

The distributional Kontorovich–Lebedev transformation with the Hankel function in the kernel

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Abstract

A version of the Kontorovich–Lebedev transformation, involving the Hankel function of second kind in its kernel and connected with the Helmholtz’s equation, has been investigated from a classical point of view by D. S. Jones. The main objective of this work is to extend this transform to certain space of generalized functions, establishing the corresponding distributional inversion formula.

1 Introduction

A. H. Zemanian affirmed that one of the integral transformations that more difficulties offered to its extension to spaces of generalized functions was the Kontorovich–Lebedev transform [19], defined by

$$(KLf)(\tau) = F(\tau) = \int_0^\infty K_{i\tau}(x)f(x)dx;$$
$$(KL^{-1}F)(x) = f(x) = \frac{2}{\pi^2x} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x)F(\tau)d\tau,$$

*Partially supported by DGI Grant MTM2004-05878, and Consejería de Educación, Gobierno de Canarias, PI042004/067 (Spain)

Received by the editors March 2005.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification* : 46F12.

Key words and phrases : Hankel function, Kontorovich–Lebedev transformation, inversion formula, distributions.

where $K_{i\tau}(x)$ is the Macdonald function ([14, p.355], [17]). Besides A. H. Zemanian, who was the first one in analysing this transformation in the space of distributions with compact supports, the same has been investigated by R. S. Pathak and J. N. Pandey [12], H.–J. Glaeske and A. Hess [5] and S. B. Yakubovich and B. Fisher [16] in other classes of generalized function spaces.

A different version of this transform is given by the pair

$$(\mathcal{J}f)(\nu) = F(\nu) = \int_0^\infty H_\nu^{(2)}(x)f(x)dx, \quad (1.1)$$

$$(\mathcal{J}^{-1}F)(x) = f(x) = -\frac{1}{2x} \int_{-i\infty}^{i\infty} \nu J_\nu(x)F(\nu)d\nu, \quad (1.2)$$

where $J_\nu(z)$ is the well-known Bessel function of the first kind and order ν , and $H_\nu^{(2)}(z)$ is the Bessel function of the third kind, named also the Hankel function of the second kind ([3], [9], [15]), introduced by means of

$$H_\nu^{(2)}(z) = \frac{e^{\nu\pi i} J_\nu(z) - J_{-\nu}(z)}{i \sin \nu\pi}. \quad (1.3)$$

Regarded as functions of the argument z both $J_\nu(z)$ and $H_\nu^{(2)}(z)$ are analytic functions in the complex plane cut along the nonpositive real axis, whereas they are entire functions of the order ν , for every $z \neq 0$. They satisfy the Bessel equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0. \quad (1.4)$$

D. S. Jones explained through a counterexample [7, §2] that the inversion formula (1.2) is, in general, faulty. By this reason he proposed to introduce a factor and replaced (1.2) by

$$(\mathcal{J}^{-1}F)(x) = f(x) = \lim_{\varepsilon \rightarrow 0^+} -\frac{1}{2x} \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu J_\nu(x)F(\nu)d\nu, \quad (1.5)$$

which will ease the deformation of the path of integration, as it will be pointed out later, in order to establish under which conditions the existence of this limit could be insured. Indeed, he proved [7] the following

Theorem 1 (D. S. Jones' theorem). *Suppose that the function f satisfies*

$$(a) \int_0^1 |f(x)| \ln x dx < \infty,$$

$$(b) \int_c^\infty x^{-\frac{1}{2}} f(x) e^{-i(x-\frac{\pi}{4})} dx \text{ is finite for any real positive number } c.$$

Then, there exists

$$F(\nu) = \int_0^\infty H_\nu^{(2)}(x)f(x)dx, \quad (\operatorname{Re} \nu = 0),$$

and we have

$$\lim_{\varepsilon \rightarrow 0^+} -\frac{1}{2x} \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu J_\nu(x)F(\nu)d\nu = \frac{f(x+0) + f(x-0)}{2},$$

whenever the function f be of bounded variation in a neighborhood of the point $x > 0$.

The main objective of this work is to give a distributional version of the Kontorovich–Lebedev transformation in the form studied by D. S. Jones [7]. We remark that M. I. Kontorovich and N. N. Lebedev [8] researched this transform in the reverse order, in other words, they started from the expression (1.2) assuming certain hypotheses on the function $F(\nu)$, to show later that the corresponding inversion formula was provided by (1.1). However, from a point of view of the applications, the sequence proposed by D. S. Jones is more logical and useful: the direct formula is given by (1.1), the conditions being imposed on the function f , and its inversion is supplied by the formula (1.2).

