Some new obstruction results for compact positively Ricci curved manifolds

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Abstract

This work considers the fundamental groups and diameters of positively Ricci curved Riemannian n-manifolds. By combining the results of equivariant Hausdorff convergence with the Ricci version of a splitting theorem, some new information on the topology of compact manifolds with positive Ricci curvature was discovered. Moreover, a weak Margulis's lemma was also obtained for Riemannian manifolds with a lower Ricci curvature bound.

1 Introduction

This investigation studied the obstruction problems for compact Riemannian n-manifolds with positive Ricci curvature. When the dimension $n = 2$, this problem is easy to understand since only the projective plan $\boldsymbol{RP}^2$ and the 2-sphere admit metrics with positive curvature. Hamilton showed in [**6**] that a compact 3-manifold with positive Ricci curvature also admits a metric with a constant sectional curvature of $+1$, and is then covered by the 3-sphere. In general, the classical Myers' theorem shows that the fundamental group of a compact positively Ricci curved manifold must be finite. Moreover, since the Ricci curvature of $SU(n)$ is positive, any finite group can be the fundamental group of some manifold with positive Ricci curvature.

Problem 5 listed in Lecture Series 4 in [[**8**], p.105] states: *Considering a compact positively Ricci curved manifold, what can be said about the fundamental group depending only on the dimension* n*, except that it is finite?* Our work partially answers this problem as indicated in [**10**] as a conjecture. Moreover, the diameter of the universal Riemannian covering space of a compact positively Ricci curved manifold M

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can not be too larger than that of M.

Theorem A. *Given $n \geq 2$, there exist constants p_n and C_n depending only on n such that if a compact Riemannian n-manifold M^n has the Ricci curvature $Ric_{M^n} > 0$, then*
(a) *the first betti number $b_1(M^n, \boldsymbol{Z}_p)$ with p-cyclic group coefficient $\boldsymbol{Z}_p$ satisfies $b_1(M^n, \boldsymbol{Z}_p) \leq n-1$ for all prime $p \geq p_n$,* and
(b) the ratio of diameters satisfies

$$\frac{diam(\tilde{M}^n)}{diam(M^n)} \;<\; C_n,$$

where $\tilde{M}^n$ denotes the universal covering of M^n.

Remark 1.1. Considering the flat n-torus T^n, $b_1(T^n, \boldsymbol{Z}_p) = n$ is obtained for all prime p, and the canonical Euclidean n-space $\boldsymbol{R}^n$ is its universal covering space. Hence Theorem A assumes an optimal. Fukaya and Yamaguchi showed in [Corollary 0.9 in [**3**]] that if a compact Riemannian n-manifold M with sectional curvature K_M and diameter $diam(M)$ satisfies $K_M diam(M)^2 > -\epsilon_n$ for some constant ϵ_n depending only on n, then $b_1(M^n, \boldsymbol{Z}_p) \leq n$ for all $p \geq p(n)$, and the maximal case $b_1(M^n, \boldsymbol{Z}_p) = n$ aries only when M^n is diffeomorphic to a torus. They also found in [Corollary 0.11 in [**3**]] that $diam(\tilde{M})/diam(M)$ is uniformly bounded by a constant depending only on n provided the fundamental group $\pi_1(M)$ is finite. Theorem A extends their results to manifolds with positive Ricci curvature. Notably, positively Ricci curved n-manifolds with $n \leq 3$ are covered by spheres as discussed above. Hence Theorem A holds for these manifolds.

Remark 1.2. An application of Theorem A is presented here. As well-known, any finite group G can be the fundamental group of a compact 4-manifold. If $G = S_m$ is taken to be the permutation group of m elements, and the 4-manifolds M^4_m are considered with fundamental group S_m, then M^4_m admits no metric with positive Ricci curvature for a large m.

The proof of Theorem A leads to the following weak Margulis's lemma under a lower Ricci curvature bound. Recall that the *length of polycyclicity* of a solvable group G is the smallest integer m for which G admits a filtration

$$\{e\} = G_m \subset G_{m-1} \subset \ldots \subset G_1 \subset G_0 = G$$

such that each G_i/G_{i-1} is cyclic.

Theorem B (A weak Margulis's Lemma). *There exists a positive number δ_n depending only on n and satisfying the following: Let (M^n, p) be a complete pointed Riemannian n-manifold with $Ric_{M^n} \geq -(n-1)$. Then there exists a point $p' \in B_p(1/2)$ such that the image of the inclusion homomorphism*

$$\Gamma' \;=\; Im[\pi_1(B_{p'}(\delta_n)) \;\rightarrow\; \pi_1(B_p(1))]$$

admits a subgroup $\Lambda' \subset \Gamma'$ with
(1) $[\Gamma' : \Lambda'] < w_n$, *where w_n depends only on n;*
(2) Λ' *is solvable with length of polycyclicity $\leq n$.*

In particular, if a complete Riemannian n-manifold M has $Ric_M diam(M)^2 > -(n-1)\delta_n$, then $\pi_1(M)$ is almost solvable. That is, $\pi_1(M)$ contains a solvable subgroup of finite index.

Remark 1.4. Gromov conjectured in [**5**] that a positive number ϵ_n exists which depends only on n such that if a compact Riemannian n-manifold with almost nonnegative Ricci curvature $Ric_M diam(M)^2 > -\epsilon_n$, then $\pi_1(M)$ is *almost nilpotent*. Fukaya and Yamaguchi have shown that Gromov's conjecture holds when the condition $K_M diam(M)^2 > -\epsilon_n$ in [**3**]. By using a solvable subgroup instead of a nilpotent subgroup, Fukaya and Yamaguchi gave a generalized Margulis's lemma in [Theorem A2.1 in [**3**]]. Some scholars suggested that establishing a splitting theorem and a volume convergence theorem under an almost nonnegative Ricci curvature bound is sufficient to extend Margulis's lemma to the Ricci case. However, it is not enough to extend Fukaya and Yamaguchi's result to the Ricci case with the original arguments, since their argument depends heavily on having a fibration with *almost Riemannian submersion*, which cannot generally be constructed under a lower Ricci curvature bound. Therefore, Fukaya and Yamaguchi's approaches cannot be applied directly to Riemannian manifolds with a lower Ricci curvature bound.

