Real hypersurfaces in complex projective space whose structure Jacobi operator is $\mathbb{D}$-parallel

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Abstract

We prove the non existence of real hypersurfaces in complex projective space whose structure Jacobi operator is parallel in any direction of the maximal holomorphic distribution.

1 Introduction.

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let J denote the complex structure of $\mathbb{C}P^m$ and N a locally defined unit normal vector field on M. Then $-JN = \xi$ is a tangent vector field to M called the structure vector field on M. We also call $\mathbb{D}$ the maximal holomorphic distribution on M, that is, the distribution on M given by all vectors orthogonal to ξ at any point of M.

The study of real hypersurfaces in nonflat complex space forms is a classical topic in Differential Geometry. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [12], [13], [14], and is given by the following list: A_1 : *Geodesic hyperspheres.* A_2 : *Tubes over totally geodesic complex projective spaces.* B : *Tubes over complex quadrics* and $\mathbb{R}P^m$. C : *Tubes over the Segre embedding of* $\mathbb{C}P^1$x$\mathbb{C}P^n$, where $2n+1 = m$ and $m \geq 5$. D : *Tubes over the Plucker embedding of the complex Grassmann manifold* $G(2,5)$. In this case $m = 9$. E : *Tubes over the*

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cannonical embedding of the Hermitian symmetric space $SO(10)/U(5)$. In this case $m = 15$.

Other examples of real hypersurfaces are *ruled real* ones, that were introduced by Kimura, [6]: Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique complex projective hyperplane cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally although globally it will in general have self-intersections and singularities. Equivalently a ruled real hypersurface is such that $\mathbb{D}$ is integrable or, equivalently, $g(A\mathbb{D}, \mathbb{D}) = 0$, where A denotes the shape operator of the immersion, see [6]. For further examples of ruled real hypersurfaces see [7].

Except these real hypersurfaces there are very few examples of real hypersurfaces in $\mathbb{C}P^n$. So we present a result about non-existence of a certain family of real hypersurfaces in complex projective space.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called *Jacobi operator*. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, and X is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to X) at $p \in M$, $\tilde{R}_X \in$End$(T_p\tilde{M})$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$ for all $Y \in T_p\tilde{M}$, being a selfadjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X.

The study of Riemannian manifolds by means of their Jacobi operators has been developed following several ideas. For instance, in [2], it is pointed out that (locally) symmetric spaces of rank 1 (among them complex space forms) satisfy that all the eigenvalues of $\tilde{R}_X$ have constant multiplicities and are independent of the point and the tangent vector X. The converse is a well-known problem that has been studied by many authors, although it is still open.

Let M be a real hypersurface in a complex projective space and let ξ be *the structure vector field* on M. We will call the Jacobi operator on M with respect to ξ the *structure Jacobi operator* on M. In [3] the authors classify, under certain additional conditions, real hypersurfaces of $\mathbb{C}P^m$ whose structure Jacobi operator is parallel, in a certain sense, in the direction of ξ, namely, they suppose that $R'_\xi = 0$, where $R'_\xi(Y) = (\nabla_\xi R)(Y, \xi)\xi$. They obtain class A_1or A_2 hypersurfaces and a non-homogeneous real hypersurface. In [4] they classify real hypersurfaces in $\mathbb{C}P^m$ whose structure Jacobi operator commutes both with the shape operator and with the restriction of the complex structure to M.

Recently, [10], we have proved the non-existence of real hypersurfaces in $\mathbb{C}P^m$ with parallel structure Jacobi operator. So it seems to be natural to study weaker conditions. In this paper we consider the parallelism of R_ξ only for directions in $\mathbb{D}$. We will say that M has $\mathbb{D}$-parallel structure Jacobi operator if $\nabla_X R_\xi = 0$ for any $X \in \mathbb{D}$. We obtain

Theorem *There exist no real hypersurfaces in* $\mathbb{C}P^m$, $m \geq 3$, *with* $\mathbb{D}$*-parallel structure Jacobi operator.*