Next, we recall some results that we need along this paper. Amongst them, the asymptotic behaviours of the functions $J_\nu(x)$ and $H_\nu^{(2)}(x)$.

$$H_\nu^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{\pi\nu}{2} - \frac{\pi}{4})}, \quad \text{as } x \rightarrow \infty \tag{1.6}$$

$$H_\nu^{(2)}(x) \approx i \left(\frac{2}{x}\right)^\nu \frac{\Gamma(\nu)}{\pi}, \quad \text{as } x \rightarrow 0+, \quad \text{Re}\nu > 0 \tag{1.7}$$

$$H_0^{(2)}(x) \approx i \frac{2}{\pi} \ln \frac{2}{x}, \quad \text{as } x \rightarrow 0+ \tag{1.8}$$

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1 + \nu)}, \quad \text{as } x \rightarrow 0+ \tag{1.9}$$

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi\nu}{2} - \frac{\pi}{4}), \quad \text{as } x \rightarrow \infty, \tag{1.10}$$

with ν fixed ([3], [9], [15]). For large values of the order ν one has [15, p. 225]

$$J_\nu(z) \approx \frac{1}{\sqrt{2\pi\nu}} e^{\nu - \nu \ln \frac{2\nu}{z}} \tag{1.11}$$

For values of the argument z greater than those ones of the order ν , we can use [15, p. 244]

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi}} (z^2 - \nu^2)^{-\frac{1}{4}} e^{-i\sqrt{z^2 - \nu^2} - i \arcsin \frac{\nu}{z}} e^{\frac{\pi i}{2}(\nu + \frac{1}{2})} \tag{1.12}$$

On the contrary, for values of the argument z less than the order ν , it turns out to be [15, p. 262]

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi}} i (\nu^2 - z^2)^{-\frac{1}{4}} e^{-\sqrt{\nu^2 - z^2} + \nu \ln \left(\frac{\nu}{z} + \sqrt{\frac{\nu^2}{z^2} - 1}\right)} \tag{1.13}$$

In the sequel I denotes the positive real interval $(0, \infty)$. $\mathcal{D}(I)$ stands for the space of infinitely differentiable functions whose supports are contained in I , being endowed with the topology of the inductive limit. Its dual $\mathcal{D}'(I)$ is the space of Schwartz distributions ([13], [18]), which is equipped with the weak topology. $\mathcal{E}(I)$ represents the space of all infinitely differentiable functions on I with the topology generated by the collection of seminorms

$$\gamma_{K,k}(\varphi) = \sup_{x \in K} |D^k \varphi(x)|, \tag{1.14}$$

where $\varphi \in \mathcal{E}(I)$, $D = \frac{d}{dx}$, K is any compact subset of I and $k = 0, 1, 2, \dots$. Its dual $\mathcal{E}'(I)$ is the space of distributions with compact supports.

Along this paper by C we will understand a positive constant not necessarily the same in each occurrence.

2 The Kontorovich–Lebedev transformation with the Hankel function in the kernel in the space $\mathcal{E}'(I)$

Through the kernel method, we will define the version of D. S. Jones of the Kontorovich–Lebedev transform of a generalized function $f \in \mathcal{E}'(I)$, directly as the application of f to $H_\nu^{(2)}(\cdot)$,

$$\left(\mathcal{J}f\right)(\nu) = F(\nu) = \langle f(x), H_\nu^{(2)}(x) \rangle \quad (2.15)$$

The right-hand side of (2.15) has a sense because, for any fixed $\nu \in \mathbb{C}$, the function $H_\nu^{(2)}(\cdot) \in C^\infty(I)$ and its derivatives of all orders are bounded on every compact subset of I . We now list some properties of this transform:

(i) F is an entire function. Indeed, for an arbitrary but fixed $\nu \in \mathbb{C}$, let $\Delta \nu$ be a non zero complex increment, and consider the expression

$$\frac{F(\nu + \Delta \nu) - F(\nu)}{\Delta \nu} - \langle f(x), \frac{\partial}{\partial \nu} H_\nu^{(2)}(x) \rangle = \langle f(x), A_{\Delta \nu}(x) \rangle \quad (2.16)$$

where

$$A_{\Delta \nu}(x) = \frac{H_{\nu + \Delta \nu}^{(2)}(x) - H_\nu^{(2)}(x)}{\Delta \nu} - \frac{\partial}{\partial \nu} H_\nu^{(2)}(x).$$

By resorting to accustomed techniques, as in the proof of analogous property concerning the Hankel transform [18, p. 145], we can verify easily that, as $\Delta \nu \rightarrow 0$, $D^k A_{\Delta \nu}(x)$ tends to zero uniformly on compact subsets of I , that is, $A_{\Delta \nu}$ converges to zero in the topology of the space $\mathcal{E}(I)$ when $\Delta \nu \rightarrow 0$. Then, $\langle f(x), A_{\Delta \nu}(x) \rangle \rightarrow 0$, as $\Delta \nu \rightarrow 0$, and, consequently

$$F'(\nu) = \langle f(x), \frac{\partial}{\partial \nu} H_\nu^{(2)}(x) \rangle.$$

