Spectral semi-norm of a p-adic Banach algebra

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Abstract

Let K be a complete ultrametric algebraically closed field, with respect to a non trivial absolute value, and let A be a commutative K-Banach algebra with identity. Let $Mult(A, \| \,.\, \|)$ be the set of continuous multiplicative semi-norms of K-algebra (with respect to the norm $\| \,.\, \|$ of A) and let $Mult_m(A, \| \,.\, \|)$ the set of the $\varphi \in Mult(A, \| \,.\, \|)$ whose kernel is a maximal ideal of A. If the norm of A is equal to its spectral semi-norm $\| \,.\, \|_{si}$ defined as $\|x\|_{si} = \lim\limits_{n\to+\infty} \| x^n \|^{\frac{1}{n}}$, we prove that $\|t\|_{si} = \sup\{\psi(t) |\ \psi \in Mult_m(A, \| \,.\, \|)\}$, without any additional condition on K. Moreover, if A has no divisors of zero, denoting by $s(x)$ the spectrum of any $x \in A$, we have $\|t\|_{si} = \sup\{|\lambda| \mid \lambda \in s(x)\}$. If $\sup\{|\lambda| \mid \lambda \in s(t)\} = \|t\|_{si}$ for every $t \in A$, then $s(t)$ is infraconnected for all $t \in A$ if and only if A has no non trivial idempotents. In particular, this applies when A has no divisors of zero. In $Mult(A, \| \,.\, \|)$ we define pseudo-dense sets, and show that a subset Σ of $Mult(A, \| \,.\, \|)$ containing $Mult_m(A, \| \,.\, \|)$ is pseudo-dense if and only if for all $t \in A$ we have $\|t\|_{si} = \sup\{\psi(t) |\ \psi \in \Sigma\}$.

1 Introduction and results

Let L be a complete ultrametric field, and let K be a complete ultrametric algebraically closed field with respect to a non trivial absolute value. L is said to be *strongly valued* if its residue class field, or if its valuation group, is not countable. As usual, given $a \in K$, $r > 0$, we put $d(a,r) = \{x \in K \mid |x-a| \le r\}$, $d(a,r^-) = \{x \in K \mid |x-a| < r\}$, $C(a,r) = \{x \in K \mid |x-a| = r\}$.
Besides, given $s > r$, we put $\Gamma(a,r,s) = d(a,s^-) \setminus d(a,r)$.

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A set D in K is said to be *infraconnected* if for every $a \in D$, the mapping I_a from D to $\mathbb{R}_+$ defined by $I_a(x) = |x-a|$ has an image whose closure in $\mathbb{R}_+$ is an interval. (In other words, a set D is not infraconnected if and only if there exist a and $b \in D$ and an annulus $\Gamma(a, r_1, r_2)$ with $0 < r_1 < r_2 < |a-b|$ such that $\Gamma(a, r_1, r_2) \cap D = \emptyset$).

Given a closed bounded set D in K, we denote by $R(D)$ the K-algebra of rational functions with no pole in D, by $\| \,.\, \|_D$ the norm of uniform convergence on D, and by $H(D)$ the completion of $R(D)$ for this norm, which is called the K-Banach algebra of analytic elements in D.

Given a ring R, $Max(R)$ denotes the set of maximal ideals of R. Let F be an algebraically closed field and let B be F-algebra with identity. Given $t \in B$, $s(t)$ will denote the spectrum of t, (i.e. the set of the $\lambda \in F$ such that $t - \lambda$is not invertible).

Let A be a commutative L-normed algebra with identity, whose norm is denoted by $\| \,.\, \|$. A norm of L-algebra φ on a L-algebra B will be said to be *semi-multiplicative* if it satisfies $\varphi(t^n) = \varphi(t)^n$ for all $t \in B$.

The map $\| \,.\, \|_{si}$ defined in A as $\|x\|_{si} = \lim\limits_{n \to +\infty} \| \, x^n \, \|^{\frac{1}{n}}$ is an ultrametric semi-norm of L-algebra called *spectral semi-norm of* A that obviously satisfies $\|x^n\|_{si} = \|x\|_{si}^n$.

Following Guennebaud's notations [6] , we denote by $Mult(A, \| \,.\, \|)$ the set of continuous multiplicative semi-norms of K-algebra (with respect to the norm $\| \,.\, \|$ of A). So, given $\varphi \in Mult(A, \| \,.\, \|)$, the set of $t \in A$ such that $\varphi(t) = 0$ is a closed prime ideal of A called *kernel of* φ, and denoted by $Ker(\varphi)$. Then, we denote by $Mult_m(A, \| \,.\, \|)$ the set of $\varphi \in Mult(A, \| \,.\, \|)$ such that $Ker(\varphi)$ is a maximal ideal of A.

Given a subset Σ of $Mult(A, \| \,.\, \|)$, the mapping $\| \,.\, \|_\Sigma$ defined as $\|t\|_\Sigma = \sup\{\psi(t) | \; \psi \in \Sigma\}$ is obviously seen to be a semi-multiplicative semi-norm of A. In particular, when $\Sigma = Mult_m(A, \| \,.\, \|)$, we denote by $\| \,.\, \|_m$ this semi-norm.

During the sixties, T.A. Springer proved that given a normed commutative L-Algebra A, for all $x \in A$, we have $\|x\|_{si} = \sup\{\psi(x) | \; \psi \in Mult(A, \| \,.\, \|)\}$ ([9], Corollary 6.25).

We will denote by (q) and (s) these properties:

(q) $\sup\{|x| \mid x \in s(t)\} = \|t\|_{si}$ for every $t \in A$.

(s) $\| \,.\, \|_{si} = \| \,.\, \|_m$

Remark: This is an opportunity to correct an inadvertance mistake in [3], Theorem 1.18. Even assuming that A is complete for $\| \,.\, \|_{si}$, one can't claim that there exists $\varphi \in Mult_m(A, \| \,.\, \|)$ such that $\varphi(x) = \|x\|_{si}$. Indeed, let D be the disk $d(0, 1^-)$ in K, and just consider the K-algebra $H(D)$. This norm obviously is the spectral norm of $H(D)$, and then the identical function x satisfies $\|x\|_D = 1$. But every maximal ideal $\mathcal{M}$ of $H(D)$ has codimension 1, and is characterized by a point $a \in D$. So, the unique φ such that $Ker(\varphi) = \mathcal{M}$ is defined as $\varphi(f) = |f(a)|$ for all $f \in H(D)$, hence of course we have $\varphi(x) < 1$. The mistakes comes from the fact that in the proof of Theorem 1.18 of [3], in general, φ does not belong to $Mult_m(A, \| \,.\, \|)$, because, (following the notations of the proof), the homomorphism θ is not necessarly surjective onto E.