The proposed approach considers strongly the induction steps to prove the Technical lemma 3.1 in section 3, which is a weaker version of [Theorem 7.1 in [**3**]]. The Margulis's lemma under a lower Ricci curvature bound cannot be obtained. We can only obtain a weaker version of the Margulis's lemma for only " one point " in the Riemannian manifold under consideration, which is sufficient to prove the solvability theorem for almost nonnegatively Ricci curved manifolds. Although the nilpotency result still cannot be obtained, Theorem B confirms, in some sense, that the almost solvability version of Gromov's conjecture.

The remainder of this paper is organized into four sections. Section 2 introduces the main theories used herein including the theory of pointed equivarient convergence and the splitting theorems for the proposed proof of Theorem A and Theorem B. In particular, Corollary 2.6 is established by merging the above two tools. Section 3 proves a Technical lemma, which extends the solvability theorem in [Theorem 7.1 in [**3**]]. Section 4 proves of Theorem B by using the Technical lemma. Then, section 5 proves Theorem A using Theorem B.

2 Equivariant Pointed Hausdorff Convergence and the Splitting Theorem

Recall the equivariant pointed Hausdorff convergence in [**3**]. Let $\mathcal{M}$ be the set of all isometry classes of pointed metric spaces (X, p) such that, for each $D > 0$, the ball $B_p(D)$ around p with radius D is relatively compact and such that X is a length space. Let $\mathcal{M}_{eq}$ be the set of triples (X, Γ, p), where $(X, p) \in \mathcal{M}$ and Γ denote a closed subgroup of isometries of X. Put $\Gamma(D) = \{\gamma \in \Gamma \mid d(\gamma p, p) < D\}$.

Definition 2.1. Let (X, Γ, p), $(Y, G, q) \in \mathcal{M}_{eq}$. An *$\varepsilon$-equivariant pointed Hausdorff approximation* is a triple (f, ϕ, ψ) of maps $f : B_p(1/\varepsilon) \to Y$, $\phi : \Gamma(1/\varepsilon) \to G(1/\varepsilon)$ and $\psi : G(1/\varepsilon) \to \Gamma(1/\varepsilon)$ such that
(2.1.1) $f(p) = q$;
(2.1.2) the ε-neighborhood of $f(B_p(1/\varepsilon))$ contains $B_q(1/\varepsilon)$;
(2.1.3) if $x\ y \in B_p(1/\varepsilon)$, then $\mid d(f(x), f(y)) - d(x, y) \mid< \varepsilon$;
(2.1.4) if $\gamma \in \Gamma(1/\varepsilon)$, $x \in B_p(1/\varepsilon)$, $\gamma x \in B_p(1/\varepsilon)$, then

$$d(f(\gamma x), \phi(\gamma)(f(x)) < \varepsilon;$$

(2.1.5) if $\lambda \in G(1/\varepsilon)$, $x \in B_p(1/\varepsilon)$, $\psi(\mu)(x) \in B_p(1/\varepsilon)$, then

$$d(f(\psi(\mu)(x)), \mu f(x)) < \varepsilon.$$

Hereafter the notion $\lim_{i\to\infty}(X_i, G_i, x_i) \ = \ (Y, G, y)$ means

$$\lim_{i\to\infty} d_{eH}((X_i, G_i, x_i), (Y, G, y)) \ = \ 0$$

, where d_{eH} denotes the *equivariant pointed Hausdorff distance.* For brevity, d_{eH} is also expressed as d_H in the remainder of this paper.

The following theorem comes from [Proposition 3.6 in **[3]**].

Theorem 2.2. *Let $(X_i, \Gamma_i, p_i) \in \mathcal{M}_{eq}$, $(Y, q) \in \mathcal{M}$. Suppose that* $\lim_{i\to\infty}(X_i, p_i) \ = \ (Y, q)$. *Then G and a subsequence k_i can be found such that $(Y, G, q) \in \mathcal{M}_{eq}$ and* $\lim_{i\to\infty}(X_{k_i}, \Gamma_{k_i}, p_{k_i}) \ = \ (Y, G, q)$.

The following theorem is shown in [Theorem 4.2 in **[4]**] by Fukaya and Yamaguchi and its proof is found in [Appendix A.1 in **[3]**].

Theorem 2.3. *Let (X_i, Γ_i, p_i), $(Y, G, q) \in \mathcal{M}_{eq}$ be such that* $\lim_{i\to\infty}(X_i, \Gamma_i, p_i) \ = \ (Y, G, q)$, *and let G' be a normal subgroup of G. Assume that*
(2.3.1) *G/G' is discrete.*
(2.3.2) *Y/G is compact.*
(2.3.3) *Γ_i is discrete and free and X_i is simply connected.*
(2.3.4) *G' is generated by $G'(R_0)$ for some $R_0 > 0$.*
Then there exists a sequence of normal subgroups Γ_i of Γ such that
(2.3.5) $\lim_{i\to\infty}(X_i, \Gamma_i', p_i) \ = \ (Y, G', q)$.
(2.3.6) *Γ_i/Γ_i' is isometric to G/G' for sufficiently large i.*
(2.3.7) *G/G' is finitely presented.*
(2.3.8) *Γ_i' is generated by $\Gamma_i'(R_0 + \varepsilon_i)$ for some $\varepsilon_i \to 0$*

Cheeger and Colding indicate in [Theorem 6.64 in **[1]**] that the limit space of a sequence of complete pointed-Riemannian n-manifolds with almost nonnegative Ricci curvature splits as long as it contains a line.

Theorem 2.4. *Let (M_i^n, p_i) be a sequence of complete pointed-Riemannian n-manifolds. Denote $B_{p_i}(R_i)$ be the open R_i-ball in M_i^n around p_i and $R_i \to \infty$*

as $i \to \infty$. Let $(X, p_\infty) \in \mathcal{M}$ with $\lim_{i\to\infty}(B_{p_i}(R_i), p_i) = (X, p_\infty)$. Suppose $Ric_{B_{p_i}(R_i)} \geq -\epsilon_i^2$, where $\epsilon_i \to 0$ as $i \to \infty$, and X contains a line. Then X splits, isometrically, $X = \boldsymbol{R} \times X'$.

Combining Theorem 2.2 with Theorem 2.4, as in [Corollary 5.3 and Theorem 5.4 in [**3**]], yields Corollary 2.5 and Corollary 2.6 respectively. Corollary 2.6 is especially important for the main proof.