2 Preliminaries.

Thoughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel we need the following results:

Theorem 2.1. *, [9], Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then the following are equivalent:*

1. *M is locally congruent to one of the homogeneous hypersurfaces of class A_1 or A_2.*
2. *$\phi A + A\phi = 0$.*

Theorem 2.2. *, [10], There exist no real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that the shape operator is given by $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$, $AX = -X$, for any tangent vector X orthogonal to Span$\{\xi, U, \phi U\}$, where U is a unit vector field in $\mathbb{D}$ and β is a nonvanishig smooth function defined on M.*

3 Some previous results.

Proposition 3.1. *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 4$, whose shape operator is given by $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $A\phi U = 0$ and there exist two nonnull holomorphic distributions $\mathbb{D}_0$ and $\mathbb{D}_1$ such that $\mathbb{D}_0 \bigoplus \mathbb{D}_1 = Span\{\xi, U, \phi U\}^\perp$, for any $Z \in \mathbb{D}_0$, $AZ = A\phi Z = 0$, for any $W \in \mathbb{D}_1$, $AW = -(1/\alpha)W$, $A\phi W = -(1/\alpha)\phi W$, where U is a unit vector field in $\mathbb{D}$, α and β are nonvanishing smooth functions defined on M and $(\phi U)(\beta) = 0$.*

Proof. For any $W \in \mathbb{D}_1$, the Codazzi equation gives $(\nabla_W A)\phi W - (\nabla_{\phi W} A)W = -2\xi$. If we develop this equation and take the scalar product with ξ we have

$$g([\phi W, W], U) = 2/\alpha^2\beta. \tag{3.1}$$

The scalar product of the same equation with U gives

$$g([\phi W, W], U) = 2\beta. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\alpha^2\beta^2 = 1. \tag{3.3}$$

As we suppose $(\phi U)(\beta) = 0$, from (3.3) $(\phi U)(\alpha) = 0$. The Codazzi equation also gives $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. If we develop it, as $(\phi U)(\beta) = (\phi U)(\alpha) = 0$ we obtain

$$\beta\nabla_{\phi U} U + A\nabla_\xi \phi U = U. \tag{3.4}$$

Taking its scalar product with U we get $1 = g(\nabla_\xi \phi U, \beta\xi) = -\beta g(\phi U, \phi A\xi) = -\beta^2$. This is impossible and finishes the proof. ■

Proposition 3.2. *Let M be a ruled real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then M has not $\mathbb{D}$-parallel structure Jacobi operator.*

Proof. We suppose $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and β a nonvanishing smooth function on M. Thus $AU = \beta\xi$, $AX = 0$ for any X orthogonal to ξ and U. The Codazzi equation gives us $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. Developing this equation and taking the scalar product with ϕU we have

$$\beta^2 - \beta g(\nabla_U U, \phi U) = 1. \tag{3.5}$$

The Codazzi equation also yields $(\nabla_{\phi U} A)U - (\nabla_U A)\phi U = 2\xi$. Taking its scalar product with ξ we obtain

$$(\phi U)(\beta) - \beta g(\nabla_U U, \phi U) = 2. \tag{3.6}$$

If $(\phi U)(\beta) = 0$, from (3.5) and (3.6) we should have $\beta^2+1=0$, which is impossible. Thus $(\phi U)(\beta) \neq 0$. We develop $(\nabla_{\phi U} R_\xi)(U)$ and obtain $-(\phi U)(\beta^2)U - \beta^2\nabla_{\phi U}U - \alpha A\nabla_{\phi U}U$. Taking its scalar product with U we get $-(\phi U)(\beta^2)$. As this does not vanish, $\nabla_{\phi U}R_\xi \neq 0$, thus M cannot have $\mathbb{D}$-parallel structure Jacobi operator. ■

4 Proof of the Theorem

As M must have $\mathbb{D}$-parallel structure Jacobi operator, $(\nabla_X R_\xi)(Y) = 0$ for any $X \in \mathbb{D}$ and $Y \in TM$. From the Gauss equation this yields

$$\begin{gathered} -g(Y,\phi AX)\xi - g(\xi,Y)\phi AX + g(\nabla_X A\xi,\xi)AY + g(A\xi,\phi AX)AY \\ +g(A\xi,\xi)(\nabla_X A)Y - g(Y,\nabla_X A\xi)A\xi - g(AY,\xi)\nabla_X A\xi = 0 \end{gathered} \tag{4.1}$$

for any $X \in \mathbb{D}$, $Y \in TM$.