Now, let A be a commutative K-Banach algebra with identity. In 1976, Escassut showed that if every maximal ideal of A has codimension 1, then Property (s) holds in A ([2], Corollary 4.4). (In particular, this applies to Tate's algebras, whose maximal ideals are of dimension 1, on an algebraically closed field [10]). Next, using the holomorphic functional calculus, in [2] Theorem 7.5, he showed that if K is strongly valued, the equality holds in any commutative K-Banach algebra with identity. But if K is not strongly valued, by Theorem 7.5 in [2], counter examples show that (s) does not hold in the general case. In particular, there exists local commutative K-Banach algebra whose spectral semi-norm is a norm.

When K is strongly valued, property (q) was proven in [2] (Theorem 7.9) , in assuming another additional hypothesis, like the integrity of A, but without assuming the norm to be the spectral norm. But counter examples given in [2] show this last equality does not hold when K is not strongly valued.

However, here we will obtain such equalities, without assuming K to be strongly valued, provided the spectral semi-norm $\| \, . \, \|_{si}$ of A is a norm equivalent to its K-Banach algebra norm. This has been made possible thanks to a recent basic result concerning a partition of any annulus by a family of disks.

In Lemma 0, we recall previous results given in [9].

Lemma 0: *Let A be a commutative normed L-algebra with identity, and let $x \in A$. Then $\| \, . \, \|_{si}$ is an ultrametric semi-multiplicative semi-norm satisfying $\|x\|_{si} = \sup\{\varphi(x) | \varphi \in Mult(A, \| \, . \, \|)\}$ and there exists $\varphi \in Mult(A, \| \, . \, \|)$ such that $\varphi(x) = \|x\|_{si}$. Further, if A is complete, for every $\mathcal{M} \in Max(A)$ there exists $\psi \in Mult_m(A, \| \, . \, \|)$ such that $Ker(\psi) = \mathcal{M}$, and if $\mathcal{M}$ has finite codimension, such a ψ is unique.*

Then we have Theorem 1:

Theorem 1: *Let A be a commutative K-Banach algebra with identity whose norm of K-Banach algebra is $\| \, . \, \|_{si}$. Then Property (s) is satisfied. Furthermore, if A has no divisors of zero, then Property (q) is satisfied.*

Corollary a: *Let A be a commutative K-Banach algebra with identity whose norm of K-Banach algebra is $\| \, . \, \|_{si}$. Then the Jacobson radical of A is null.*

However Theorem 2 shows that, even assuming the norm to be the spectral norm, Properties (s) and (q) are not equivalent.

Theorem 2: *There exists a commutative K-Banach algebra with identity whose norm is $\| \, . \, \|_{si}$ which satisfies Property (s) but not Property (q).*

Theorem 3: *Let A be a K-Banach algebra with identity, satisfying Property (q). Then A has no non trivial idempotents, if and only if for every $x \in A$, $s(x)$ is infraconnected.*

Corollary b: *Let A be a commutative K-Banach algebra with identity, with no divisors of zero, whose norm of K-Banach algebra is $\| \, . \, \|_{si}$. Then for every $x \in A$, $s(x)$ is infraconnected.*

Theorem 4 shows that Theorem 3 couldn't be much generalized.

Theorem 4: *There exists a commutative K-Banach algebra with identity whose norm is $\| \,.\, \|_{si}$ but does not satisfy Property (q), which has no non trivial idempotent, but has an element x such that $s(x)$ is not infraconnected.*

Now, as the set $Mult(A, \| \,.\, \|)$ is provided with the topology of simple convergence, and is compact for it, given a subset Σ in $Mult(A, \| \,.\, \|)$, one can try to compare the properties:

"Σ is dense in $Mult(A, \| \,.\, \|)$", and

" $\|x\|_{si} = \sup\{\varphi(x) |\ \varphi \in \Sigma\}$ for every $x \in A$".

In fact, in the general case, this seems far from easy, due to the various forms of the neighborhoods of any point, with respect to the topology of simple convergence. So we define a notion of pseudo-density.

Notation: Given $\psi \in Mult(A, \| \,.\, \|)$, $f \in A$, $\epsilon > 0$, we denote by $V(\psi,\ f,\ \epsilon)$ the set of the $\varphi \in Mult(A, \| \,.\, \|)$ such that $|\varphi(f) - \psi(f)| \leq \epsilon$.

Remark: So, we have a basis of neighborhoods of any $\psi \in Mult(A, \| \,.\, \|)$ by taking the sets of the form $\bigcap_{j=1}^{q} V(\psi,\ f_j,\ \epsilon_j)$, $q \in \mathbb{N}^*$.

Definition: A subset Σ of $Mult(A, \| \,.\, \|)$ will be said to be *pseudo-dense in* $Mult(A, \| \,.\, \|)$ if for every $\psi \in Mult(A, \| \,.\, \|)$, for every $f \in A$, for every $\epsilon > 0$, we have $V(\psi,\ f,\ \epsilon) \cap \Sigma \neq \emptyset$.

Remark: By definition, if Σ is dense in $Mult(A, \| \,.\, \|)$, it is pseudo-dense in $Mult(A, \| \,.\, \|)$. The converse seems unlikely, though we don't know any counter examples.

Theorem 5: *Let A be a commutative K-Banach algebra with identity, and let Σ be a subset of $Mult(A, \| \,.\, \|)$ that contains $Mult_m(A, \| \,.\, \|)$. Then Σ is pseudo-dense in $Mult(A, \| \,.\, \|)$ if and only if it satisfies $\|x\|_{si} = \sup\{\varphi(x) | \varphi \in \Sigma\}$ for every $x \in A$.*

Corollary c: *Let A be a commutative K-Banach algebra with identity. Then A satisfies Property (s) if and only if $Mult_m(A, \| \,.\, \|)$ is pseudo-dense in $Mult(A, \| \,.\, \|)$.*

Corollary d: *Let A be a commutative K-Banach algebra with identity whose Banach algebra norm is $\| \,.\, \|_{si}$. Then $Mult_m(A, \| \,.\, \|_{si})$ is pseudo-dense in $Mult(A, \| \,.\, \|_{si})$.*

2 Proofs of the theorems

Definitions and notations: Let D be set in K. An annnulus $\Gamma(a, r, l)$ is called an *empty annulus of* D if it satisfies $\Gamma(a, r, l) \cap D = \emptyset$, $r = \sup\{|\lambda| \mid \lambda \in D \cap d(a, r)\}$, and $l = \inf\{|\lambda| \mid \lambda \in D \setminus d(a, l^-)\}$.

Circular filters are defined in [1], [3], [4]. A circular filter is said to be *large* if its diameter is different from zero. Large circular filters are known to characterize the absolute values on $K(x)$ in this way:

For each large circular filter $\mathcal{F}$ on K, for each $h \in K(x)$, $|h(x)|$ admits a limit along $\mathcal{F}$ denoted by $\varphi_{\mathcal{F}}(h)$, and then, $\varphi_{\mathcal{F}}$ defines an absolute value on $K(x)$, extending this of K, i.e. a multiplicative norm of $K(x)$ [1], [3], [4]. Then, the mapping that associates a multiplicative norm of $K(x)$ to a large circular filter $\mathcal{F}$ on K, in this way, is a bijection from the set of large circular filter $\mathcal{F}$ on K.