Corollary 2.5. *Let (M_i, p_i), $B_{p_i}(R_i)$ and (X, p_∞) be as in Theorem 2.4. Suppose G_i is a closed subgroup of $Isom(B_{p_i}(R_i))$ such that $diam(B_{p_i}(R_i)/G_i) \leq D$ for some constant D. Then there exists a subsequence k_i that*

$$\lim_{i\to\infty}(B_{p_{k_i}}(R_{k_i}), G_{k_i}, p_{k_i}) = (\boldsymbol{R}^\ell \times Y, G, p_\infty)$$

, where Y is a compact metric space and G is a closed subgroup of $Isom(\boldsymbol{R}^\ell \times Y)$ with $\ell \leq n$.

Corollary 2.6. *Let (M_i, p_i) be a sequence of complete pointed-Riemannian n-manifolds with $Ric_{M_i} \geq -(n-1)$. Suppose $\lim_{i\to\infty}(M_i, p_i) = (X, p_\infty)$, where $(X, p_\infty) \in \mathcal{M}$. Then for every $x \in X$ there exists sequences $y_i \in X$, $q_i \in M_i$ and $r_i \to \infty$ as $i \to \infty$ such that*
(2.6.1) $y_i \to x$, $q_i \to x$ *as* $i \to \infty$,
(2.6.2) $\lim_{i\to\infty}((X, r_i d_X), y_i) = (\boldsymbol{R}^k, can, 0) = \lim_{i\to\infty}((M_i, r_i g_i), q_i)$,
(2.6.3) $k \leq n$,
where d_X and g_i are the original metric of X and M_i respectively.

Remark 2.7 As in the proof of [Theorem 5.4 in [**3**]], Corollary 2.6 can be demonstrated by blowing up the metrics at most finite times and using Theorem 2.4. Notably, for a given convergent sequence $\delta_i \to 0$, a sequence $r_i \to \infty$ in Corollary 2.6 can always be found such that $r_i\delta_i \to 0$.

Let Y be a compact metric space, and let G a closed subgroup of $Isom(\boldsymbol{R}^\ell \times Y)$. Since G preserves the splitting $\boldsymbol{R}^\ell \times Y$, the projection $\phi : G \to Isom(\boldsymbol{R}^\ell)$ is well defined. The following theorem was shown in [Lemma 6.1 in [**3**]].

Theorem 2.8. *For each $\varepsilon > 0$ there exists a normal subgroup G_ε of G such that*
(2.8.1) *G/G_ε is discrete;*
(2.8.2) *there exists an exact sequence $1 \to G_\varepsilon \to G \to \Lambda \to 1$,*
where Λ contains a finite-index free abelian subgroup of rank not greater than $\dim(\boldsymbol{R}^\ell/\phi(G))$;
(2.8.3) *for every $g \in G_\varepsilon$ and every $x \in \boldsymbol{R}^\ell \times Y$ there exists $g_1, \ldots, g_s \in G_\varepsilon$ satisfying*
(i) $g = g_s \ldots g_1$,
(ii) $d(g_i g_{i-1} \ldots g_1(x), g_{i-1} \ldots g_1(x)) < \varepsilon$ for all $1 \leq i \leq s$.

The group G_ε was constructed in [**3**] as follows: Let $K = Ker(\phi)$, which acts on Y. Set $\hat{K}_\varepsilon = \{g \in K \mid d(g(x), x) < \varepsilon \, \forall \, x \in Y\}$. Let K_ε be the group generated by $\hat{K}_\varepsilon$. Since K_ε is normal in G, the natural projection $\pi : G \to G/K_\varepsilon$ is defined.

Define $G_\varepsilon = \pi^{-1}((G/K_\varepsilon)_0)$, where $(G/K_\varepsilon)_0$ is the identity component of G/K_ε.

Remark 2.9. Fukaya and Yamaguchi showed in [**4**] that if the limit space Y is an Alexandrov space, then $Isom(Y)$ is in fact a Lie group. Thus G is a Lie group and G_0 can be treated as G_ε for every ε. Moreover, Cheeger and Colding announced a result that if $Ric_{M_i^n} \geq -(n-1)$ and $Vol(B_{p_i}(1)) \geq v > 0$ for all i and all $p_i \in M_i$, then the isometry group of the limit space is a Lie group. Siganificantly, G_ϵ in Theorem 2.8 is independent of curvature and volume.

3 A Technical Lemma

The following Technical lemma plays a very important role in the proposed approach to proving Theorem A and Theorem B, and it has its own interest for investigating manifolds with lower Ricci curvature bounds. The proposed lemma can be viewed as a *weak* Ricci version of [Theorem 7.1 in [**3**]].

Technical Lemma 3.1. *For given positive integers n and k, $n \geq k$ and a positive number μ_0, there exist positive numbers $\epsilon = \epsilon_{n,k}(\mu_0)$, $w = w_{n,k}$ and a function $\tau(\epsilon) = \tau_{n,k,\mu_0}(\epsilon)$ with $\lim_{\epsilon\to 0}\tau(\epsilon) = 0$ such that if (M^n, p) and (N^k, q) are pointed-Riemannian manifolds of dimension n and k respectively such that*
(3.1.1) $Ric_M \geq -(n-1)$, $Ric_N \geq -(n-1)$ *and* $inj(N) > \mu_0 > 0$,
(3.1.2) $d_{GH}((M,p),(N,q)) < \epsilon$,
where d_{GH} denotes the Gromov-Hausdorff distance, then there exists a map $f : M \to N$ with $f(p) = q$ satisfying the following:
(3.1.3) *f is a continuous $\tau(\epsilon)$-Hausdorff approximation such that $f_* : \pi_1(M,p) \to \pi_1(N,q)$ is surjective;*
(3.1.4) *Let $V = B_q(\frac{\mu_0}{2})$ be the ball around q with radius $\frac{\mu_0}{2}$. Set $U = f^{-1}(V)$. Then there is a normal subgroup H of the fundamental group $\Gamma = \pi_1(U)$ of U such that*
(i) *H is a solvable subgroup of Γ with length of polycyclicity $\leq n-k$,*
(ii) $[\Gamma : H] \leq w_{n,k}$.