If we suppose that M is Hopf, that is, $A\xi = \alpha\xi$, see [8], α is locally constant and (4.1) gives

$$\begin{gathered} -g(Y,\phi AX)\xi - g(Y,\xi)\phi AX + \alpha(\nabla_X A)Y \\ -\alpha^2 g(Y,\phi AX)\xi - \alpha^2 g(Y,\xi)\phi AX = 0 \end{gathered} \tag{4.2}$$

for any $X \in \mathbb{D}$, $Y \in TM$. Taking the scalar product of (4.2) with ξ we obtain

$$g(Y,\phi AX) + \alpha g(AY,\phi AX) = 0. \tag{4.3}$$

Thus for any $X \in \mathbb{D}$ we get

$$\phi AX + \alpha A\phi AX = 0. \tag{4.4}$$

Therefore for any $X, Y \in \mathbb{D}$ we have $g(\phi AY + \alpha A\phi AY, X) = 0 = -g(Y, (A\phi + \alpha A\phi A)X)$. Then

$$A\phi X + \alpha A\phi AX = 0 \tag{4.5}$$

for any $X \in \mathbb{D}$. From (4.4) and (4.5) we obtain $\phi AX = A\phi X$ for any $X \in \mathbb{D}$. As $\phi A\xi = A\phi\xi = 0$, we have $A\phi = \phi A$. Thus from Theorem 2.1, M must be locally congruent to a real hypersurface of type A_1 or A_2. In both cases, see [8], we can take $X \in \mathbb{D}$ such that $AX = cot(r)X$, $A\xi = 2cot(2r)\xi$, r being the radius of the tube, $0 < r < \pi/2$. If we compute $(\nabla_X R_\xi)(\xi)$ we obtain $-cot^3(r)\phi X \neq 0$. Thus we get

Proposition 4.1. *There exist no Hopf real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, whose structure Jacobi operator is $\mathbb{D}$-parallel.*

From now on we suppose that our real hypersurface is not Hopf. That is, there exist a unit $U \in \mathbb{D}$ and a nonvanishing smooth function β on M such that $A\xi = \alpha\xi + \beta U$. Now we take $Y = \phi U$ in (4.1). For any $X \in \mathbb{D}$ we have

$$\begin{gathered} -g(U, AX)\xi + g(\nabla_X A\xi, \xi)A\phi U + g(A\xi, \phi AX)A\phi U \\ +\alpha(\nabla_X A)\phi U - g(\phi U, \nabla_X A\xi)A\xi = 0. \end{gathered} \tag{4.6}$$

Taking the scalar product of (4.6) with ξ we obtain

$$g(U, AX) + \alpha g(A\phi U, \phi AX) = 0 \tag{4.7}$$

for any $X \in \mathbb{D}$. Taking $X = \phi U$ in (4.7) we have

$$g(AU, \phi U) = 0. \tag{4.8}$$

From (4.7), $AU - \alpha A\phi A\phi U$ has not component in $\mathbb{D}$. Thus

$$AU - \alpha A\phi A\phi U = (\beta + \alpha\beta g(A\phi U, \phi U))\xi. \tag{4.9}$$