Given such a multiplicative norm ψ of $K(x)$, we will denote by $\mathcal{G}_\psi$ the large circular filter that defines ψ. Then, each multiplicative semi-norm ψ of $R(D)$,

either it is a norm, and then it has continuation to $K(x)$, and is defined by a large circular filter on K that we will denote again by $\mathcal{G}_\psi$,

or it is not a norm, and then, there exists $a \in D$ such that $\psi(h) = |h(a)|$ for every $h \in R(D)$ [1], [3], [4] and we will denote by $\mathcal{G}_\psi$ the filter of neighborhoods of the point a.

Given $a \in K$ and $r > 0$, we call *a classic partition of* $d(a, r)$ a partition of the form $\left(d(b_j, r_j^-)\right)_{j \in I}$. The disks $d(b_j, r_j^-)$ are called *the holes* of the partition.

Let $\mathcal{P} = \left(d(b_j, r_j^-)\right)_{j \in I}$ be a classic partition of $d(a, r)$. An annulus $\Gamma(b, r', r'')$ included in $d(a, r)$ will be said to be *$\mathcal{P}$-minorated* if there exists $\delta > 0$ such that $r_j \geq \delta$ for every $j \in I$ such that $d(b_j, r_j^-) \subset \Gamma(b, r', r'')$.

Given a closed bounded set E in K, we denote by $\widetilde{E}$ the smallest disk of the form $d(\alpha, \rho)$ that contains E (i.e. ρ is the diameter of E, and α may be taken in E). Besides, $\widetilde{E} \setminus E$ admits a unique partition of the form $(d(\alpha_j, \rho_j^-))_{j \in J}$, such that for each $j \in J$, ρ_j is the distance from α_j to E. Then each disk $d(\alpha_j, \rho_j^-)$ is called *a hole of* E. A closed infraconnected set E included in $d(a, r)$, will be said to be a *$\mathcal{P}$-set* if $\widetilde{E} = d(a, r)$, and if every hole of E is a hole of $\mathcal{P}$.

For each $j \in I$, we denote by $\mathcal{F}_j$ the circular filter of center b_j and diameter r_j, and for every $h \in K(x)$ we put $\|h\|_{\mathcal{P}} = \sup_{j \in I} \varphi_{\mathcal{F}_j}(h)$. Then, by [8] we know that $\|\,.\,\|_{\mathcal{P}}$ is a semi-multiplicative norm of K-algebra on $K(x)$.

Next, $H(\mathcal{P})$ will denote the completion of $K(x)$ for this norm. Hence $H(\mathcal{P})$ is a K-Banach algebra provided with a semi-multiplicative norm.

Let F be an algebraically closed field, let A be a F-algebra, let $t \in A$, and let $\mathcal{I}$ be the ideal of the $G(X) \in F[X]$ such that $G(t) = 0$. If $\mathcal{I} = \{0\}$, we call 0 *the minimal polynomial of* t. If $\mathcal{I} \neq \{0\}$, we call *minimal polynomial of* t the unique monic polynomial that generates $\mathcal{I}$. Lemma 1 is given in [8]:

Lemma 1: *Let $\mathcal{P}$ be a classic partition of a disk $d(a, r)$, and let E be a $\mathcal{P}$-set. Then we have $\|h\|_E = \|h\|_{\mathcal{P}}$ for every $h \in R(E)$.*

Corollary: *Let $\mathcal{P}$ be a classic partition of a disk $d(a, r)$, and let E be a $\mathcal{P}$-set. Then $H(E)$ is isometrically isomorphic to a K-subalgebra of $H(\mathcal{P})$.*

Henceforth, given a classic partition $\mathcal{P}$ of a disk $d(a, r)$, and a $\mathcal{P}$-set E, we will consider $H(E)$ as a K-subalgebra of $H(\mathcal{P})$.

Lemma 2: *Let A be a K-algebra with identity and let $t \in A$. There exists a homomorphism Θ from $R(s(t))$ into A such that $\Theta(P) = P(t)$ for all $P \in K[x]$. Moreover, Θ is injective if and only if t has a null minimal polynomial. Besides, for every $h \in R(s(t))$, we have $s(h(t)) = h(s(t))$.*

Proof: Let $D = s(t)$. We may obviously define Θ from $K[x]$ to A as $\Theta(P) = P(t)$. Now let $Q \in K[x]$ have its zeros in $K \setminus D$. Then $Q(t)$ is invertible in A, so we may extend to $R(D)$ the definition of Θ, as $\Theta(\frac{P}{Q}) = P(t)\,Q(t)^{-1}$, for all rational function $\frac{P}{Q} \in R(D)$ (with $(P,Q) = 1$). Next, $Ker(\Theta)$ is an ideal of $R(D)$ which is obviously generated by a polynomial G. Then $G = 0$ if and only if t has a null minimal polynomial.

Now, let $h = \dfrac{P}{Q} \in R(s(t))$, (with $(P,Q) = 1$). Let $\lambda \in s(t))$, and let χ be a homomorphism from A onto a field extension of K such that $\chi(t) = \lambda$. It is easily seen that $h(s(t)) \subset s(h(t))$, because $\chi(h(t)) = h(\lambda)$. Now, let $\mu \in s(h(t))$, let τ a homomorphism from A onto a field extension of K such that $\tau(h(t)) = \mu$, and let $\sigma = \tau(t)$. Then, we have $\tau(P(t)) - \mu\tau(Q(t)) = 0$, hence σ is a zero of the polynomial $P(X) - \sigma Q(X)$, and therefore, σ lies in K (because K is algebraically closed). But then, as $t - \sigma$ belongs to the kernel of τ, σ does lie in $s(t)$. Hence we have $s(h(t)) = h(s(t))$.

Remark: When the homomorphism Θ in Lemma 2 is injective, the K-subalgebra $B = \Theta(R(D))$ is isomorphic to $R(D)$, and in fact is the full subalgebra generated by t in A. So, in such a case, we may consider $R(D)$ as a K-subalgebra of A.

By results of [1], [4], also given in [3], we have Lemma 3.