Remark 3.2. Sormani and Wei considered in [**9**] the group $\bar{\pi}_1(Y)$ of deck transforms in the universal cover Y of the Gromov-Hausdorff limit of compact manifolds $\{M_i^n\}$ with $Ric_{M_i^n} \geq (n-1)H$ and $diam(M_i^n) \leq D$ for some $H \in \boldsymbol{R}$ and $D > 0$. They showed that for sufficient large n_0 depending on Y, a surjective homeomorphism $\Phi_i : \pi_1(M_i) \to \bar{\pi}_1(Y)$, for $i \geq n_0$ can be found. Technical Lemma 3.1 considers Gromov-Hausdorff convergence to LGC(ρ)-space to obtain a similar result.

The proof of Technical Lemma 3.1 was divided into two parts as follows. The first part reveals the existence of the map f satisfying (3.1.3).

Proof of (3.1.3). The construction of such a Hausdorff approximation f depends heavily on the assumption that $inj_q(N) \geq \mu_0$. A function $\rho : [0, r) \to [0, \infty)$ is said to be a contractibility function provided:(i) $\rho(0) = 0$, (ii) $\rho(\epsilon) \geq \epsilon$, (iii) $\rho(\epsilon) \to 0$ as $\epsilon \to 0$, (iv) ρ is non-decreasing. Then, a metric space X is said to be an *LGC(ρ)-space with a contractibility function ρ* if for every $\epsilon \in [0, r]$ and $x \in X$, the ball $B_x(\epsilon)$ is contractible inside $B_x(\rho(\epsilon))$. Since $inj_q(N) > \mu_0 > 0$, then (N, q) is an

LGC(ρ)-space with contractibility function $\rho(s) = s$ defined in $[0, \mu_0/2]$.

Choose ϵ be such that $8(n+3)^2\epsilon < \mu_0$. Since $d_{GH}((M,p),(N,q)) < \epsilon$, a metric d is fixed on the disjoint union space $M \amalg N$ such that $d((M,p),(N,q)) < \epsilon$. For each $x \in M$, a map $h : M \to N$ can be found such that $h(x)$ is a point in N with $d(h(x), x) < \epsilon$. Thus, the triangle inequality shows that h is 4ϵ-continuous (cf. [**7**]). Hence, [Main obstruction result 3 in [**7**]] indicates a continuous map $f : M \to N$ with $d(h(x), f(x)) \leq (n+2)\epsilon$ for all $x \in M$. Take $\tau(\epsilon) = 4(n+3)\epsilon$. Clearly, $f : M \to N$ is an $\tau(\epsilon)$-Hausdorff approximation with $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$. Moreover, [Corollary 4.6 in [**7**]] and [**10**] show that the induced map $f_* : \pi_1(M,p) \to \pi_1(N,q)$ is surjective, and hence (3.1.3) is established. ■

The proof of (3.1.4) is similarly to Fukaya and Yamaguchi's proof in [Theorem 7.1 in [**3**]]. The following proof indicates how the original process can work under the proposed settings. For brevity, we say that a group has **property (*)** if it containes a subgroup satisfying (i) and (ii) in (3.1.4).

Proof of (3.1.4). This proof is achieved by induction on $\dim N$ and by contradiction. When $\dim N = n$, by [Theorem A1.12 in [**2**]], a value of ϵ can be chosen that is small enough such to form a diffeomorphism from (M^n, p) to (N^n, q), which is taken as f. Since $\pi_1(V)$ is trivial, Γ is also trivial, and the theorem holds. Now suppose that (3.1.4) holds for $k < \dim N < n$ with fixed k but not for $\dim N = k$. Then, for sequences $\epsilon_i \to 0$ and $w_i \to \infty$ as $i \to \infty$, sequences (M_i^n, p_i) and (N_i^k, q_i) exist satisfying (3.1.1) and (3.1.2), but no map $(M_i, p_i) \to (N_i, q_i)$ satisfies (3.1.4) for $\epsilon = \epsilon_i$ and $w = w_i$ simultaneously. Notable, a continuous $\tau(\epsilon_i)$-Hausdorff approximation map $f_i : M_i \to N_i$ with $f_i(p_i) = q_i$ always exists for sufficiently large i.

Define $V_i = B_{q_i}(\frac{\mu_0}{2})$, $U_i = f_i^{-1}(V_i)$ and $\Gamma_i = \pi_1(U_i)$ as above. To utilize the induction hypothesis, the metrics need to be blowed-up by the technique shown in [**3**]. However, since f_i in this case is only a continuous map, the metrics can be scaled as follows. Since $\dim(M_i^n) > \dim(N_i^k)$, there exists $q_i' \in B_{q_i}(\frac{\tau(\epsilon_i)}{10})$ such that $diam(f_i^{-1}(q_i')) > 0$. Let $p_i' \in f_i^{-1}(q_i')$. Then it can be shown that

$$d_H((M_i^n, p_i'), (N_i^k, q_i')) < 2\tau(\epsilon_i).$$

Indeed, since $d_H((M_i^n, p_i), (N_i^k, q_i)) < \epsilon$, there exist ϵ_i-pointed Hausdorff approximations h_i and $\tilde{h}_i$ such that the map f_i is induced by h_i as in the proof of (3.1.3) with $h_i(p_i) = q_i$, $\tilde{h}_i(q_i) = p_i$, $h_i(B_{p_i}(\frac{1}{\epsilon_i})) \subseteq B_{q_i}(\frac{1}{\epsilon_i} + \epsilon_i)$ and $\tilde{h}_i(B_{q_i}(\frac{1}{\epsilon_i})) \subseteq B_{p_i}(\frac{1}{\epsilon_i} + \epsilon_i)$. Moreover, $d(p_i, p_i') < \frac{6}{5}\tau(\epsilon_i)$ since f_i is a $\tau(\epsilon_i)$-Hausdorff approximation. Notably, $\epsilon_i < \frac{\tau(\epsilon)}{10}$ and for $x \in B_{p_i'}(\frac{1}{2\tau(\epsilon_i)})$ we have

$$\begin{aligned} d(h_i(x), q_i') &\leq d(h_i(x), h_i(p_i)) + d(q_i, q_i') \\ &< d(x, p_i) + 2\epsilon_i + d(q_i, q_i') \\ &\leq d(x, p_i) + d(p_i, p_i') + d(q_i, q_i') + 2\epsilon_i \\ &< \frac{1}{2\tau(\epsilon_i)} + 2\tau(\epsilon_i) \end{aligned}$$

Thus $h_i(B_{p_i'}(\frac{1}{2\tau(\epsilon_i)})) \subseteq B_{q_i'}(\frac{1}{2\tau(\epsilon_i)} + 2\tau(\epsilon_i))$. Similarly, $\tilde{h}_i(B_{q_i'}(\frac{1}{2\tau(\epsilon_i)})) \subseteq B_{p_i'}(\frac{1}{2\tau(\epsilon_i)} + 2\tau(\epsilon_i))$ and then $d_H((M_i^n, p_i'), (N_i^k, q_i')) < 2\tau(\epsilon_i)$ can be demonstrated. Therefore, for brevity, the map $f_i : M_i^n \to N_i^k$ with $f_i(p_i) = q_i$ is assumed to satisfy the condition as in (3.1.3)and $diam(f_i^{-1}(q_i)) > 0$ for large enough i.