(ii) Now we equip the space $\mathcal{E}(I)$ with the topology generated by the separating family of seminorms $\{\lambda_{K,k}\}_{k \in \mathbb{N}}$, defined by

$$\lambda_{K,k}(\varphi) = \sup_{x \in K} |\Delta^k \varphi(x)|, \quad \varphi \in \mathcal{E}(I),$$

where $\Delta = \Delta_x = x^2 D^2 + xD + x^2$ and K is a compact of I . This topology is equivalent to that one given rise by the collection of seminorms (1.14). In effect, suppose that the sequence (φ_n) converges to zero in the space $\mathcal{E}(I)$ with the topology due to the family of seminorms $\{\lambda_{K,k}\}_{k \in \mathbb{N}}$. Then, $(\Delta^\ell \varphi_n)$ converges uniformly to zero on K , for every $\ell = 0, 1, 2, \dots$. From here and following a standard method, it is concluded that $(D^k \varphi_n)$ converges uniformly to zero on K , for all $k = 0, 1, 2, \dots$; that is, (φ_n) converges to zero in the space $\mathcal{E}(I)$ endowed with the topology assigned by

the collection of seminorms (1.14) ([18, p.177],[19]). And conversely, assume the last statement. By an inductive argument, we get easily

$$\Delta^k \varphi(x) = \sum_{j=0}^{2k} P_j(x) D^{2k-j} \varphi(x), \quad k = 0, 1, 2, \dots, \tag{2.17}$$

where $P_j(x)$ denote polynomials of order less or equal to $2k$ with positive integer coefficients. So, it is quickly inferred that

$$\lambda_{K,k}(\varphi_n) \leq \sum_{j=0}^{2k} C_j \gamma_{K,2k-j}(\varphi_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

where $C_j = \sup_{x \in K} |P_j(x)|$. This implies that (φ_n) tends towards zero in $\mathcal{E}(I)$ equipped with the second topology. Consequently, with the topology supplied by the family of seminorms $\{\lambda_{K,k}\}_{k \in \mathbb{N}}$, $\mathcal{E}(I)$ is also a Fréchet space [18, p.37].

Recall finally that $\mathcal{D}(I) \subset \mathcal{E}(I)$ ([13],[18]), the dual space $\mathcal{E}'(I)$ being a subspace of $\mathcal{D}'(I)$.

(iii) By virtue of [18, theorem 1.8-1] and the fact that the kernel is a solution of the Bessel equation (1.4), there exist a positive constant C and a nonnegative integer number r such that

$$|F(\nu)| \leq C \max_{0 \leq k \leq r} \sup_{x \in K} |\Delta_x^k H_\nu^{(2)}(x)| = C \max_{0 \leq k \leq r} \sup_{x \in K} |\nu^{2k} H_\nu^{(2)}(x)|.$$

Bearing in mind asymptotic expansions (1.8) and (1.13) and the facts that x belongs to the compact set $K \subset [x_0, y_0]$ with $0 < x_0 < y_0$ and $\ln \frac{2}{x}$ is bounded on K , we are led to

$$F(\nu) = \begin{cases} O(1), & \text{as } \nu \longrightarrow 0 \\ O\left(\nu^{2r-\frac{1}{2}} e^{\nu(\ln \frac{2\nu}{x_0} - 1)}\right), & \text{as } \nu \longrightarrow \infty \end{cases}$$

3 The inversion formula.

In this paragraph we establish the main result of this work

Theorem 2. *Let $f \in \mathcal{E}'(I)$. We define its Kontorovich–Lebedev transform by*

$$\left(\mathcal{J}f\right)(\nu) = F(\nu) = \langle f(x), H_\nu^{(2)}(x) \rangle \tag{3.18}$$

Then, we have

$$\lim_{\substack{N \rightarrow \infty \\ \varepsilon \rightarrow 0+}} -\frac{1}{2x} \int_{-iN}^{iN} e^{\varepsilon \nu^2} \nu J_\nu(x) F(\nu) d\nu = f(x),$$

in the sense of the convergence in the space $\mathcal{D}'(I)$.

Proof: We have to prove that, for every $\varphi \in \mathcal{D}(I)$,

$$\left\langle -\frac{1}{2x} \int_{-iN}^{iN} e^{\varepsilon \nu^2} \nu J_\nu(x) F(\nu) d\nu, \varphi(x) \right\rangle \longrightarrow \langle f, \varphi \rangle \tag{3.19}$$

when $N \rightarrow \infty$ and $\varepsilon \rightarrow 0+$. We assume that support of φ is contained in $[a, b]$, where $0 < a < b < \infty$.