If we take $Y = U$ in (4.1) and the scalar product with ξ we obtain

$$(1 - \beta^2)g(\phi U, AX) + \alpha g(A\phi AU, X) = 0 \tag{4.10}$$

for any $X \in \mathbb{D}$. Therefore $(1-\beta^2)A\phi U + \alpha A\phi AU = -\alpha\beta g(AU, \phi U)\xi$ and from (4.8),

$$(1 - \beta^2)A\phi U + \alpha A\phi AU = 0. \tag{4.11}$$

Let us call $\mathbb{D}_U = \mathbb{D} \cap \mathrm{Span}\{U, \phi U\}^\perp$. Then we take $Y \in \mathbb{D}_U$, $X \in \mathbb{D}$ in (4.1) and the scalar product with ξ. We obtain $g(\phi Y, AX) - \alpha g(Y, A\phi AX) = 0$. Taking $X = Y$ we get

$$g(\phi X, AX) = 0 \tag{4.12}$$

for any $X \in \mathbb{D}_U$. Moreover

$$A\phi X + \alpha A\phi AX = -\alpha\beta g(AX, \phi U)\xi \tag{4.13}$$

for any $X \in \mathbb{D}_U$. Taking the scalar product of (4.9) with U and the scalar product of (4.11) with ϕU it follows

$$g(AU, U) = (1 - \beta^2)g(A\phi U, \phi U). \tag{4.14}$$

If we take $Y \in \mathbb{D}_U$, $X = \phi U$ in (4.1), taking its scalar product with ξ, from (4.9) it follows

$$g(\phi Y, A\phi U) = g(AY, U) \tag{4.15}$$

for any $Y \in \mathbb{D}_U$. Similarly, for any $Y \in \mathbb{D}_U$, we have

$$g(Y, AY) = g(\phi Y, A\phi Y). \tag{4.16}$$

If we change Y by ϕY in (4.15) it follows $-g(AY, \phi U) = g(A\phi Y, U)$, for any $Y \in \mathbb{D}_U$. This equality, (4.8) and (4.14) yield

$$A\phi U - \phi AU = \beta^2 g(A\phi U, \phi U)\phi U. \tag{4.17}$$

We want to prove that AU and $A\phi U$ have no component in $\mathbb{D}_U$. Thus from (4.8) we can suppose

$$\begin{aligned} AU &= \beta\xi + g(AU, U)U + \mu Z \\ A\phi U &= g(A\phi U, \phi U)\phi U + \epsilon W \end{aligned} \tag{4.18}$$

where μ, ϵ are smooth functions on M and Z, W unit vector fields in $\mathbb{D}_U$. Now from (4.14), (4.17) and (4.18) we have $\epsilon W = \mu\phi Z$. That is, $A\phi U = g(A\phi U, \phi U)\phi U + \mu\phi Z$. Taking $Y = \phi Z$, $X = U$ in (4.1) and its scalar product with ξ we obtain

$$\mu + \alpha\mu g(AU, U) + \alpha\mu g(A\phi Z, \phi Z) = 0. \tag{4.19}$$

From (4.19) we have either $\mu = 0$ or $1 + \alpha g(AU, U) + \alpha g(A\phi Z, \phi Z) = 0$. Taking $Y = Z$, $X = \phi U$ in (4.1) and its scalar product with ξ we get

$$\mu + \alpha\mu g(A\phi U, \phi U) + \alpha\mu g(AZ, Z) = 0. \tag{4.20}$$

From (4.16) and (4.20) we obtain either $\mu = 0$ or $1 + \alpha g(A\phi U, \phi U) + \alpha g(A\phi Z, \phi Z) = 0$. From (4.14), (4.19) and (4.20), if $\mu \neq 0$, it follows $\alpha \neq 0$, $g(A\phi U, \phi U) = 0$ and $g(A\phi Z, \phi Z) = g(AZ, Z) = -(1/\alpha)$. Thus we have two possibilities:

1. $\mu \neq 0$. Then $AU = \beta\xi + \mu Z$, $A\phi U = \mu\phi Z$, $g(AZ, Z) = g(A\phi Z, \phi Z) = -(1/\alpha)$.
2. $\mu = 0$. Then $AU = \beta\xi + \delta(1 - \beta^2)U$, $A\phi U = \delta\phi U$, where we have called $\delta = g(A\phi U, \phi U)$.