For each i, set δ_i to a positive number such that

$$\delta_i = (\frac{1}{10} \sup\{d(x,y)|x,y \in f_i^{-1}(q_i)\})^2 \leq \frac{1}{4}(\tau(\epsilon_i))^2,$$

then $\delta_i \to 0$ as $i \to \infty$. Blow-up the original metrics g_{M_i} and g_{N_i} of M_i and N_i respectively by

$$g_i = \frac{1}{\delta_i} g_{M_i} \quad and \quad h_i = \frac{1}{\delta_i} g_{N_i}.$$

Then, by taking a subsequence if necessary,

$$d_H((V_i, h_i), q_i), ((\boldsymbol{R}^k, can), 0) < 2\sqrt{\delta_i} \leq \tau(\epsilon_i)$$

since $Ric_{N_i} \geq -(n-1)$ and $inj_{q_i}(N_i) > \mu_0$. Assume that $((U_i, g_i), p_i)$ converges to a pointed metric space (X, x_0). Moreover, we have that the sequence $\{f_i\}$ is an *almost equicontinuous family* and is convergent.

Sublemma 3.3. *f_i converges to a continuous map $f : X \to \boldsymbol{R}^k$ with*
(3.3.1) *For every $x, y \in \boldsymbol{R}^k$, $d(f^{-1}(x), f^{-1}(y)) = d(x,y)$.*
(3.3.2) *For every $x' \in f^{-1}(x)$ there exists a point $y' \in f^{-1}(y)$ such that $d(x', y') = d(x,y)$.*
(3.3.3) $f(x_0) = 0$.

Proof of Sublemma 3.3. Assume that $\tau(\epsilon_i)$ decreases monotonically to 0. Fix i_0 and select $\eta_{i_0} = \frac{3}{2}\tau(\epsilon_{i_0})$. Then, given $\eta > \eta_{i_0}$, $d(f_i(x_i), f_i(y_i)) < \eta$ for all $x_i, y_i \in U_i$ holds provided $d(x_i, y_i) < \frac{1}{2}\tau(\epsilon_i)$ and $i \geq i_0$. Take a dense subset $A_i = \{a_1^i, a_2^i, \ldots\} \subset U_i$ for each i and let $a_j^i \to a_j \in X$ as $i \to \infty$. Then, the set $A = \{a_j\}_{j=1}^{\infty} \subset X$ is dense in X. Since $((V_i, h_i), q_i)$ converges to $((\boldsymbol{R}^k, can), 0)$, by using the diagonal process, $\{f_i(a_j^i)\}$ converges for each fixed j. Define the map $f : A \to \boldsymbol{R}^k$ by $f(a_j) \equiv \lim_{i\to\infty} f_i(a_j^i)$. Then, for given $\xi > 0$, choose i_0 large enough such that, for $i > i_0$, $10\tau(\epsilon_{i_0}) < \xi$. Now, for $x \in X$, it can be assumed that $a_j \to x$ as $j \to \infty$ and $d(a_j, a_m) < \frac{1}{10}\tau(\epsilon_{i_0})$ for $j,\ m \geq j_0$, where j_0 depends on i_0 and $j_0 \to \infty$ as $i_0 \to \infty$. Then

$$d(f(a_j), f(a_m)) \leq d(f_i(a_j^i), f_i(a_m^i)) + 4\tau(\epsilon_i) < 2\eta_{i_0} + 4\tau(\epsilon_i) < \xi.$$

Therefore, $\{f(a_j)\}$ is a Cauchy sequence in $\boldsymbol{R}^k$ and hence $f : X \to \boldsymbol{R}^k$ defined by $f(x) \equiv \lim_{j\to\infty} f(a_j)$ is well-defined and continuous. Clearly f satisfies equations (3.3.1)-(3.3.3). ∎

From Theorem 2.4 and (3.3.1)-(3.3.3), [Lemma 7.4 and 7.5 in [**3**]] is applied to show that X is isometric to a product $\boldsymbol{R}^k \times Z$, where Z is compact and not a single point, and that the map $f : \boldsymbol{R}^k \times Z \to \boldsymbol{R}^k$ is the projection.

Let $B^k(r_0)$ denote the metric r_0-ball in $\boldsymbol{R}^k$ around the origin. Denote $U_i(r_0) \equiv f_i^{-1}(B_{q_i}(r_0))$, where $B_{q_i}(r_0) \subseteq (N_i, h_i)$, then

$$\lim_{i\to\infty} U_i(r_0) = B^k(r_0) \times Z. \tag{3.4}$$

Let d_0 be the distance of $B^k(r_0) \times Z$. Corollary 2.6 gives the sequences $y_j \in B^k(r_0) \times Z$ and $r_j \to \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty}((B^k(r_0) \times Z, r_j d_0), y_j) \;=\; ((\boldsymbol{R}^m, can), 0)) \tag{3.5}$$

, where $m > k$ since Z is not one point. Combining (3.4) and (3.5) shows that for given $\epsilon > 0$, i_0, j_0 and $\hat{p}_i \in U_i(r_0)$ exist such that for $i \geq i_0$,

$$d_H((U_i(r_0), r_{j_0} g_i), \hat{p}_i), ((\boldsymbol{R}^m, can), 0)) \;<\; \epsilon. \tag{3.6}$$

Therefore, induction hypothesis gives, for sufficiently large i, a map $\Phi_i : (U_i(r_0), r_{j_0} g_i), \hat{p}_i) \to ((\boldsymbol{R}^m, can), 0)$ such that Φ_i is a $\tau(\epsilon)$-Hausdorff approximation, and the fundamental group $\pi_1(\Phi_i^{-1}(B^m(\frac{\mu_0}{2})))$ satisfying property (*) for $w = w_{n,m}$.