In the sequel we will denote

$$\Phi(\nu) = -\frac{1}{2}e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\varphi(y)}{y} dy \quad (3.20)$$

and

$$\Omega_{N,\varepsilon}(x) = -\frac{1}{2} \int_{-iN}^{iN} H_\nu^{(2)}(x) e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\varphi(y)}{y} dy d\nu = \int_{-iN}^{iN} \Phi(\nu) H_\nu^{(2)}(x) d\nu \quad (3.21)$$

Observe that the integral in the first functional of (3.19) defines a continuous function of x . This allows us to write

$$\begin{aligned} \int_a^b -\frac{1}{2} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu J_\nu(y) F(\nu) d\nu \frac{\varphi(y)}{y} dy = \\ -\frac{1}{2} \int_{-iN}^{iN} \langle f(x), H_\nu^{(2)}(x) \rangle e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\varphi(y)}{y} dy d\nu \\ = \int_{-iN}^{iN} \langle f(x), H_\nu^{(2)}(x) \rangle \Phi(\nu) d\nu, \end{aligned} \quad (3.22)$$

in accordance with definition (2.15) and notation (3.20).

We will proceed in two steps:

(a) Firstly, by using the Riemann sums techniques [18, p. 187], we obtain

$$\int_{-iN}^{iN} \langle f(x), H_\nu^{(2)}(x) \rangle \Phi(\nu) d\nu = \left\langle f(x), \int_{-iN}^{iN} H_\nu^{(2)}(x) \Phi(\nu) d\nu \right\rangle, \quad (3.23)$$

in other words, the integral operator interchanges with the functional. Notice that, according to (3.21), the right-hand side in (3.23) adopts the form

$$\left\langle f(x), \Omega_{N,\varepsilon}(x) \right\rangle. \quad (3.24)$$

(b) In the second step we have to prove that $\Omega_{N,\varepsilon}(x) \rightarrow \varphi(x)$ in the topology of the space $\mathcal{E}(I)$, when $N \rightarrow \infty$, $\varepsilon \rightarrow 0+$. To do it, we shall need some previous results.

On the one hand, an integration by parts two times yields

$$\int_a^b \Delta_y \{J_\nu(y)\} \frac{\varphi(y)}{y} dy = \int_a^b J_\nu(y) \frac{\Delta_y \varphi(y)}{y} dy \quad (3.25)$$

On the other hand, it is well-known that $J_\nu(x)$ and $H_\nu^{(2)}(x)$ are solutions of the differential equation (1.4), and, consequently,

$$\begin{aligned} \Delta_x H_\nu^{(2)}(x) &= \nu^2 H_\nu^{(2)}(x) \\ \Delta_x J_\nu(x) &= \nu^2 J_\nu(x) \end{aligned} \quad (3.26)$$

We can introduce the differential operator Δ under the integral sign and use (3.26) and (3.25) to derive

$$\begin{aligned}
 \Delta_x^k \Omega_{N,\varepsilon}(x) &= -\frac{1}{2} \int_{-iN}^{iN} \Delta_x^k \{H_\nu^{(2)}(x)\} e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\varphi(y)}{y} dy d\nu \\
 &= -\frac{1}{2} \int_{-iN}^{iN} \nu^{2k} H_\nu^{(2)}(x) e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\varphi(y)}{y} dy d\nu \\
 &= -\frac{1}{2} \int_{-iN}^{iN} H_\nu^{(2)}(x) e^{\varepsilon\nu^2} \nu \int_a^b \{\Delta_y^k J_\nu(y)\} \frac{\varphi(y)}{y} dy d\nu \\
 &= -\frac{1}{2} \int_{-iN}^{iN} H_\nu^{(2)}(x) e^{\varepsilon\nu^2} \nu \int_a^b J_\nu(y) \frac{\Delta_y^k \varphi(y)}{y} dy d\nu \tag{3.27}
 \end{aligned}$$

Now the change of the order of integration is valid in (3.27) and we can break the integral into three parts

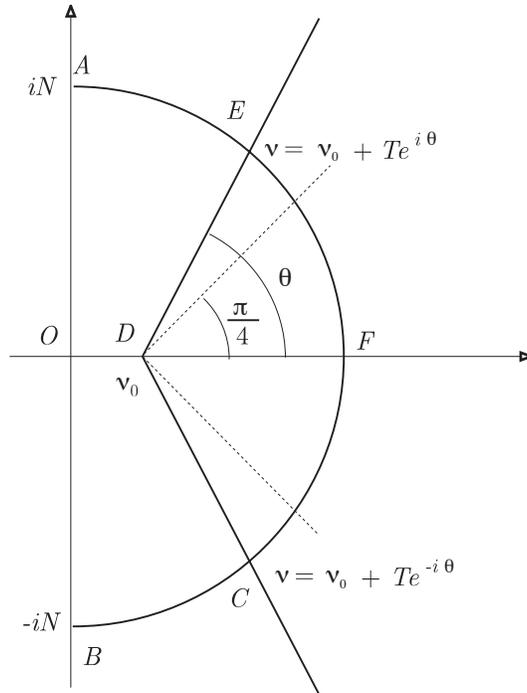
$$\begin{aligned}
 &-\frac{1}{2} \int_a^b \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \\
 &= -\frac{1}{2} \int_0^\infty \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \\
 &= -\frac{1}{2} \left\{ \int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^\infty \right\} \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \tag{3.28}
 \end{aligned}$$

where $\delta > 0$ will be fixed farther on.

