m})^{\frac{1}{2}}][\sum_{m=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1}{2}}a_m^2] \\
\le\ & [\int_0^\infty k(n,x)[(\frac{n}{x})^{\frac{1}{2}}dx][\sum_{m=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1}{2}}a_m^2] \\
=\ & k\sum_{m=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1}{2}}a_m^2;
\end{aligned}$$

$$\begin{aligned}
||Ta||^2 &= (Ta,Ta) = \sum_{n=1}^\infty \left(\sum_{m=1}^\infty k(m,n)a_m\right)^2 \\
&\le k\sum_{n=1}^\infty[\sum_{m=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1}{2}}a_m^2] = k\sum_{m=1}^\infty [\sum_{n=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1}{2}}]a_m^2 \\
&\le k\sum_{m=1}^\infty [\int_0^\infty k(m,y)(\frac{m}{y})^{\frac{1}{2}}dy]a_m^2 = k^2||a||^2,
\end{aligned} \tag{17}$$

and then $||Ta|| \le k||a||$. It follows that $Ta \in l^2$ and $||T|| \le k$.

For $0 < \varepsilon < \varepsilon_0$, setting $\widetilde{a}$ as: $\widetilde{a} = \{m^{-\frac{1+\varepsilon}{2}}\}_{m=1}^\infty \in l^2$, then by (ii) and (iii), we have

$$\begin{aligned}
(T\widetilde{a},\widetilde{a}) &= \sum_{m=1}^\infty\sum_{n=1}^\infty k(m,n)\ (\frac{1}{mn})^{\frac{1+\varepsilon}{2}} \\
&= \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ \sum_{n=1}^\infty k(m,n)(\frac{m}{n})^{\frac{1+\varepsilon}{2}} \ge \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ \int_1^\infty k(m,y)(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy \\
&= \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ [\int_0^\infty k(m,y)(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy - \int_0^1 k(m,y)(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy] \\
&= \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ k(\varepsilon) - \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ \int_0^1 k(m,y)(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy \\
&= \sum_{m=1}^\infty m^{-(1+\varepsilon)}\ k(\varepsilon) - O(1) = ||\widetilde{a}||^2(k+o(1))\ \ (\varepsilon\to 0^+),
\end{aligned}$$

and then

$$||T||||\widetilde{a}||^2 \ge ||T\widetilde{a}||||\widetilde{a}|| \ge (T\widetilde{a},\widetilde{a}) \ge ||\widetilde{a}||^2(k+o(1)).$$

Hence $||T|| \ge k$ $(\varepsilon \to 0^+)$, and $||T|| = k$. Since

$$(Ta,b) = \sum_{n=1}^\infty\sum_{m=1}^\infty k(m,n)a_m b_n = \sum_{m=1}^\infty a_m\sum_{n=1}^\infty k(m,n)\ b_n = (a,Tb).$$

It follows that $T = T^*$ and T is a bounded self-adjoint operator. The theorem is proved.

By using (2) and Theorem 1, we have

Theorem 2. *If l^2 is a real inner-product space, $a = \{a_m\}_{m=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in l^2$, and $k(x,y)$ is defined by Theorem 1, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)a_m b_n \le k||a||||b||, \tag{18}$$

where the constant factor k is the best possible and $k = k(0) = \int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{2}}dy$.

Note 2. If $k = k(0) = \int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{2}}dy$ is a constant but the integral $\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}}dy$ $(0 < \varepsilon < \varepsilon_0)$ is dependent on x and ε, then (17) is still valid, and we have $||T|| \le k$. In this case, by (2), we still have (18), but we can't affirm that the constant factor k in (18) is still the best possible.

In the following, we need the formula of the Beta function B(u,v) as (cf. Wang et al. [9]):

$$B(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1}dt = B(v,u)\ (u,v>0). \tag{19}$$

Lemma 1. *If $\lambda > 0$, define the function $f(u) := \frac{\ln u}{u^\lambda - 1}, u \in (0,\infty)$ $(f(1) := \frac{1}{\lambda} = \lim_{u\to 1} f(u))$, then $f(u)$ is decreasing in $(0,\infty)$.*

Proof. Setting $g(u) = u^\lambda - 1 - \lambda u^\lambda \ln u$, then $f'(u) = \frac{g(u)}{(u^\lambda-1)^2 u}$. Since $g'(u) = -\lambda^2 u^{\lambda-1}\ln u$, we have $g'(u) > 0, u \in (0,1); g'(u) < 0, u \in (1,\infty)$, and then $g(1) = 0 = \max_{u>0}\{g(u)\} \ge g(u)$ $(u > 0)$. Hence $f'(u) \le 0$ and $f(u)$ is decreasing in $(0,\infty)$. The lemma is proved.