Let $\Gamma_i(\hat{p}_i, \mu_0) = Im[i_* : \pi_1(\Phi_i^{-1}(B^m(\frac{\mu_0}{2}))) \to \pi_1(U_i)]$ be the image of the induced map i_* of the inclusion map $i : \Phi_i^{-1}(B^m(\frac{\mu_0}{2})) \to U_i$. Then $\Gamma_i(\hat{p}_i, \mu_0)$ naturally has the property (*).

Let $(\tilde{U}_i, \tilde{g}_i, \tilde{p}_i)$ be the universal Riemannian covering space of (U_i, g_i, p_i) with covering map $\Pi_i : (\tilde{U}_i, \tilde{g}_i, \tilde{p}_i) \to (U_i, g_i, p_i)$. Then, $\Gamma_i = \pi_1(U_i, p_i)$ is the deck transformation group. By taking a subsequence if necessary, a triple $(W, G, \tilde{p}_\infty) \in \mathcal{M}_{eq}$ can be assumed such that

$$\lim_{i\to\infty}(\tilde{U}_i, \Gamma_i, \tilde{p}_i) \;=\; (W, G, \tilde{p}_\infty), \tag{3.7}$$

$$\Pi_i \textit{ converges to a map } \Pi_\infty : W \to \boldsymbol{R}^k \times Z. \tag{3.8}$$

Since Π_∞ also fulfills (3.3.1) and (3.3.2), Theorem 2.4 and Corollary 2.5 imply that W is isometric to $\boldsymbol{R}^k \times W'$ and W' is isometric to $(\boldsymbol{R}^\ell \times Y)$, where Y is a compact metric space.

Let $\tilde{U}_i(r_0) = \Pi_i^{-1}(U_i(r_0))$. Then

$$\lim_{i\to\infty}(\tilde{U}_i(r_0), \Gamma_i, \tilde{p}_i) \;=\; (B^k(r_0) \times \boldsymbol{R}^\ell \times Y, G, \tilde{p}_\infty) \tag{3.9}$$

Applying Theorem 2.8 gives, for each $\varepsilon > 0$, a normal subgroup G_ε of G such that (2.8.1)-(2.8.3) hold. Therefore, Theorem 2.3 gives a sequence of normal subgroups $\Gamma_{i,\varepsilon}$ of Γ_i with

$$\lim_{i\to\infty}(\tilde{U}_i(r_0), \Gamma_{i,\varepsilon}, \tilde{p}_i) \;=\; (B^k(r_0) \times \boldsymbol{R}^\ell \times Y, G_\varepsilon, \tilde{p}_\infty), \textit{ and} \tag{3.10}$$

$$\Gamma_i/\Gamma_{i,\varepsilon} \cong G/G_\varepsilon \tag{3.11}$$

for each sufficiently large i.

Now we investigate the relationship between $\Gamma_{i,\varepsilon}$ and $\Gamma_i(x, \varepsilon)$ for some $\varepsilon > 0$.

Sublemma 3.12. *For every $x \in U_i(r_0)$, $\Gamma_{i,\varepsilon} \subset \Gamma_i(x, 3\varepsilon)$ for sufficiently large i.*

Proof of Sublemma 3.12. Let τ_i be the equivarient pointed Hausdorff distance between $(\tilde{U}_i(r_0), \Gamma_{i,\varepsilon}, \tilde{p}_i)$ and $(B^k(r_0) \times \boldsymbol{R}^\ell \times Y, G_\varepsilon, q)$. To prove this lemma, we shall show that, for sufficiently large i, each element $\gamma \in \Gamma_{i,\varepsilon}$ can be generated by geodesic loops of length less than $C\tau_i + \epsilon$ at x, where $C = C(G, \varepsilon) > 0$.

As in Definition 2.1, let $\varphi_i : (\tilde{U}_i(r_0), \Gamma_{i,\varepsilon}, \tilde{p}_i) \to (B^k(r_0) \times \boldsymbol{R}^\ell \times Y, G_\varepsilon, q)$ be a τ_i-Hausdorff approximation, and let $\lambda_i : \Gamma_{i,\varepsilon}(\frac{1}{\tau_i}) \to G_\varepsilon(\frac{1}{\tau_i})$, $\lambda_i' : G_\varepsilon(\frac{1}{\tau_i}) \to \Gamma_{i,\varepsilon}(\frac{1}{\tau_i})$ be the corresponding maps. A point $\tilde{x} \in \tilde{M}_i$ over x can be taken such that $d(\tilde{x}, \tilde{p}_i)$ is uniformly bounded, say $2r_0$. Condition (2.3.8) implies that the length of γ is uniformly bounded by a constant. Hence, it can be assumed that $\bar{\gamma} = \lambda_i(\gamma)$ for each sufficiently large i. By (2.8.3), there are $\bar{\gamma}_1, \dots, \bar{\gamma}_s \in G_\varepsilon$ such that

$$\bar{\gamma} = \bar{\gamma}_s \bar{\gamma}_{s-1} \dots \bar{\gamma}_1, \tag{3.12.1}$$

$$d(\bar{\gamma}_j \bar{\gamma}_{j-1} \dots \bar{\gamma}_1(\varphi(\tilde{x})), \bar{\gamma}_{j-1} \dots \bar{\gamma}_1(\varphi(\tilde{x})) < \varepsilon, \ for\ 1 \le j \le s. \tag{3.12.2}$$

For each j, write $\gamma_j = \lambda_i'(\bar{\gamma}_j)$. Then, γ has the expression $\gamma \ = \ \gamma_s \dots \gamma_1 (\gamma_s \dots \gamma_1)^{-1}\gamma$. and therefore,

$$d(\gamma_j \gamma_{j-1} \dots \gamma_1(\tilde{x}), \gamma_{j-1} \dots \gamma_1(\tilde{x})) < 2(j+1)\tau_i + \varepsilon \tag{3.12.3}$$

$$d(\gamma(\tilde{x}), \gamma_s \dots \gamma_1(\tilde{x}) < 2(s+1)\tau_i. \tag{3.12.4}$$

Notably, s depends on G, ε and μ_0. Thus, the sublemma holds. ∎

Up to now, we know that Γ_{i,μ_0} has a subgroup H_{i,μ_0} with the property (*). Then, the argument in [[**3**], p.288 and p.289] can be used to conclude that $\pi_1(U_i)$ has a subgroup H with property (*) for i large enough, which contradicts the previous assumption. Hence, for $\dim N = k$, $\epsilon_{n,k}(\mu_0)$, $w_{n,k}$ and a map $f_i : M_i{}^n \to N_i{}^k$ exist such that (3.1.3) and (3.1.4) hold under the assumption of (3.1.1) and (3.1.2). Thus, by induction, the proof of the Technical Lemma 3.1 is finished. ∎

4 A weak Margulis's Lemma

This section presents a proof of Theorem B. First, by Technical lemma 3.1, the following lemma is presented.