We investigate first the integral

$$\int_{x+\delta}^\infty \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \tag{3.29}$$

For $x + \delta \geq b$ this integral is identically zero, since the support of φ is contained in $[a, b]$. When $x + \delta < b$, the path of the inner integral in (3.29) can be deformed moving it from $\text{Re}\nu = 0$ (the integral is calculated over the imaginary axis) to a region of the ν -plane where $\text{Re}\nu > 0$. If we choose $\nu_0 \in \mathbb{R}$, $\nu_0 > b > 0$, the new path of integration is composed of two semi-straight lines starting from ν_0 and making angles $\pm\theta$ with the positive real axis, as it has been drawn in the figure



The angle θ is fixed and slightly greater than $\frac{\pi}{4}$. This path is denoted by Γ , whereas Γ_T stands for the part of it constituted by the segments CD and DE , that is to say, by the points $\nu \in \mathbb{C}$ such that $\nu = \nu_0 + te^{-i\theta}$ and $\nu = \nu_0 + te^{i\theta}$, with $0 \leq t \leq T$, respectively.

By means of definition (1.3) and some straightforward manipulations, D. S. Jones realized that

$$\int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu = \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu$$

Hence, instead of (3.29) we can consider

$$\int_{x+\delta}^{\infty} \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu dy \tag{3.30}$$

and demonstrate that this expression converges uniformly to zero, when $N \rightarrow \infty$ and $\varepsilon \rightarrow 0+$, for every compact $K \subset I$. Inasmuch as the function $y^{-1} \Delta^k \varphi(y)$, $\varphi \in \mathcal{D}(I)$, satisfies the hypotheses assumed by D. S. Jones in [7, p. 135], the integral along the path $(-i\infty, i\infty)$ can be replaced by the integral throughout the path Γ described formerly. Thus, to prove that (3.30) converges uniformly to zero, as $N \rightarrow \infty$ and $\varepsilon \rightarrow 0+$, is equivalent to verify that

$$\int_{x+\delta}^{\infty} \frac{\Delta_y^k \varphi(y)}{y} \int_{\Gamma_T} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu dy \tag{3.31}$$

converges uniformly to zero if $T \rightarrow \infty$ and $\varepsilon \rightarrow 0+$. The last integral converges absolutely. Certainly, along the straight line DE we have $\nu = \nu_0 + te^{i\theta}$, $0 \leq t < T$, $\theta > \frac{\pi}{4}$, and when $T \rightarrow \infty$ we are led to

$$\begin{aligned} \left| \int_{DE} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu \right| &\leq \int_0^\infty |e^{\varepsilon\nu^2}| |\nu H_\nu^{(2)}(y) J_\nu(x)| dt \\ &\leq C \int_0^\infty e^{\varepsilon(\nu_0^2 + 2\nu_0 t \cos \theta + t^2 \cos 2\theta)} e^{(\nu_0 + t \cos \theta) \ln \frac{x}{y}} dt, \end{aligned}$$

owing to asymptotic formulas (1.11) and (1.13) (Let see also [7, (16)]). Due to the special choice of θ , we see that $\cos 2\theta < 0$ and consequently this integral exists. But we can assure more, the aforesaid integral converges absolutely even in the case $\varepsilon = 0$, because

$$C \int_0^\infty e^{(\nu_0+t \cos \theta) \ln \frac{x}{y}} dt \leq C \int_0^\infty e^{(\nu_0+t \cos \theta) \ln \frac{x}{x+\delta}} dt = C e^{\nu_0 \ln \frac{x}{x+\delta}} \left(\ln \frac{x+\delta}{x} \right)^{-1} \sec \theta,$$

which is uniformly bounded on every compact $K \subset I$. Through an argument similar to the one given, we arrive at the same conclusion over the semi-straight line DC . By dominated convergence theorem we can put $\varepsilon = 0$ in (3.31), that is, we get

$$\int_{x+\delta}^\infty \frac{\Delta_y^k \varphi(y)}{y} \int_{\Gamma_T} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu dy \tag{3.32}$$