Lemma 3: *Let A be a commutative K-Banach algebra with identity and let $t \in A$ have a null minimal polynomial. Let $a \in K$, let $\psi \in Mult(A, \|\,.\|)$, let $\widetilde{\psi}$ be the restriction of ψ to $R(s(t))$, and let $r = \psi(t-a)$. Then $\mathcal{G}_{\widetilde{\psi}}$ is secant with $C(a,r)$.*

Proposition A: *Let A be a commutative K-Banach algebra with identity. Let $t \in A$ be such that the mapping Θ from $K[x]$ into A defined as $\Theta(P) = P(t)$ is injective. Let $a \in K \setminus s(t)$, and let $r = \|(t-a)^{-1}\|_{si}^{-1}$. There exists $\theta \in Mult(A, \|\,.\,\|)$ whose restriction to $R(s(t))$ has a circular filter secant with $C(a,r)$.*

Proof: We consider $R(s(t))$ as a K-subalgebra of A. For all $\phi \in Mult(A, \|\,.\,\|)$ we denote by $\widetilde{\phi}$ the restriction of ϕ to $R(s(t))$. Let $\psi \in Mult(A, \|\,.\,\|)$. If $\mathcal{G}_{\widetilde{\psi}}$ is secant with a disk $d(a,\rho)$ for some $\rho \in]0,r[$, then clearly we have $\psi(t-a) \leq \rho$ hence $\psi((t-a)^{-1}) > \dfrac{1}{r}$ and therefore $\|(t-a)^{-1}\|_{si} > \dfrac{1}{r}$ which contradicts the hypothesis. So $\mathcal{G}_{\widetilde{\psi}}$ is secant with $K \setminus d(a, r^-)$.

Suppose that there exists $\rho > r$ such that, for every $\phi \in Mult(A, \|\,.\,\|)$, $\mathcal{G}_\phi$ is not secant with $d(a,\rho)$. Clearly we have $\phi(t-a) \geq \rho$ for all $\phi \in Mult(A, \|\,.\,\|)$ and therefore $\|(t-a)^{-1}\|_{si} < \dfrac{1}{r}$. As a consequence, for each $n \in \mathbb{N}^*$ we can find $\psi_n \in Mult(A, \|\,.\,\|)$ such that $\mathcal{G}_{\widetilde{\psi_n}}$ is secant with $d(a, r + \dfrac{1}{n})$, and since it is also secant with $K \setminus d(a, r^-)$, finally, it is secant with $\Gamma(a, r, r + \frac{1}{n})$. Since $Mult(A, \|\,.\,\|)$ is compact [6], we can extract from the sequence $(\psi_n)_{n\in\mathbb{N}}$ a subsequence $(\psi_{n_q})_{q\in\mathbb{N}}$ which converges in $Mult(A, \|\,.\,\|)$. So, without loss of generality, we may directly assume that the sequence is convergent. Let θ be its limit. For each $n \in \mathbb{N}^*$, $\mathcal{G}_{\widetilde{\psi_n}}$ is secant with a circle $C(a, r_n)$ with $r \leq r_n \leq r + \dfrac{1}{n}$. But putting $s_n = \widetilde{\psi}_n(t-a)$, by lemma 3, it is secant with $C(a, s_n)$. Suppose $s_n \neq r_n$. Clearly $\mathcal{G}_{\widetilde{\psi_n}}$ may not be secant with both $C(a,r_n)$ and $C(a,s_n)$. Hence we have $\widetilde{\psi}_n(t-a)=r_n$. Since $\lim\limits_{n\to\infty} r_n=r$, we have $\widetilde{\theta}(t-a)=r$, hence by Lemma 3, $\mathcal{G}_{\widetilde{\theta}}$ is secant with $C(a,r)$. This completes the proof.

Notations and definitions: Let A be a K- normed algebra, and suppose that an element $x\in A$ has a null minimimal polynomial and is such that $s(x)$ admits an empty annulus $\Gamma(a,r,l)$. Such an empty annulus is said to be *x-cleaved* if for every $r',\ r''\in]r,l[$, with $r'<r''$, there exists $\psi\in Mult(A,\|\ .\ \|)$, such that the circular filter of the restriction of ψ to $R(s(x))$ is secant with $\Gamma(a,r',r'')$.

Let A be a commutative K-Banach algebra with identity. Let $t\in A$ be such that the mapping Θ from $K[x]$ into A defined as $\Theta(P)=P(t)$ is injective. Let $a\in s(t)$, and let $r=\|(t-a)\|_{si}$. For each $b\in d(a,r)\setminus s(t)$ we put $r_b=\dfrac{1}{\|\frac{1}{x-b}\|}$, $\Lambda_b=d(b,r_b^-)$. By Lemma 3.1 of [2], we know that if $c\in\Lambda_b$, then $\Lambda_c=\Lambda_b$. For every $b\in d(a,r)\setminus s(t)$, we denote by ψ_b the element of $Mult(R(D))$ whose circular filter has center b and diameter r_b, so ψ_b satisfies $\psi_b(h)=\lim\limits_{|x-b|\to r_b,|x-b|\neq r_b}|h(x)|\ \forall h\in R(D)$.

For every $h\in R(D))$, we put $\|h\|_t=\max(\|h\|_D,\sup\{\psi_b(h)|\ b\in d(a,r)\setminus s(t)\})$.

As the Λ_b form a partition of $d(a,r)\setminus s(t)$, by [8], and Proposition 3.3 of [2], we have Proposition B:

Proposition B: *Let A be a commutative K-Banach algebra with identity. Let $t\in A$ be such that the mapping Θ from $K[x]$ into A defined as $\Theta(P)=P(t)$ is injective. Let $a\in K$, and let $r=\|(t-a)\|_{si}$. Then $\|\ .\ \|_t$ defines on $R(s(t))$ a semi-multiplicative norm satisfying $\|h\|_t\geq\|h(t)\|_{si}$ for every $h\in R(s(t))$.*

Proposition C: *Let $\mathcal{P}$ be a classic partition of a disk $d(a,r)$, let $\Gamma(b,r',r'')$ be a $\mathcal{P}$-minorated annulus included in $d(a,r)$, let $l\in]r',r''[$. There exist a $\mathcal{P}$-set E containing $d(b,r'^-)\cup K\setminus d(b,r''^-)$ together with elements $f,\ g\in H(E)$ such that $|f(x)|=1$ for all $x\in K\setminus d(b,r'^-)$ and $f(x)=0$ for all $x\in d(b,l)\cap E$, and $g(x)=0$ for all $x\in E\setminus d(b,l)$, and $|g(x)|=1$ for all $x\in d(b,r')$.*