Lemma 4.1. *For given integer n and k with $n \ge k$ there exist $\delta_{n,k}$, $I_{n,k}$ and $w_{n,k}$ depending only on n and k satisfying the following: Let (M_i^n, p_i) be a sequence of complete pointed Riemannian n-manifold with $Ric_{M_i^n} \ge -(n-1)$. Suppose (M_i^n, p_i) converges to a metric space (X, p_∞) of dimension k in the pointed Hausdorff convergence. Then there exist $p_i' \in B_{p_i}(1/2)$ such that, for $\delta < \delta_{n,k}$ and $i \ge I_{n,k}$, the image of the inclusion homomorphism*

$$\Gamma' \ = \ Im[\pi_1(B_{p_i'}(\delta)) \ \to \ \pi_1(B_{p_i}(1))]$$

admits a subgroup $\Lambda' \subset \Gamma'$ with
(4.1.1) $[\Gamma' : \Lambda'] < w_{n,k}$;
(4.1.2) Λ' *is solvable with length of polycyclicity* $\le n - k$.

Proof of Lemma 4.1. The lemma can be proven by contradiction. Assume that there exit $\delta_i \to 0$, $w_i \to \infty$ and M_i with $Ric_{M_i} \geq -(n-1)$ satisfying the following: For each $p'_i \in B_{p_i}(1/2)$ and each $I > 0$, there exists an $i \geq I$ such that

$$\Gamma'_i \ = \ Im[\pi_1(B_{p'_i}(\delta_i)) \ \to \ \pi_1(B_{p_i}(1))]$$

never admits a subgroup with properties (4.1.1) and (4.1.2) in Lemma 4.1 for $w = w_i$.

By Corollary 2.6, there are sequences $q'_i \in B_{p_i}(1/2)$ and $r_i \to \infty$ such that $\lim_{i\to\infty}((M_i, r_i g_i), q'_i) = (\boldsymbol{R}^m, can, 0)$, where $m \geq k = \dim(X)$. The sequence r_i can be selected such that $r_i\delta_i \to 0$ as $i \to \infty$. From Technical lemma 3.1, there exist an i_0 large enough and a Hausdorff approximation $f_{i_0} : (M^n_{i_0}, q'_{i_0}) \to (\boldsymbol{R}^k, 0)$ with $f_{i_0}(q'_{i_0}) = 0$ so that the fundamental group $\Gamma''_{i_0} = \pi_1(f^{-1}_{i_0}(B^k(10)))$ admits a subgroup $\Lambda''_{i_0} \subset \Gamma''_{i_0}$ satisfying properties (4.1.1) and (4.1.2) of Lemma 4.1 for $w_{n,k}$ independent of i. By $r_i\delta_i \to 0$,

$$\Gamma'_{i_0} \ \subset \ Im[\Gamma''_{i_0} \to \pi_1(M_{i_0})].$$

Then, Γ'_{i_0} admits a subgroup Λ'_{i_0} satisfying properties (4.1.1) and (4.1.2) of Lemma 4.1 creating a contradiction. ■

Proof of Theorem B. Now, a proof of Theorem B is presented. Given a divergent sequence $r_i \to \infty$, there exists a metric space (X, p_∞) of dimension $k \leq n$ such that $((M^n, r_i g_M), p)$ converges to (X, p_∞). By Lemma 4.1, choose $\delta_n = \min_{0\leq k\leq n}\delta_{n,k}$ and $w_n = \max_{0\leq k\leq n} w_{n,k}$, and then Theorem B holds. ■

Remark 4.2. Consider a compact Riemannian n-manifold M^n with $Ric_{M^n} \geq 0$. Scaling the metric of M so that $diam(M) \leq \frac{\delta_n}{2}$ still leaves $Ric_M \geq 0$, thus showing the fundamental group $\pi_1(M)$ of M admits a subgroup H such that

(4.2.1) H is solvable with length of polycyclicity $\leq n$;

(4.2.2) $[H : \pi_1(M)] \ < \ w_n$.

In the next section, this result is applied to present a proof of Theorem A.

5 Proof of Theorem A

The proof of Theorem A followa similar methods to the proofs in [Corollary 7.20 in [**3**], p.289] and [Corollary 0.11 in [**3**], p.290 and p.291] respectively. However, the results are extended to manifolds with positive Ricci curvature.

Proof of part (a) in Theorem A. Let M^n be a compact Riemannian n-manifold with $Ric \geq 0$. By Remark 4.2 (weak Margulis's lemma), the fundamental group $\pi_1(M)$ of M admits a subgroup H satisfying (4.2.1) and (4.2.2) and then $b_1(M, \boldsymbol{Z}_p) \leq n$ for all prime $p \geq w_n \equiv p_n$. This proof shows by contradiction that $b_1(M, \boldsymbol{Z}_p) \leq n-1$, as long as $Ric > 0$.

This proof uses the notations as the proof of Technical Lemma 3.1. Consider a compact Riemannian n-manifold (M, g_M) with positive Ricci curvature and $b_1(M, Z_P) = n$. Since M is compact, the metric g_M can be scaled such that $diam(M)$ is small enough. So, M can be assumed to be Hausdorff close to a point. Define g_i as in the proof of (3.1.4). Then, for each i, $U_i = (M, g_i)$, and hence $Z = \lim_{i\to\infty} U_i$ is in fact a compact n-manifold (cf. (3.4)). Equations (3.7), (3.8) and (3.9) indicate that the universal Riemannian covering $\tilde{U}_i = \tilde{M}_i$, $\Gamma_i = \pi_1(M)$, and therefore $(\tilde{M}_i, \pi_1(M)$ converges to $(\boldsymbol{R}^\ell \times Y, G)$. Hence, $(\boldsymbol{R}^\ell \times Y)/G \ = \ Z$.

