Unordered Baire-like vector-valued function spaces.

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Abstract

In this paper we show that if I is an index set and X_i a normed space for each $i \in I$, then the ℓ_p-direct sum $(\oplus_{i\in I} X_i)_p, 1 \leq p \leq \infty$, is UBL (unordered Baire-like) if and only if $X_i, i \in I$, is UBL. If X is a normed UBL space and (Ω, Σ, μ) is a finite measure space we also investigate the UBL property of the Lebesgue-Bochner spaces $L_p(\mu, X)$, with $1 \leq p < \infty$.

In what follows (Ω, Σ, μ) will be a finite measure space and X a normed space. As usual, $L_p(\mu, X), 1 \leq p < \infty$, will denote the linear space over the field $\mathbb{K}$ of the real or complex numbers of all X-valued μ-measurable p-Bochner integrable (classes of) functions defined on Ω, provided with the norm

$$\|f\| = \left\{ \int_\Omega \|f(\omega)\|^p \, d\mu(\omega) \right\}^{1/p}$$

When $A \in \Sigma, \chi_A$ will denote the indicator function of the set A.

On the other hand, if $\{X_i, i \in I\}$ is a family of normed spaces, we denote by $(\oplus_{i\in I} X_i)_p$, with $1 \leq p < \infty$, the ℓ_p-direct sum of the spaces X_i, that is to say :

$$(\bigoplus_{i\in I} X_i)_p = \{\mathbf{x} = (x_i) \in \prod\{X_j, j \in I\} : (\|x_i\|) \in \ell_p\}$$

provided with the norm $\|(x_i)\| = \|(\|x_i\|)\|_p$. If $p = \infty$, then

$$(\bigoplus_{i\in I} X_i)_\infty = \{\mathbf{x} = (x_i) \in l_\infty\left((X_i)\right) : \text{ card (supp } \mathbf{x}) \leq \aleph_0\}$$

*This paper has been partially supported by DGICYT grant PB91-0407.
Received by the editors March 1994
Communicated by J. Schmets
AMS Mathematics Subject Classification : 46A08, 46E40.
Keywords : Unordered Baire-like (UBL) space, Lebesgue-Bochner space, ℓ_p-direct sum.

equipped with the norm $\|(x_i)\| = \sup\{\|x_n\|, n \in \mathbb{N}\}$.

A Hausdorff locally convex space E over $\mathbb{K}$ is said to be unordered Baire-like, [6] (also called UBL in [5]) if given a sequence of closed absolutely convex sets of E covering E, there is one of them which is a neigbourhood of the origin. When E is metrizable, E is said to be totally barrelled (also called TB in [5]) if given a sequence of linear subspaces of E covering E there is one which is barrelled. This last definition coincides with the one given in [5] and [8] for the general locally convex case.

It is known that if μ is atomless, $L_p(\mu, X)$ enjoys very good strong barrelledness properties (even if $p = \infty$) ([1] and [2]). If μ has some atom, then X must share the same strong barrelledness property than $L_p(\mu, X)$ do. On the other hand, by a well-known result of Lurje, $(\oplus_{i\in\mathbb{N}} X_i)_p$ is barrelled (and hence, Baire-like) if and only if each X_i is barrelled (see [5], 4.9.17). This result has been extended independently in [3] and [4] by showing that, whenever each X_i is seminormed, $(\oplus_{i\in I} X_i)_p$ is barrelled (ultrabarrelled) if and only each X_i is barrelled (ultrabarrelled). For the definitions of Baire-like and ultrabarrelled spaces see [5] (pp. 333 and 366).

In this paper we are going to investigate for a general positive μ the UBL property of the space $L_p(\mu, X), 1 \leq p < \infty$, whenever X is UBL. We will also prove that $(\oplus_{i\in I} X_i)_p$, with $1 \leq p \leq \infty$, is UBL if and only if each X_i is UBL.