Proof: Let $(r_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ be sequences in $|K|$ satisfying $r''>r_n>r_{n+1}>l$, for all $n\in\mathbb{N}$, $r'<s_n<s_{n+1}<l$, $\lim\limits_{n\to\infty}r_n=\lim\limits_{n\to\infty}s_n=l$ and (1) $\prod\limits_{n=0}^{\infty}\dfrac{l}{r_n}=\prod\limits_{n=0}^{\infty}\dfrac{s_n}{l}=0$.
For each $n\in\mathbb{N}$, let $b_n\in C(b,r_n)$, $c_n\in C(b,s_n)$, let T_n, be a hole of $\mathcal{P}$ that contains b_n, and let V_n be a hole of $\mathcal{P}$ that contains c_n. Then we set $E=d(a,r)\setminus\Big(\big(\bigcup\limits_{n=0}^{\infty}T_n\big)\cup\big(\bigcup\limits_{n=0}^{\infty}V_n\big)\Big)$. Since $\Gamma(b,r',r'')$ is $\mathcal{P}$-minorated, there exists $\rho>0$ such that $diam(T_n)\geq\rho$ and $diam(V_n)\geq\rho$ for all $n\in\mathbb{N}$. Therefore, thanks to (1) and Proposition 36.6 in [3], the sequence $(T_n,1)_{n\in\mathbb{N}}$ is a decreasing idempotent T-sequence of E, of center b and diameter l, and the sequence $(V_n,1)_{n\in\mathbb{N}}$ is an increasing idempotent T-sequence of E, of center b and diameter l. Hence by Proposition 45.3 in [3], there exists $f\in H(E)$, strictly vanishing along the decreasing T-filter of center b and diameter l, satisfying further $|f(x)|=1$ for all $x\in D\setminus d(b,r''^-)$, and $f(x)=0$ for all $x\in d(b,l)$. In the same way, exists $g\in H(E)$, strictly vanishing along the increasing T-filter of center b and diameter l, satisfying further $|g(x)|=1$ for all $x\in d(b,r''^-)$, and $g(x)=0$ for all $x\in D\setminus d(b,l^-)$.

Proof of Theorem 1: Suppose that there exists $t \in A$ such that $\|t\|_m < \|t\|_{si}$, (resp. such that $\sup\{|x| \mid x \in s(t)\} < \|t\|_{si}$).

By Lemma 2 there exists a K-algebra homomorphism Θ from $R(D)$ into A such that $\Theta(P) = P(t)$ for all $P \in K[x]$. Let $B = \Theta(R(D))$.

First we suppose that $Ker(\Theta) \neq \{0\}$, and therefore is an ideal of $R(D)$ generated by a monic polynomial $G(x) = \prod_{i=1}^{q}(x - a_i)$. Since $G(t) = 0$, for every $\psi \in Mult(A, \|\,.\,\|)$ we have $\psi(G(t)) = 0$, hence there exists $l(\psi) \in \{1, \dots, q\}$ such that $\psi(t - a_{l(\psi)}) = 0$, hence $\psi(t) = |a_{l(\psi)}|$. Then $t - a_{l(\psi)}$ lies in $Ker(\psi)$ and therefore belongs to a maximal ideal $\mathcal{M}$ of A. But there exists $\theta_\psi \in Mult_m(A, \|\,.\,\|)$ such that $Ker(\theta_\psi) = \mathcal{M}$ (Theorem 1.16 of [3]). Hence we have $\theta_\psi(t) = |a_{l(\psi)}| = \psi(t)$. Thus we have shown that $\psi(t) \leq \|t\|_m$. But this is true for all $\psi \in Mult(A, \|\,.\,\|)$. So, as the norm $\|\,.\,\|$ of A is $\|\,.\,\|_{si}$, we have $\|t\| \leq \|t\|_m$ and therefore $\|t\| = \|t\|_m$. Besides, we notice that $Ker(\Theta)$ admits a generator $G(x) \in K[x]$ whose zeros lie in $s(t)$. If $deg(G) = 1$, then t lies in K (considered as a K-subalgebra of A), and obviously we have $\psi(t) = |t|\ \forall \psi \in Mult(A, \|\,.\,\|)$, and therefore, this contradicts the hypothesis that there exists $t \in A$ such that $\sup\{|x| \mid x \in s(t)\} < \|t\|_{si}$. Next, if $deg(G) > 1$, then $Ker(\Theta)$ is not prime, hence A contains divisors of zero, so this case does not concern the second statement.

Now we suppose $Ker(\Theta) = \{0\}$. Hence B is isomorphic to $R(D)$. Furthermore, by Proposition B, once $R(D)$ is provided with the norm $\|\,.\,\|_t$, Θ is continuous. Therefore, denoting by $H(s(t), \|\,.\,\|_t)$ the completion of $R(D)$ with respect to $\|\,.\,\|_t$, Θ has continuation to a continuous homomorphism Θ' from $H(s(t), \|\,.\,\|_t)$ into the closure $\overline{B}$ of B in A. For each $\psi \in Mult(A, \|\,.\,\|)$ we denote by $\widetilde{\psi}$ the restriction of ψ to B, and by $\mathcal{G}_\psi$ the circular filter of $\widetilde{\psi}$. We put $r = \|t\|_{si}$ and $r'' = \sup\{|x| \mid x \in s(t)\}$, and let $r' = \|t\|_m$. We will suppose $r' < r$,(resp. $r'' < r$).

Let $W = d(0, r)$, and let $s' \in]r', r[$, (resp. and let $s'' \in]r', r[$). Let $W' = d(0, s')$ (resp. let $W'' = d(0, s'')$). And for each $\alpha \in W \setminus W'$ (resp. $\alpha \in W \setminus W''$) we put $r_\alpha = \dfrac{1}{\|\frac{1}{x} - \alpha\|}$, and $\Lambda_\alpha = d(\alpha, r_\alpha^-)$. So, $(\Lambda_\alpha)_{\alpha \in W \setminus W'}$ (resp. $(\Lambda_\alpha)_{\alpha \in W \setminus W''}$ is a partition $\mathcal{T}'$ (resp. $\mathcal{T}''$) of $W \setminus W'$ (resp. $W \setminus W''$).

Let $a \in s(t)$. In particular, the annulus $\Gamma(a, s', r)$, (resp. $\Gamma(a, s'', r)$) admits a partition by a subfamily $\mathcal{S}$ of $\mathcal{T}'$ (resp. of $\mathcal{T}''$). Hence by [8], $\Gamma(a, s', r)$, (resp. $\Gamma(a, s'', r)$) contains a $\mathcal{P}$-minorated annulus $\Gamma(b, \rho, \sigma)$. Of course, we may choose σ as close as we want to ρ. Then, if $|a - b| > \rho$, we take $\sigma \in]\rho, |a - b|[$. Next, we take $\lambda \in]\rho, \sigma[$. Clearly b does not lie in $s(t)$, hence we may apply Proposition A to b and to the circle $C(b, r_b)$. So, there exists $\varphi_1 \in Mult(A, \|\,.\,\|)$ such that $\mathcal{G}_{\varphi_1}$ is secant with $C(b, r_b)$. In fact, by definition, we have $r_b \leq \rho$, hence $C(b, r_b)$ is included in $d(b, \rho)$, hence $\mathcal{G}_{\varphi_1}$ is secant with $d(b, \rho)$. On the other hand, there certainly exists $\varphi_2 \in Mult(A, \|\,.\,\|)$ such that $\mathcal{G}_{\varphi_2}$ is secant with $C(a, r)$. Then by Proposition C there exists a $\mathcal{P}$-set E containing $(K \setminus d(b, \sigma)) \cup d(b, \rho)$, together with elements $f,\ g \in H(E)$ satisfying:
$|f(x)| = 1$ for all $x \in K \setminus d(b, \sigma)$, $f(x) = 0$ for all $x \in d(b, \lambda)$, and
$|g(x)| = 1$ for all $x \in d(b, \rho)$, $|g(x)| = 0$ for all $x \in K \setminus d(b, \lambda)$.