Notably, $\boldsymbol{R}^\ell/\phi(G)$ is compact, and the projection $\phi : G \to Isom(\boldsymbol{R}^\ell)$ is defined as in Theorem 2.8. Moreover, since $b_1(M, Z_P) = n$, the estimation of the length of polycyclicity in [[**3**], p.288-289] gives the following equality for Hausdorff dimensions:

$$dim\frac{\boldsymbol{R}^\ell}{\phi(G)} \ = \ dimZ.$$

Then, the generalized Bieberbach's theorem [Corollary 4.2 in [**3**], p.273], reveals a finite-index normal subgroup G' of $\phi(G)$ such that $\boldsymbol{R}^\ell/G'$ is a flat s-torus T^s for some $s > 0$. Futhermore, Theorem 2.3 gives a finite-index normal subgroup $\hat{\Gamma}_i$ of $\Gamma_i = \pi_1(M)$ converging to $\phi^{-1}(G')$. Thus, the compact manifold $\hat{M}_i \equiv \tilde{M}_i/\hat{\Gamma}_i$ converges to T^s. Additionally, $Ric_{\hat{M}_i} > 0$ and $\hat{M}_i$ is a covering space of M. Then, $\pi_1(\hat{M}_i)$ is a subgroup of $\pi_1(M_i)$. Using (3.1.3), a surjection between $\pi_1(\hat{M}_i)$ and $\pi_1(T^s)$ is found for i large enough. Thus, $\pi_1(\hat{M}_i)$ has infinite order and contradicts to the fact that $\pi_1(M)$ is finite. Hence, $b_1(M, \boldsymbol{Z}_p) \leq n-1$ for all prime $p \geq w_n \equiv p_n$.∎

Proof of part (b) in Theorem A. Part (b) is also proven by contradiction. Consider a sequence of Riemannian manifolds M_i with $Ric_{M_i} > 0$, and universal Riemannian covering $\tilde{M}_i$ with,

$$\lim_{i\to\infty} \frac{diam(\tilde{M}_i)}{diam(M_i)} \ = \ \infty.$$

Scaling the metric gives $diam(\tilde{M}_i) = 1$ for all i yielding a sequence ϵ_i with $\lim_{i\to\infty} \epsilon_i = 0$ such that $diam(M_i) < \epsilon_i$. Notably, for each i, $\pi_1(M_i)$ admits a subgroup H_i which satisfies (4.2.1) and (4.2.2).

Define $H_i^0 = H_i$, $H_i^j = [H_i^{j-1}, H_i^{j-1}]$, the commutator of H_i^{j-1}, and then $H_i^n = 0$. Choose $\tilde{p}_i \in \Pi_i^{-1}(p_i)$. Since $diam(\tilde{M}_i/H_i^j) \leq 1$ for all i and j, Corollary 2.5 implies a triple (X, G_j, x_0) such that

$$\lim_{i\to\infty}(\tilde{M}_i, H_i^j, \tilde{p}_i) \ = \ (X, G_j, x_0)$$

for each $0 \leq j \leq n$. Note that G_0 acts on X transitively. Take a number j_0 such that X/G_{j_0-1} is a point and X/G_{j_0} is not a point.

Put

$$\bar{M}_i \ = \ \tilde{M}_i/H_i^{j_0}, \ \ \Lambda_i \ = \ H_i^{j_0}/H_i^{j_0-1}.$$

Let $\bar{p}_i \in \bar{M}_i$ be the point corresponding to $\tilde{p}_i$. By applying Corollary 2.6 and repeating the blow-up arguments (at most finite times), $(\bar{M}_i, \bar{p}_i)$ can be assumed to converge to $(\boldsymbol{R}^\ell, 0)$ for some $\ell > 0$. Let Λ be the group such that $\lim_{i\to\infty}(\bar{M}_i, \Lambda_i, \bar{p}_i) = (\boldsymbol{R}^\ell, \Lambda, 0)$. Then, the abelian group Λ acts on $\boldsymbol{R}^\ell$ transitively ,and Λ is indeed the vector group $\boldsymbol{R}^\ell$.

Now, the *pseudogroup technique* is applied as in [**11**]. Let

$$\Lambda_i' = \{\gamma \in \Lambda_i \mid d(\gamma(\bar{p}_i), \bar{p}_i) < 10\ell\}.$$

Consider $B^\ell(10\ell) \subset \boldsymbol{R}^\ell$ and Λ_i' to be the pseudogroups of isometric embeddings of $B^\ell(10\ell)$ to $B^\ell(20\ell)$ and $B_{\bar{p}_i}(10\ell)$ to $B_{\bar{p}_i}(20\ell)$ in $\bar{M}_i$, respectively. Consider the lattice

$$E_\infty \ = \ B^\ell(10\ell) \cap \boldsymbol{Z}^\ell$$

and take $\gamma_{1_i}, \ldots, \gamma_{\ell_i} \in \Lambda_i'$ such that γ_{j_i} converges to $e_j \in E_\infty$, where $e_1, \ldots, e_\ell$ denote the canonical basis of $\boldsymbol{Z}^\ell$. Let E_i be the pseudogroup of Λ_i' generated by $\gamma_{1_i}, \ldots, \gamma_{\ell_i}$. Since E_i is abelian, $(B_{\bar{p}_i}(10\ell), E_i)$ converges to $(B^\ell(10\ell), E_\infty)$. Hence, $\hat{M}_i \equiv B_{\bar{p}_i}(10\ell)/E_i$ converges to the flat torus $T^\ell = B^\ell(10\ell)/E_\infty$ with respect to the Hausdorff distance. As in the proof of part (a), $Ric_{\hat{M}_i} > 0$ and $\hat{M}_i$ is also a covering of M_i. Again as in (3.1.3), a surjection from $\pi_1(\hat{M}_i)$ to $\pi_1(T^\ell)$ is obtained for i large enough, and group $\pi_1(\hat{M}_i)$ thus has infinite order. Therefore, $\pi_1(M_i)$ also has infinite order, giving a contradiction. Hence, the proof is complete. ■

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