(e) Setting $k(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}(xy)^{\frac{\lambda-1}{2}}$ $(0 < \lambda \le 2)$, then by Lemma 1, for fixed $x > 0, x^\lambda f(\frac{y}{x}) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ is decreasing in $y \in (0,\infty)$, and for $x > 0, \varepsilon \ge 0$ and $0 < \lambda \le 2$,

$$k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}} = \frac{\ln(x/y)}{x^\lambda - y^\lambda}(\frac{1}{y})^{\frac{2-\lambda+\varepsilon}{2}} x^{\frac{\lambda+\varepsilon}{2}}$$

is decreasing in $y \in (0,\infty)$. For $0 < \varepsilon < \lambda/2$, we obtain that

$$\begin{aligned} k(\varepsilon) &= \int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda}(xy)^{\frac{\lambda-1}{2}}(\frac{x}{y})^{\frac{1+\varepsilon}{2}}dy = \frac{1}{\lambda^2}\int_0^\infty \frac{\ln u}{u-1}u^{\frac{\varepsilon-\lambda}{2\lambda}}du \\ &\to \frac{1}{\lambda^2}\int_0^\infty \frac{\ln u}{u-1}u^{-\frac{1}{2}}du = (\frac{\pi}{\lambda})^2 = k \quad (\varepsilon \to 0^+). \end{aligned}$$

Since $\frac{\ln(m/y)}{m^\lambda - y^\lambda}$ is decreasing in $y \in (0,\infty)$, then for $0 < \varepsilon < \lambda/2$ $(0 < \lambda \le 2)$, we have

$$\begin{aligned} 0 &< A(m,\varepsilon) := \sum_{m=1}^{\infty} m^{-(1+\varepsilon)} \int_0^1 \frac{\ln(m/y)}{m^\lambda - y^\lambda}(my)^{\frac{\lambda-1}{2}}(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy \\ &\le \sum_{m=1}^{\infty} m^{-1}\int_0^1 \frac{\ln m}{m^\lambda - 1}(my)^{\frac{\lambda-1}{2}}(\frac{m}{y})^{\frac{1+\varepsilon}{2}}dy \\ &= \frac{2}{\lambda-\varepsilon}\sum_{m=1}^{\infty}\frac{\ln m}{m^\lambda - 1}m^{\frac{\lambda+\varepsilon}{2}-1} \le \frac{4}{\lambda}\sum_{m=1}^{\infty}\frac{\ln m}{(m^\lambda - 1)m^{1-\frac{3\lambda}{4}}} < \infty, \end{aligned}$$

and then $A(m,\varepsilon) = O(1)$. Hence $k(x,y)$ possesses the conditions of (i),(ii) and (iii).

If l^2 is a real space, define the operator $T : l^2 \to l^2$ with the kernel $k(m,n) = \frac{\ln(m/n)}{m^\lambda - n^\lambda}(mn)^{\frac{\lambda-1}{2}}$ $(0 < \lambda \le 2)$ as: for $n \in N$,

$$Ta := b = \left\{ \sum_{m=1}^{\infty} \frac{\ln(m/n)}{m^\lambda - n^\lambda}(mn)^{\frac{\lambda-1}{2}} a_m \right\}_{n=1}^{\infty}, \quad a = \{a_m\}_{m=1}^{\infty} \in l^2.$$

Then by Theorem 1, T is a bounded self-adjoint operator and

$$||T|| = k := k(0) = \int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{2}} dy = (\frac{\pi}{\lambda})^2.$$

By Theorem 2, we have

Corollary 1. If l^2 is a real space, $a = \{a_m\}_{m=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in l^2$, then for $0 < \lambda \le 2$,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}} \ln(\frac{m}{n})}{m^\lambda - n^\lambda} a_m b_n \le (\frac{\pi}{\lambda})^2 ||a|| ||b||, \tag{20}$$

where the constant factor $(\frac{\pi}{\lambda})^2$ is the best possible. In particular, for $\lambda = 1$, we have the following Hilbert's type inequality (see [6]) :

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})}{m - n} a_m b_n \le \pi^2 ||a|| ||b||, \tag{21}$$