We put $\overline{f} = \Theta'(f)$, and $\overline{g} = \Theta'(g)$. Hence, in $H(E)$ we have $fg = 0$, and therefore $\overline{f}\overline{g} = 0$. But since $\varphi_1(\overline{f})\varphi_2(\overline{g}) \neq 0$, $\overline{f}$, $\overline{g}$ are divizors of zero in A. Thus, if A has no divizors of zero, then we have $r'' = r$.

Now, suppose $r' < r$.
We first assume $|a-b| \leq \rho$, hence $d(a,r')$ is included in $K \setminus d(b,\lambda)$. It is seen that for every $\phi \in Mult_m(A, \| \, . \, \|)$, we have $\phi(g) = 1$, because $\phi(t-a) \leq r'$ and therefore, g is invertible in A. But, as we saw, $\mathcal{G}_{\varphi_2}$ is secant with $C(a,r)$, and then, $\varphi_2(\overline{g}) = 0$, which contradicts the property $\overline{g}$ invertible.
Finally , we assume $|a-b| > \rho$. Then we have $d(a,r') \subset K \setminus d(b,\lambda)$, and therefore, $\overline{f}$ satisfies $\psi(\overline{f}) = 1$ for all $\psi \in Mult_m(A, \| \, . \, \|)$, and $\psi(\overline{f}) = 0$ for all ψ such that $\mathcal{G}_\psi$ is secant with $d(b,\lambda)$. So, $\overline{f}$ is invertible. But we have seen that the set of $\psi \in Mult(A, \| \, . \, \|)$ such that $\mathcal{G}_\psi$ is secant with $d(b,\lambda)$ is not empty, and therefore, such a ψ satisfies $\psi(\overline{f}) = 0$, which contradicts the property $\overline{f}$ invertible. This ends the proof of Theorem 1.

Notation: Let $h \in K(x)$. For every large circular filter on K, we put $\varphi_{\mathcal{F}}(h) = \lim_{\mathcal{F}} |h(x)|$. In particular, for each $r > 0$, we denote by $\mathcal{F}_r$ the circular filter of center 0, and diameter r, and we put $|h|(r) = \varphi_{\mathcal{F}_r}(h) = \lim\limits_{|x| \to r, \; |x| \neq r} |h(x)|$.

Proof of Theorem 2: Let $l \in]0,1[$, let $D = d(0,l^-)$ and let $\mathcal{P}$ be the partition of $d(0,1)$ that consists of the disks $d(a,|a|^-)$ for $a \in d(0,1)$ and $l \leq |a| \leq 1$. By definition of circular filters, it is easily seen that, for every $h \in R(D)$, we have $\|h\|_{\mathcal{P}} = \sup\{|h|(r), | \; l \leq r \leq 1\}$. Let A be the completion of $R(D)$ for the norm $\| \, . \, \|_{\mathcal{P}}$. By construction, A is complete for its norm $\| \, . \, \|_{si}$ and is isometrically isomorphic to a K-subalgebra of $H(\mathcal{P})$. Given any $\mathcal{P}$-set E containing D, it is seen that $H(E)$ is isometrically isomorphic to a K-subalgebra of A. More, and by definition, the identical mapping from the normed K-algebra $(R(D), \| \, . \, \|_{\mathcal{P}})$ onto the normed K-algebra $(R(D), \| \, . \, \|_D)$ is continuous and enables us to consider A as a K-subalgebra of $H(D)$. Each element of $Mult(R(D)), \| \, . \|_D)$ has continuation to an element of $Mult(H(D), \| \, . \|_D)$, and is of the form $\varphi_{\mathcal{F}}$, with $\mathcal{F}$ a circular filter on K secant with D [3]. In particular such a $\varphi_{\mathcal{F}}$ has continuation to A, and belongs to $Mult(A, \| \, . \, \|)$.

Now, we consider a circular filter $\mathcal{G}$ on K that is not secant with D. First, we will show that $\mathcal{G}$ is secant with a unique circle $C(0,r)$. Indeed, suppose it is not secant with any circle $C(0,r)$. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ in K, thinner than $\mathcal{G}$, such that $|a_{n+1} - a_n|$ is a strictly decreasing sequence, of limit r. Let $|a_n| = r_n$. Since the sequence $|a_{n+1} - a_n|$ is a strictly decreasing, it is easily seen that for n great enough, the sequence r_n is decreasing. If $r_n = r_{n+1}$, clearly we have $|a_n - a_{n+1}| = r_n$, hence $|a_{m+1} - a_m| < r_n$ for every $m > n$, and therefore a_m belongs to $C(0,r_n)$ for every $m > n$, hence $\mathcal{G}$ is secant with $C(0,r_n)$.
Else, the sequence $(r_n)_{n \in \mathbb{N}}$ is strictly decreasing, (for n big enough, of limit $r \geq \ell$), and finally, $\mathcal{G}$ is the circular filter of center 0 and diameter r. So, we have proven that, anyway, $\mathcal{G}$ is secant with a circle $C(0,r)$. Then such a circle is unique, because a circular filter cannot be secant with two different circles of same center.

Now, we will show that $\varphi_{\mathcal{G}}$ belongs to $Mult(R(D), \| \, . \, \|)$ if and only if $\mathcal{G}$ is of the form $\mathcal{F}_\rho$, with $\ell < \rho \leq 1$. Indeed, let $\mathcal{G}$ be secant with $C(0,r)$. If $r > 1$, it is seen that a non constant polynomial P having no zero in $C(0,r)$ satisfies $\lim\limits_{\mathcal{F}} |P(x)| = |P|(r) > \|P\|$, hence $\varphi_{\mathcal{G}}$ does not belong to $Mult(A, \| \, . \, \|)$. So, we have $\ell < r \leq 1$. Besides, if $\mathcal{G}$ is not of the form $\mathcal{F}_r$, then it is secant with a disk

$d(b,s)$, included in $C(0,r)$, with $s<r$, and then, putting $g(x)=\dfrac{1}{x-b}$, it is seen that $\varphi_{\mathcal{G}}(g)\geq \dfrac{1}{s}>\dfrac{1}{r}=\|g\|$, so $\varphi_{\mathcal{G}}$ does not belong to $Mult(A,\|\ .\ \|)$. Finally, we see that the only elements $Mult(A,\|\ .\ \|)$ are the $\varphi_{\mathcal{F}}$ when $\mathcal{F}$ is either a circular filter secant with D or a large circular filter of the form $\mathcal{F}_r$, with $\ell<r\leq 1$.