Proposition 1 *If X is an UBL space, then $L_1(\mu, X)$ is UBL.*

Proof. Our argument is based upon the proof of the Proposition 2 of [7]. So, let $\{W_n, n \in \mathbb{N}\}$ be a sequence of closed absolutely convex subsets of $L_1(\mu, X)$ covering $L_1(\mu, X)$. It suffices to show that there is an $i \in \mathbb{N}$ such that W_i absorbs the family

$$\{\chi_A x/\{\mu(A)\}, \|x\| = 1, A \in \Sigma, \mu(A) \neq 0\}$$

since, if $\chi_A x/\{\mu(A)\} \in qW_i$ for some $q \in \mathbb{N}$, each $x \in X$ of norm one and each $A \in \Sigma$ with $\mu(A) \neq 0$, given any simple function $s = \sum_{1\leq j\leq n} y_j \chi_{C_j}$ of $L_1(\mu, X)$, with $C_j \in \Sigma$, $\mu(C_j) \neq 0$, $\|y_j\| \neq 0$ for $1 \leq j \leq n$ and $C_i \cap C_j = \emptyset$ if $i \neq j$, so that $\|s\|_1 \leq 1$, then $\sum_{1\leq j\leq n} \|y_j\|\mu(C_j) = \|s\|_1 \leq 1$, and since W_i is absolutely convex,

$$\sum_{1\leq j\leq n} y_j \chi_{C_j} = \sum_{1\leq j\leq n} \|y_j\|\mu(C_j)\chi_{C_j}(y_j/\|y_j\|)/\{\mu(C_j)\} \in qW_i$$

Hence, W_i, being closed, absorbs the closed unit ball of $L_1(\mu, X)$.

Let us define the closed absolutely convex subsets of X

$$V_{nm} = \{x \in X : \chi_A x/\{\mu(A)\} \in mW_n \text{ for each } A \in \Sigma \text{ with } \mu(A) \neq 0\}$$

for each $n, m \in \mathbb{N}$.

Given $z \in X, z \neq 0$, then $L(z) := \{f(\cdot)z : f \in L_1(\mu)\}$ is a closed subspace of $L_1(\mu, X)$ isomorphic to $L_1(\mu)$ and therefore there are $r, s \in \mathbb{N}$ such that $\chi_A z/\{\mu(A)\} \in sW_r$ for each $A \in \Sigma$ with $\mu(A) \neq 0$. This implies that $z \in V_{rs}$ and, consequently, that $\bigcup\{V_{nm} : n, m \in \mathbb{N}\} = X$. As X is UBL there are $i, j, k \in \mathbb{N}$ so that kV_{ij} contains the unit sphere of X. Hence $\chi_A x/\{\mu(A)\} \in jkW_i$ for each $x \in X$ so that $\|x\| = 1$ and each $A \in \Sigma$ with $\mu(A) \neq 0$. This completes the proof. $\square$

Proposition 2 *Let X be an UBL space. If $L_p(\mu, X), 1 < p < \infty$, is a TB space, then $L_p(\mu, X)$ is UBL.*

Proof. If $\{W_n, n \in \mathbb{N}\}$ is a sequence of closed absolutely convex subsets of $L_p(\mu, X)$ covering $L_p(\mu, X)$, a similar argument to the proof of the previous proposition shows that there exists an index $j \in \mathbb{N}$ such that W_j absorbs the family

$$\{\chi_A x / \{\mu(A)\}^{1/p}, \|x\| = 1, A \in \Sigma, \mu(A) \neq 0\}.$$

This implies that the linear span of W_j contains the subspace of the simple functions. Hence $\mathrm{span}(W_j)$ is a dense subspace of $L_p(\mu, X)$ and thus ([6], Theorem 4.1) there is no loss of generality by assuming that $\mathrm{span}(W_n)$ is dense in $L_p(\mu, X)$ for each $n \in \mathbb{N}$.