Next, if the region corresponding to the angular vector of amplitude 2θ is closed by an arc of circumference of radius T and center in ν_0 , the integral along the closed path $DCFED$ is equal to zero by virtue of the Cauchy Theorem [1], since the only singularities of the function $\nu H_\nu^{(2)}(y) J_\nu(x)$ are simple poles in $\nu = 1, 2, 3, \dots$, but they are removable in agreement with the elementary result $J_{-n}(y) = (-1)^n J_n(y)$, $n \in \mathbb{N}$ [9, (5.3.3)]; in other words, $\nu H_\nu^{(2)}(y) J_\nu(x)$ is a holomorphic function in that region. Note that the points of the arc of circumference EFC adopt the form $\nu = \nu_0 + T e^{i\phi}$, $-\theta < \phi < \theta$. Then, by resorting once more to (1.11) and (1.13), making the change of variable $\alpha = \frac{\pi}{2} - \phi$ and using the fact that $\sin \alpha \geq \frac{2\alpha}{\pi}$ for all $\alpha \in [0, \frac{\pi}{2}]$, it is inferred

$$\begin{aligned} \left| \int_{EFC} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu \right| &\leq C \int_{-\theta}^\theta e^{(T \cos \phi + \nu_0) \ln \frac{x}{x+\delta}} T d\phi \\ &= CT e^{\nu_0 \ln \frac{x}{x+\delta}} \int_0^\theta e^{T \cos \phi \ln \frac{x}{x+\delta}} d\phi \\ &= CT e^{\nu_0 \ln \frac{x}{x+\delta}} \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} e^{T \sin \alpha \ln \frac{x}{x+\delta}} d\alpha \\ &\leq CT e^{\nu_0 \ln \frac{x}{x+\delta}} \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} e^{\frac{2T}{\pi} (\ln \frac{x}{x+\delta}) \alpha} d\alpha \\ &= C \frac{e^{\nu_0 \ln \frac{x}{x+\delta}}}{\ln \frac{x}{x+\delta}} \left[e^{T \ln(\frac{x}{x+\delta})} - e^{\frac{2T}{\pi} (\ln \frac{x}{x+\delta}) (\frac{\pi}{2}-\theta)} \right], \end{aligned}$$

that tends uniformly to zero, as $T \rightarrow \infty$, on every compact $K \subset I$, because $\ln \frac{x}{x+\delta} < \ln \frac{b-\delta}{b} < 0$. From all the above considerations, we deduce

$$\int_\Gamma \nu H_\nu^{(2)}(y) J_\nu(x) d\nu = 0$$

This result permits to conclude, starting from (3.32), that

$$\left| \int_{x+\delta}^\infty \frac{\Delta_y^k \varphi(y)}{y} \int_{\Gamma_T} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu dy \right|$$

$$\leq \sup_{y \in [a, b]} |y \Delta_y^k \varphi(y)| \frac{1}{a + \delta} \sup_{(x, y) \in K \times [a, b]} \left| \int_{\Gamma_T} \nu H_\nu^{(2)}(y) J_\nu(x) d\nu \right| \longrightarrow 0,$$

if $T \longrightarrow \infty$, uniformly on every compact $K \subset I$.

In like manner we can make $\varepsilon = 0$ and assure that the integral

$$\int_0^{x-\delta} \frac{\Delta_y^k \varphi(y)}{y} \int_{\Gamma_T} e^{\varepsilon \nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \quad (3.33)$$

converges uniformly to zero when $T \longrightarrow \infty$, on any compact $K \subset I$.

Finally, we discuss

$$-\frac{1}{2} \int_{x-\delta}^{x+\delta} \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon \nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \quad (3.34)$$

If either $b \leq x - \delta$ or $x + \delta \leq a$, the integral (3.34) is identically equal to zero. Therefore, we need only consider the range $a - \delta < x < b + \delta$. Henceforth we fix $0 < \delta < \frac{a}{2}$.

We next recall the inversion formula of the Laplace transformation \mathcal{L} . For every $\varphi \in \mathcal{D}(I)$

$$\begin{aligned} \frac{\Delta_y^k \varphi(y)}{y} &= \mathcal{L}^{-1} \left(\mathcal{L} \left(\frac{\Delta_x^k \varphi}{x} \right) \right) (y) \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iN}^{c+iN} e^{sy} \int_0^\infty e^{-sx} \frac{\Delta_x^k \varphi(x)}{x} dx ds \\ &= \lim_{N \rightarrow \infty} \frac{1}{\pi} \int_0^\infty \frac{\Delta_x^k \varphi(x)}{x} \frac{\sin N(y-x)}{y-x} dx \end{aligned}$$

holds. In the corresponding proof of this inversion formula (see G. Doetsch, [2, pp. 148-151]), the integrals on ranges $(0, x - \delta)$ and $(x + \delta, \infty)$ vanish when $N \longrightarrow \infty$, the integral on the interval $(x - \delta, x + \delta)$ being the only one which is meaningful.