(f) Setting $k(x,y) = \frac{(xy)^{(\lambda-1)/2}}{(1+xy)^\lambda}$ $(0 < \lambda \le 2)$, then for $x > 0, \varepsilon \ge 0$,

$$k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}} = \frac{1}{(1+xy)^\lambda}(\frac{1}{y})^{\frac{2-\lambda+\varepsilon}{2}} x^{\frac{\lambda+\varepsilon}{2}}$$

is decreasing in $y \in (0,\infty)$. For $0 < \varepsilon < \lambda$, setting $u = xy$, we obtain from (19) that

$$\begin{aligned} k_x(\varepsilon) \quad : \quad &= \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{(1+xy)^\lambda} (\frac{x}{y})^{\frac{1+\varepsilon}{2}} dy = x^\varepsilon \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\varepsilon}{2}-1} du \\ &= x^\varepsilon B(\frac{\lambda-\varepsilon}{2}, \frac{\lambda+\varepsilon}{2}) \to B(\frac{\lambda}{2}, \frac{\lambda}{2}) = k \quad (\varepsilon \to 0^+). \end{aligned}$$

If l^2 is a real space, define the operator $T : l^2 \to l^2$ with the kernel $k(m,n) = \frac{(mn)^{(\lambda-1)/2}}{(1+mn)^\lambda}$ $(0 < \lambda \le 2)$ as: for $n \in N$,

$$Ta := b = \left\{ \sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}}}{(1+mn)^\lambda} a_m \right\}_{n=1}^{\infty}, \quad a = \{a_m\}_{m=1}^{\infty} \in l^2.$$

Then T is a self-adjoint operator and by Note 2, $||T|| \le B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Hence by (18), we have

Corollary 2. If l^2 is a real space, $a = \{a_m\}_{m=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in l^2$, then for $0 < \lambda \le 2$,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(mn)^{(\lambda-1)/2}}{(1+mn)^\lambda} a_m b_n \le B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||a|| ||b||. \tag{22}$$

(g) Setting $k(x,y) = \frac{(xy)^{(\lambda-1)/2}}{1+(xy)^\lambda}$ $(0 < \lambda \leq 2)$, then for $x > 0, \varepsilon \geq 0$,

$$k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{2}} = \frac{1}{1+(xy)^\lambda}(\frac{1}{y})^{\frac{2-\lambda+\varepsilon}{2}} x^{\frac{\lambda+\varepsilon}{2}}$$

is decreasing in $y \in (0,\infty)$. For $0 < \varepsilon < \lambda$, setting $u = (xy)^\lambda$, we obtain from (19) that

$$\begin{aligned} k_x(\varepsilon) \quad : \quad &= \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{1+(xy)^\lambda} (\frac{x}{y})^{\frac{1+\varepsilon}{2}} dy = x^\varepsilon \frac{1}{\lambda} \int_0^\infty \frac{1}{1+u} u^{\frac{\lambda-\varepsilon}{2\lambda}-1} du \\ &= x^\varepsilon \frac{1}{\lambda} B(\frac{\lambda-\varepsilon}{2\lambda}, \frac{\lambda+\varepsilon}{2\lambda}) \to \frac{\pi}{\lambda} = k \quad (\varepsilon \to 0^+). \end{aligned}$$

If l^2 is a real space, define the operator $T : l^2 \to l^2$ with the kernel $k(m,n) = \frac{(mn)^{(\lambda-1)/2}}{1+(mn)^\lambda}$ $(0 < \lambda \leq 2)$ as: for $n \in N$,

$$Ta := b = \left\{ \sum_{m=1}^\infty \frac{(mn)^{(\lambda-1)/2}}{1+(mn)^\lambda} a_m \right\}_{n=1}^\infty, \ a = \{a_m\}_{m=1}^\infty \in l^2.$$

Then T is self-adjoint operator and by Note 2, $||T|| \leq \frac{\pi}{\lambda}$. By (18), we have

Corollary 3. If l^2 is a real space, $a = \{a_m\}_{m=1}^\infty, b = \{b_n\}_{n=1}^\infty \in l^2$, then for $0 < \lambda \leq 2$,

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(mn)^{(mn)^{(\lambda-1)/2}}}{1+(mn)^\lambda} a_m b_n \leq \frac{\pi}{\lambda} ||a|| ||b||. \tag{23}$$

Remarks. (i) For $\lambda = 1$, both (5) and (7) reduce to the following improved Hilbert's inequality:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sqrt{2}} \left\{ \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 + \sum_{n=1}^\infty a_n b_n \right\}^{\frac{1}{2}}. \tag{24}$$

(ii) For $\lambda = 1$, both (22) and (23) reduce to the following new Hilbert's type inequality:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{1+mn} \leq \pi ||a|| ||b||. \tag{25}$$

(iii) By using Theorem 2 and Note 2, we can build some new Hilbert's type inequalities such as (20), (22) and (23).

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