First case is impossible: From (4.9) we should have $AU - \alpha A\phi A\phi U = \beta\xi$. Introducing in this equation the values of AU and $A\phi U$ we get $\beta\xi + \mu Z - \alpha\mu A\phi^2 Z = \beta\xi$. That is, $\mu Z + \alpha\mu AZ = 0$. Taking its scalar product with U it follows $\alpha\mu^2 = 0$, which is impossible.

Now we consider the second case. Take $Z \in \mathbb{D}_U$ such that $AZ = \lambda Z$. From (4.13) it follows $A\phi Z + \alpha A\phi AZ = 0$. This gives $(1 + \alpha\lambda)A\phi Z = 0$. Thus either $A\phi Z = 0$ or $1+\alpha\lambda = 0$. If $A\phi Z = 0$, taking $X = \phi Z$ in (4.13) we get $AZ = 0$. Thus $\lambda = 0$. Thus the unique eigenvalues of A that could appear in $\mathbb{D}_U$ are either 0 or $-(1/\alpha)$. We also can conclude that the corresponding eigenspaces are holomorphic, that is, they are invariant by ϕ.

Suppose firstly that there exists $Z \in \mathbb{D}_U$ such that $AZ = A\phi Z = 0$. The Codazzi equation gives $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$. Developing this equation and taking its scalar product with ϕZ we get

$$g(\nabla_Z U, \phi Z) = -(1/\beta). \tag{4.21}$$

Again the Codazzi equation implies $(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$. Developing it and taking its scalar product with Z we have

$$\delta g(\nabla_Z U, \phi Z) = 0. \tag{4.22}$$

If $\delta \neq 0$, (4.21) and (4.22) give a contradiction. Thus we suppose $\delta = 0$. In this case, if for any $Z \in \mathbb{D}_U$, $AZ = 0$, remind that we have $AU = \beta\xi$, $A\phi U = 0$. Thus we obtain a ruled real hypersurface. Proposition 3.2 implies that this case does not occur.

Now we suppose that there exists $Z \in \mathbb{D}_U$ such that $AZ = A\phi Z = 0$, that is $Z \in \mathbb{D}_0$ as in Proposition 3.1, and there exists $W \in \mathbb{D}_U$ such that $AW = -(1/\alpha)W$, $A\phi W = -(1/\alpha)\phi W$, that is, $W \in \mathbb{D}_1$. From Proposition 3.1 we have $(\phi U)(\beta) \neq 0$. Now we develop $(\nabla_{\phi U} R_\xi)(U)$ and take its scalar product with U. We obtain $-(\phi U)(\beta^2) \neq 0$. Thus this kind of real hypersurfaces does not satisfy our condition.

Therefore we must suppose that $AU = \beta\xi + \delta(1 - \beta^2)U$, $A\phi U = \delta\phi U$, $AZ = -(1/\alpha)\phi Z$ for any $Z \in \mathbb{D}_U$. From the Codazzi equation $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$. Developing it and taking its scalar product with ξ we get

$$(\alpha + (1/\alpha))g([\phi Z, Z], \xi) + \beta g([\phi Z, Z], U) = -2 \tag{4.23}$$

and its scalar product with U yields

$$\beta g([\phi Z, Z], \xi) + (\delta(1 - \beta^2) + (1/\alpha))g([\phi Z, Z], U) = 0. \tag{4.24}$$

As $g([\phi Z, Z], \xi) = -(2/\alpha)$, from (4.23) and (4.24) we have

$$\alpha\delta(1 - \beta^2) + 1 = \alpha^2\beta^2. \tag{4.25}$$

On the other hand, if these real hypersurfaces satisfy our condition, $(\nabla_{\phi U} R_\xi)(U) = 0$. Developing this we get

$$\begin{aligned}(\phi U)(\alpha\delta(1 - \beta^2) - \beta^2)U + (\alpha\delta(1 - \beta^2) - \beta^2)\nabla_{\phi U} U \\ +\delta\xi - \alpha A\nabla_{\phi U} U + \alpha^2\delta\xi + \alpha\delta\beta U = 0.\end{aligned} \tag{4.26}$$