It is well known that ([7], [4, p. 188 (55)])

$$H_\nu^{(2)}(y) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{(s^2 + 1)^{\frac{1}{2}} \sin \nu\pi} e^{sy} ds$$

where $b = s + (s^2 + 1)^{\frac{1}{2}}$, $y > 0$ and $|\operatorname{Re} \nu| < 1$. From here, using similar reasoning as in [7, p. 139], we can come to

$$\begin{aligned} \int_{x-\delta}^{x+\delta} H_\nu^{(2)}(y) \frac{\Delta_y^k \varphi(y)}{y} dy &= \int_{x-\delta}^{x+\delta} \frac{\Delta_y^k \varphi(y)}{y} \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{(s^2 + 1)^{\frac{1}{2}} \sin \nu\pi} e^{sy} ds dy \\ &= \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{(s^2 + 1)^{\frac{1}{2}} \sin \nu\pi} \int_{x-\delta}^{x+\delta} e^{sy} \frac{\Delta_y^k \varphi(y)}{y} dy ds \quad (3.35) \end{aligned}$$

Hence, by multiplying by $-\frac{1}{2} e^{\varepsilon \nu^2} \nu J_\nu(x)$ and integrating from $-i\infty$ to $i\infty$, we obtain from (3.35)

$$\lim_{\varepsilon \rightarrow 0^+} -\frac{1}{2} \int_{-i\infty}^{i\infty} e^{\varepsilon \nu^2} \nu J_\nu(y) \int_{x-\delta}^{x+\delta} H_\nu^{(2)}(y) \frac{\Delta_y^k \varphi(y)}{y} dy d\nu$$

$$\begin{aligned}
 &= \lim_{\varepsilon \rightarrow 0+} -\frac{1}{4\pi} \int_{-i\infty}^{i\infty} \frac{e^{\varepsilon\nu^2} \nu J_\nu(y)}{\sin \nu\pi} \int_{c-i\infty}^{c+i\infty} \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{(s^2 + 1)^{\frac{1}{2}}} \int_{x-\delta}^{x+\delta} e^{sy} \frac{\Delta_y^k \varphi(y)}{y} dy ds d\nu \\
 &= \lim_{\varepsilon \rightarrow 0+} -\frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} \frac{1}{(s^2 + 1)^{\frac{1}{2}}} \int_{x-\delta}^{x+\delta} e^{sy} \frac{\Delta_y^k \varphi(y)}{y} \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu J_\nu(x) \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{\sin \nu\pi} d\nu dy ds
 \end{aligned} \tag{3.35}$$

But the path of the integral in the variable ν can be deformed by employing the same argument applied to preceding cases and it is licit to make $\varepsilon = 0$. The resultant integral along the path Γ can be evaluated by means of Cauchy’s residue theorem [1], as it is made by D. S. Jones in [7, p. 139], giving rise to

$$\int_\Gamma \nu J_\nu(x) \frac{b^\nu - e^{i\nu\pi} b^{-\nu}}{\sin \nu\pi} d\nu = 2ix(s^2 + 1)^{\frac{1}{2}} e^{-sx}$$

If we take account of this and of the above remark concerning the integral Laplace transform, the right-hand side of (3.35) can be written

$$\begin{aligned}
 &-\frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} 2ixe^{-sx} \int_{x-\delta}^{x+\delta} e^{sy} \frac{\Delta_y^k \varphi(y)}{y} dy ds \\
 &= \frac{x}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}\left(\frac{\Delta_y^k \varphi(y)}{y}\right)(-s) e^{-sx} ds = \Delta_x^k \varphi(x)
 \end{aligned}$$

So we have proved that

$$-\frac{1}{2} \int_{x-\delta}^{x+\delta} \frac{\Delta_y^k \varphi(y)}{y} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu H_\nu^{(2)}(x) J_\nu(y) d\nu dy \longrightarrow \Delta_x^k \varphi(x),$$

as $N \rightarrow \infty$ ($T \rightarrow \infty$) and $\varepsilon \rightarrow 0+$, uniformly on every compact $K \subset I$.

Finally, by combining all the above results, we can conclude that

$$\lambda_{K,k}(\Omega_{N,\varepsilon} - \varphi(x)) \longrightarrow 0,$$

as $N \rightarrow \infty$, $\varepsilon \rightarrow 0+$, that is to say, $\Omega_{N,\varepsilon}(\cdot) \rightarrow \varphi(\cdot)$ in the topology of the space $\mathcal{E}(I)$, when $N \rightarrow \infty$, $\varepsilon \rightarrow 0+$. We need merely invoke the last conclusion and the linearity of f in (3.24) to finish the proof of our assertion.