For every $a\in D$, we denote by $\mathcal{M}_a$ the set of $f\in A$ such that $f(a)=0$. It is obviously seen that the maximal ideals of codimension 1 of A are the $\mathcal{M}_a$, with $a\in D$. For all $a\in D$, we denote by φ_a the element of $Mult_m(A,\|\ .\ \|)$ defined as $\varphi_a(f)=|f(a)|$.

Now, we will show that for each $r\in]\ell,1]$, the ideal $\mathcal{J}=Ker(\varphi_{\mathcal{F}_r})$ is a maximal ideal of infinite codimension of A. By Proposition C, there exists a $\mathcal{P}$-set E containing D, and $f\in H(E)$ such that $\varphi_{\mathcal{F}_r}(f)=0$, and

(1) $|f(x)|=1$ for all $x\in D$.

Now, by (1) $\mathcal{J}$ is included in a maximal ideal $\mathcal{M}$ different from $\mathcal{M}_a$, whenever $a\in D$. Hence $\mathcal{M}$ has infinite codimension. Further, it is the kernel of a certain $\psi\in Mult_m(A,\|\ .\ \|)$. Let $\mathcal{G}$ be the circular filter such that $\psi=\varphi_{\mathcal{G}}$. Then, by (1) $\mathcal{G}$ is not secant with D, and therefore is of the form $\mathcal{F}_t$, with $t\in]\ell,1]$. Suppose $t<r$, (resp. $t>r$). By Proposition C, there exists a $\mathcal{P}$-set F containing $d(0,t)$, (resp. $D\cup K\setminus d(0,t^-)$), and $f\in H(F)$ such that $\varphi_{\mathcal{F}_r}(f)=0$, and $|f(x)|=1$ for all $x\in F\cap(K\setminus d(0,t^-))$, (resp. for all $x\in d(0,t)$), hence in particular $|f|(t)=1$, which contradicts the hypothesis $f\in\mathcal{M}$. So, we have $t=r$, and therefore $\mathcal{J}=\mathcal{M}$.

Now, it is clearly seen that $Mult_m(A,\|\ .\ \|)$ is dense in $Mult(A,\|\ .\ \|)$, because the only elements of $Mult(A,\|\ .\ \|)\setminus Mult_m(A,\|\ .\ \|)$ are the $\varphi_{\mathcal{F}}$, with $\mathcal{F}$ a large circular filter on K secant with D. And such a $\varphi_{\mathcal{F}}$ is known to be the limit of a sequence φ_{a_n}, with $(a_n)_{n\in\mathbb{N}}$ a sequence thinner than $\mathcal{F}$ ([3], Lemma 12.2). As a consequence, A satisfies Property (s).

Finally we shortly check that A does not satisfies Property (q) because for every $\lambda\in s(x)$, we have $|\lambda|<\ell$, whereas $\|x\|_m=|x|(1)=1$.

In the proof of Theorem 3, we will need this basic lemma:

Lemma 4: *Let F be an algebraically closed field, and let $P\in F[X]$ be a polynomial of degree strictly greater than 1. Let $B=\dfrac{F[X]}{P(X)F[X]}$, let θ be the canonical homomorphism from $F[x]$ onto B, and let $x=\theta(X)$. If the spectrum of x is not reduced to a singleton, then B admits non trivial idempotents.*

Proof: Let $P(X)=\prod_{j=1}^{q}(X-a_j)^{n_j}$, (with $a_i\neq a_j\ \forall i\neq j$). It is known that B admits non trivial idempotents if and only if $q>1$. But on the other hand, the spectrum of x is clearly equal to $\{a_1,...a_q\}$, so the conclusion follows.

Proof of Theorem 3: Obviously, if A admits an idempotent u different from 0 and 1, we have $s(u)=\{0,1\}$ which is not infraconnected. Now, we suppose that A admits no non trivial idempotent, and that there exists $x\in A$ such that $s(x)$ is not infraconnected, and we consider an annnulus $\Gamma(a,r,l)$ which is an empty annulus of $s(x)$. Let Θ be the canonical homomorphism from $R(s(x))$ into A, and let $B=\Theta(R(s(x)))$. Let P be the minimal polynomial of x. Then, B is isomorphic to $\dfrac{K[X]}{P(X)K[X]}$. Since A has no non trivial idempotents, neither has B. Hence by Lemma 4, if $P \neq 0$, $s(x)$ is reduced to one point, which is a contradiction with $s(x)$ not infraconnected. So, we may assume that the minimal polynomial of x is null. Let $b \in \Gamma(a, r, l)$, and let $t = \dfrac{x}{(x-b)^2}$. In $K(X)$, we put $h(X) = \dfrac{X}{(X-b)^2}$. Let $\sigma = \sup\{|x| \mid x \in s(t)\}$, and let $\beta = |a - b|$. By Lemma 2 we know that $s(t) = h(s(x))$. Hence it is easily seen that we have

(1) $\sigma \leq \max(\dfrac{r}{\beta^2}, \dfrac{1}{s})$. Now, since a has no non trivial idempotent, by Theorem 5.10 of [2], the empty annulus Λ is not x-cleaved, then for every $\epsilon > 0$, there exists $\varphi_\epsilon \in Mult(A, \| \, . \, \|)$ such that the circular filter of the restriction of φ to $R(s(x))$ is secant with $\Gamma(a, \beta, \beta + \epsilon)$. As a consequence, we have $\varphi_\epsilon(t) \geq \dfrac{1}{\beta + \epsilon}$. Now, we can choose ϵ such that $\dfrac{1}{\beta + \epsilon} > \max(\dfrac{r}{\beta^2}, \dfrac{1}{l})$. As a consequence, we have $\|t\|_{si} \geq \max(\dfrac{r}{\beta^2}, \dfrac{1}{l})$, and then by (1), this is a contradiction of Property (q).

Proof of Theorem 4: As in Proof of Theorem 2, we take $l \in]0, 1[$ and denote by $\mathcal{P}$ be the partition of $d(0, 1)$ that consists of the disks $d(\alpha, |\alpha|^-)$ for $\alpha \in d(0, 1)$ and $l \leq |\alpha| \leq 1$. Let $a \in C(0, 1)$ and let $D = d(0, l^-) \cup d(a, 1^-)$. It is clear that for every $h \in R(D)$, we have $\|h\|_{\mathcal{P}} = \sup\{|h|(r), l \leq r \leq 1\}$. Let A be the completion of $R(D)$ for the norm $\| \, . \, \|_{\mathcal{P}}$. By construction, A is complete for its norm $\| \, . \, \|_{si}$. Now, denoting by x the identical mapping on D, $s(x)$ is equal to D, and therefore is not infraconnected. Now, suppose that A has a non trivial idempotent u. Since $\|h\|_{\mathcal{P}} \geq \|h\|_D$ for all $h \in R(D)$, A is clearly isomorphic to a subalgebra of $H(D)$. As a consequence, as a function in D, u is a constant equal to 0 or 1 in each set $d(0, l^-)$, and $d(a, 1^-)$. Without loss of generality, we may clearly assume that $u(\zeta) = 0 \ \forall \zeta \in d(0, l^-)$ (because else, we consider $1 - u$ instead of u).