The scalar product of (4.26) with ξ gives $-(\alpha\delta(1-\beta^2)-\beta^2)g(U,\phi A\phi U)+\delta-\alpha^2 g(\nabla_{\phi U}U,\xi)+\alpha^2\delta=0$. From (4.25) this yields

$$(\alpha^2-1)\beta^2\delta=0. \tag{4.27}$$

We have two possibilities: either $\delta=0$ or $\alpha^2=1$. In this second case, changing, if necessary, ξ by $-\xi$, we can suppose $\alpha=1$. Now fron (4.25) we obtain two new possibilities: either $\beta^2=1$ or $\delta=-1$.

If we suppose $\delta=0$, $\alpha^2\beta^2=1$, $A\xi=\alpha\xi+\beta U$, $AU=\beta\xi$, $A\phi U=0$, $AZ=-(1/\alpha)Z$, for any $Z\in\mathbb{D}_U$. From the Codazzi equation $(\nabla_{\phi U}A)\xi-(\nabla_\xi A)\phi U=U$. Developing this equality and taking its scalar product with U we obtain $(\phi U)(\beta)-\beta^2=1$. If we suppose $(\phi U)(\beta)=0$ we have a contradiction. Thus we must have $(\phi U)(\beta)\neq 0$. But we have $(\nabla_{\phi U}R_\xi)(U)=0$. Developing it and taking its scalar product with U we get $-(\phi U)(\beta^2)=0$, which is impossible.

Thus $\delta\neq 0$. The possiblity of being $\alpha=1$, $\delta=-1$ cannot appear by Theorem 2.2.

Thus the unique possibility is $\alpha=1$, $\beta^2=1$. If we change U by $-U$, if necessary, we can suppose $\beta=1$. We should have $(\nabla_U R_\xi)(\phi U)=0$. Developing this equation and taking its scalar product with ϕU we should obtain

$$U(\delta)=0. \tag{4.28}$$

Developing now $(\nabla_{\phi U}R_\xi)(\phi U)=0$ and taking its scalar product with ϕU we get

$$(\phi U)(\delta)=0. \tag{4.29}$$

Now, for any $Z\in\mathbb{D}_U$, $(\nabla_Z R_\xi)(\phi U)=0$ and its scalar product with ϕU yields

$$Z(\delta)=0. \tag{4.30}$$

The Codazzi equation gives $(\nabla_\xi A)\phi U-(\nabla_{\phi U}A)\xi=-U$. Its scalar product with ϕU implies

$$\xi(\delta)=g(\nabla_{\phi U}U,\phi U). \tag{4.31}$$

Again the Codazzi equation implies $(\nabla_U A)\phi U-(\nabla_{\phi U}A)U=-2\xi$. Its scalar product with ϕU yields $\delta g(\nabla_{\phi U}U,\phi U)=0$. As we suppose $\delta\neq 0$, from (4.31) we get

$$\xi(\delta)=0. \tag{4.32}$$

From (4.28), (4.29), (4.30) and (4.32), we conclude that δ is constant.

The Codazzi equation yields $(\nabla_\xi A)\phi U-(\nabla_{\phi U}A)\xi=-U$ and its scalar product with ξ gives

$$g(\nabla_\xi \phi U, U) = -3\delta + 1. \tag{4.33}$$

Its scalar product with U implies

$$\delta g(\nabla_\xi \phi U, U) = -2 - \delta. \tag{4.34}$$

From (4.33) and (4.34) we get

$$3\delta^2 - 2\delta - 2 = 0. \tag{4.35}$$

But from the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$, and its scalar product with U yields

$$g(\nabla_U \phi U, U) = -2. \tag{4.36}$$

Taking the scalar product of the above Codazzi equation and ξ we get

$$g(\nabla_U \phi U, U) = \delta + 2. \tag{4.37}$$

From (4.35), (4.36) and (4.37) we arrive to a contradiction, and this finishes the proof.

□

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