4 Applications

Regarding the applications, it is more convenient to use the Kontorovich–Lebedev transformation given by

$$(\mathcal{J}^*g)(\nu) = G(\nu) = \int_0^\infty \frac{H_\nu^{(2)}(x)}{x} g(x) dx \tag{4.36}$$

$$(\mathcal{J}^{*-1}G)(x) = g(x) = -\frac{1}{2} \int_{-i\infty}^{i\infty} \nu J_\nu(x) G(\nu) d\nu \tag{4.37}$$

instead of the pair (1.1)-(1.2). The consideration of this new version produces scarcely any substantial modification in the theoretic study carried out before. Thus, if $g \in \mathcal{E}'(I)$, its generalized Kontorovich–Lebedev transform will be defined by

$$(\mathcal{J}^*g)(\nu) = G(\nu) = \left\langle g(x), \frac{H_\nu^{(2)}(x)}{x} \right\rangle. \tag{4.38}$$

Theorem 2 is now stated as follows

Theorem 3. Let $g \in \mathcal{E}'(I)$ and suppose that \mathcal{J}^*g is given by (4.38). Then,

$$\lim_{\substack{N \rightarrow \infty \\ \varepsilon \rightarrow 0^+}} -\frac{1}{2} \int_{-iN}^{iN} e^{\varepsilon\nu^2} \nu J_\nu(x) G(\nu) d\nu = g(x). \quad (4.39)$$

in the sense of the convergence in the space of distributions $\mathcal{D}'(I)$.

If $g \in \mathcal{E}'(I)$, we have $\Delta g \in \mathcal{E}'(I)$ as well, due to the usual manner of defining the generalized differentiation. In our case, we wish to evaluate $\mathcal{J}^*(\Delta g)$. By invoking (3.26) some manipulations yield

$$\begin{aligned} \mathcal{J}^*(\Delta g)(\nu) &= \left\langle \Delta g(x), \frac{H_\nu^{(2)}(x)}{x} \right\rangle = \left\langle g(x), \frac{\Delta_x H_\nu^{(2)}(x)}{x} \right\rangle \\ &= \nu^2 \left\langle g(x), \frac{H_\nu^{(2)}(x)}{x} \right\rangle = \nu^2 \mathcal{J}^*(g)(\nu). \end{aligned}$$

In a word, the operational rule

$$\mathcal{J}^*(\Delta g)(\nu) = \nu^2 \mathcal{J}^*(g)(\nu), \quad (4.40)$$

holds for every $g \in \mathcal{E}'(I)$.

By way of illustration the applications of the transform (4.36)-(4.37), we propose to find the solution $u(r, \theta)$ of the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + r^2 u = -r \delta(r - r_0) \quad (4.41)$$

in the wedge $0 < \theta < \theta_0$, where the constants θ_0 and r_0 are such that $0 < \theta_0 < \pi$ and $r_0 > 0$. The corresponding boundary conditions, $u(r, 0) = 0$ and $u(r, \theta_0) = 0$, are fixed on the sides of the wedge. Here $\delta(\cdot)$ stands for the Dirac functional, which belongs to $\mathcal{E}'(I)$. If we set $U(\nu, \theta) = \mathcal{J}^*(u(r, \theta))$ and apply the transform \mathcal{J}^* to (4.41), by using (4.40) and the fact that $\mathcal{J}^*(-r\delta(r - r_0)) = -H_\nu^{(2)}(r_0)$, the boundary value problem for the partial differential equation (4.41) is converted into the ordinary problem

$$\begin{cases} \frac{\partial^2 U(\nu, \theta)}{\partial \theta^2} + \nu^2 U(\nu, \theta) = -H_\nu^{(2)}(r_0) \\ U(\nu, 0) = 0, \quad U(\nu, \theta_0) = 0, \end{cases}$$

whose solution is given by

$$U(\nu, \theta) = 2 \frac{H_\nu^{(2)}(r_0)}{\nu^2} \sin \frac{\nu\theta}{2} \sin \frac{\nu(\theta_0 - \theta)}{2} \sec \frac{\nu\theta_0}{2}.$$

Lastly, the inversion formula (4.39) supplies the formal solution

$$u(r, \theta) = \lim_{\varepsilon \rightarrow 0^+} \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \frac{J_\nu(r) H_\nu^{(2)}(r_0)}{\nu} \sin \frac{\nu\theta}{2} \sin \frac{\nu(\theta - \theta_0)}{2} \sec \frac{\nu\theta_0}{2} d\nu$$

Some problems of grand physical interest and connected with the Helmholtz's equation have been analysed in great detail in [6], [7] and [10].

Acknowledgment We are indebted to Prof. J.J. Betancor, who revised early versions of the manuscript, and the referee, who made several suggestions to improve this work.

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