For every $r \in [l, 1[$, we denote by $\mathcal{F}_r$ the circular filter of center 0, and diameter r, and we put $g(r) = \varphi_{\mathcal{F}_r}(u)$. Then g is known to be a continuous function of r that satisfies $g(l) = \lim\limits_{|\zeta| \to l, |\zeta| < l} |u(\zeta)|$, hence $g(l) = 0$, and of course, $g(r) = 0$ or 1 for all $r \in]l, 1]$. As a consequence, we have $g(r) = 0$ for all $r \in [l, 1]$. So, $\|u\|_{\mathcal{P}} = 0$, and therefore $u = 0$. This proves that A only has trivial idempotents. Finally, the fact that A does not satisfy Property (q) is an obvious consequence of Theorem 3, but may also be directly cheked, just by considering any $b \in \Gamma(0, r, 1)$, and $t = \dfrac{x}{(x-b)^2}$ and this ends the proof.

Proof of Theorem 5: On one hand, it is obviously seen that if for every $\psi \in Mult(A, \| \, . \, \|)$, $f \in A$, and $\epsilon > 0$, we have $V(\psi, f, \epsilon) \cap \Sigma \neq \emptyset$ then, for all $x \in A$, we have $\sup\{\varphi(x) | \ \varphi \in \Sigma\} = \sup\{\varphi(x) | \varphi \in Mult(A, \| \, . \, \|)\}$ and therefore $\|x\|_\Sigma = \|x\|_{si}$.

On the other hand we suppose that $\|x\|_\Sigma = \|x\|_{si}$ for all $x \in A$, and that there exists $\psi \in Mult(A, \| \, . \, \|)$, $t \in A$ and $\epsilon > 0$ such that $V(\psi, t, \epsilon) \cap \Sigma = \emptyset$. We put $\psi(t) = r$.

First, suppose $r = 0$. Since $\Sigma \supset Mult_m(A, \| \, . \, \|)$, 0 does not lie in $s(t)$, hence t is invertible in A and satisfies $\psi(t)\psi(t^{-1}) = 1$, which contradicts $\psi(t) = 0$. So we

have $r > 0$. Now we can take $\delta \in]0, r[$ such that $|\varphi(t) - r| \geq \delta$ for every $\varphi \in S$. Since $Mult_m(A, \|\,.\,\|) \subset \Sigma$, it is seen that $s(t) \cap \Gamma(0, r - \delta, r + \delta) = \emptyset$. Let $\omega \in]0, \frac{\delta}{4}[$, and let $a, b \in K$ satisfy $r - \omega < |a| < r$ and $r + \delta - \omega < |b| < r + \delta$. Since a lies in $\Gamma(0, r - \delta, r + \delta)$, $x - a$ is invertible in A. We put $u = \frac{t(t-b)}{(t-a)^2}$ and then we have $\psi(t - a) = \psi(t)$, $\psi(t - b) = |b|$, hence we have
(1) $\psi(u) = \frac{|b|}{r}$.
First, we suppose $\varphi(t) \geq \psi(t) + \delta$. Then we have $\varphi(t - a) = \varphi(t - b) = \varphi(t)$, hence $\varphi(u) = 1$, and therefore $\psi(u) - \varphi(u) = \frac{|b|}{r} - 1 \geq \frac{r + \delta - \omega}{r} - 1 = \frac{\delta - \omega}{r}$. Since $\omega < \frac{\delta}{4}$, we obtain $\psi(u) - \varphi(u) \geq \frac{\delta}{2r}$.
Now, we suppose $\varphi(t) \leq \psi(t) - \delta$. Then we have $\varphi(t - a) = |a|$, $\varphi(t - b) = |b|$, hence $\varphi(u) = \frac{\varphi(t)|b|}{|a|^2}$, and then by (1) we have

$$\psi(u) - \varphi(u) = |b|(\frac{1}{r} - \frac{\varphi(t)}{|a|^2}) \geq |b|(\frac{1}{r} - \frac{r - \delta}{|a|^2}) \geq (r + \delta - \omega)(\frac{1}{r} - \frac{r - \delta}{(r - \omega)^2}).$$

But since $\omega < \frac{\delta}{4}$, we obtain

$$(r + \delta - \omega)(\frac{1}{r} - \frac{r - \delta}{(r - \omega)^2}) \geq \frac{(r + \delta - \omega)(\delta - 2\omega)}{r^2} \geq \frac{\delta - 2\omega}{r} \geq \frac{\delta}{2r}.$$

Thus we have proven that $\varphi(u) \leq \psi(u) - \frac{\delta}{2r}$ for every $\varphi \in \Sigma$, and therefore we have $\|u\|_\Sigma < \|u\|_{si}$. This completes the proof of Theorem 5.

References

[1] Escassut, A. *Elements analytiques et filtres percés sur un ensemble infraconnexe*, Ann. Mat. Pura Appl. t.110 p. 335-352 (1976).

[2] Escassut, A. *The ultrametric spectral theory*, Periodica Mathematica Hungarica, Vol.11, (1), p7-60, (1980).

[3] Escassut, A. *Analytic Elements in p-adic analysis* World Scientific Publishing, Singapore, (1995).

[4] Garandel, G. *Les semi-normes multiplicatives sur les algèbres d'élé ments analytiques au sens de Krasner*, Indag. Math., 37, n4, p.327-341, (1975).

[5] Guennebaud, B. *Algèbres localement convexes sur les corps valués*, Bull. Sci. Math. 91, p.75-96, (1967).

[6] Guennebaud, B. *Sur une notion de spectre pour les algèbres normées ultramétriques*, thèse, Université de Poitiers, (1973).

[7] Krasner, M. *Prolongement analytique uniforme et multiforme dans les corps valués complets: éléments analytiques, préliminaires du théorème d'unicité.* C.R.A.S. Paris, A 239, p.468-470, (1954).

[8] Mainetti, N. *Algebras of Abstract Analytic Elements.* To appear in Lecture Notes in Pure and Applied Mathematics (Proceedings of the 4-th International Conference in p-adic Functional Analysis, Nijmegen, 1996).

[9] Van Rooij, A.C.M. *Non-Archimedean Functional Analysis*, Marcel Dekker, inc. (1978).

[10] Tate, J. *Rigid Analytic Spaces.* Inventiones mathematicae t.12, fasc.4, p 257-289, (1971).

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