Since we have supposed that $L_p(\mu, X)$ is TB, it follows that there exists an $i \in \mathbb{N}$ such that $\mathrm{span}(W_i)$ is barrelled. This ensures, W_i being closed in $L_p(\mu, X)$, that $\mathrm{span}(W_i)$ is closed. Consequently, one has that $\mathrm{span}(W_i) = L_p(\mu, X)$. This implies that W_i is absorbent in $L_p(\mu, X)$. Since W_i was absolutely convex and closed by hypothesis, we have that W_i is a barrel in $L_p(\mu, X)$ and hence a zero-neighbourhood because $L_p(\mu, X)$ is always barrelled ([2]). □

Lemma 1 *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of normed spaces and assume that $\{W_n, n \in \mathbb{N}\}$ is a sequence of closed absolutely convex subsets of $(\oplus_{n=1}^{\infty} X_n)_p$ covering $(\oplus_{n=1}^{\infty} X_n)_p$, $1 \leq p \leq \infty$. Then there is $m \in \mathbb{N}$ such that*

$$\mathit{span}(W_m) \supseteq (\oplus_{n>m} X_n)_p.$$

Proof. If this is not the case, for each $n \in \mathbb{N}$ there is

$$\mathbf{x}_n \in (\oplus_{k>n} X_k)_p \setminus \mathrm{span}(W_n)$$

with $\|x_n\| = 1$. Since the sequence $(\mathbf{x}_n)$ is bounded in $(\oplus_{n=1}^{\infty} X_n)_p$, then for each $\xi \in \ell_1$ the series $\sum_n \xi_n \mathbf{x}_n$ converges to some $\mathbf{x}(\xi)$ in the completion $(\oplus_{n=1}^{\infty} \hat{X}_n)_p$ of $(\oplus_{n=1}^{\infty} X_n)_p$. Since $x(\xi)_j = \sum_n \xi_n x_{nj} = \sum_{1 \leq n \leq j-1} \xi_n x_{nj} \in X_j$, it follows that $\mathbf{x}(\xi) \in (\oplus_{n=1}^{\infty} X_n)_p$ and then $D = \{\sum_n \xi_n \mathbf{x}_n, \xi \in \ell_1, \|\xi\|_1 \leq 1\}$ is a Banach disk in $(\oplus_{n=1}^{\infty} X_n)_p$. Consequently, there must be some $m \in \mathbb{N}$ such that W_m absorbs D and hence $\mathbf{x}_m \in \mathrm{span}(W_m)$, a contradiction. □

Theorem 1 *If X_n is UBL for each $n \in \mathbb{N}$, then $(\oplus_{n=1}^{\infty} X_n)_p, 1 \leq p \leq \infty$, is UBL.*

Proof. Our argument adapts some methods of [8] to our convenience.

If $(\oplus_{n=1}^{\infty} X_n)_p$ is not UBL, there exists a sequence $\{W_n, n \in \mathbb{N}\}$ of closed absolutely convex subsets of $(\oplus_{n=1}^{\infty} X_n)_p$ covering $(\oplus_{n=1}^{\infty} X_n)_p$ such that no W_n is a neighbourhood of the origin in $(\oplus_{n=1}^{\infty} X_n)_p$.

Define $\mathcal{F} = \{F \in \{\mathrm{span}(W_n), n \in \mathbb{N}\} : \exists m \in \mathbb{N} \text{ with } F \supseteq (\oplus_{n>m} X_n)_p\}$. If $\mathcal{F}$ does not cover $(\oplus_{n=1}^{\infty} X_n)_p$ then $(\oplus_{n=1}^{\infty} X_n)_p$ is covered by all those subspaces $\mathrm{span}(W_n)$ that do not belong to $\mathcal{F}$, as a consequence of the Theorem 4.1 of [6]. But this contradicts the previous lemma. Hence $\mathcal{F}$ covers the whole space.

Let $\mathcal{F}_n := \{F \in \mathcal{F} : F$ does not contain $X_n\}$, where we consider X_n as a subspace of $(\oplus_{n=1}^{\infty} X_n)_p$. Let us see first that $\mathcal{F} = \cup\{\mathcal{F}_n, n \in \mathbb{N}\}$. Indeed, if $G \in \mathcal{F}$, there is a $n(G) \in \mathbb{N}$ with $G \supseteq (\oplus_{m>n(G)} X_m)_p$. Hence, there must be $r \leq n(G)$ such that G does not contain X_r, otherwise $G = (\oplus_{n=1}^{\infty} X_n)_p$, which is a contradiction because $G = \operatorname{span}(W_p)$ for some p and we would have that W_p is a barrel, hence a zero-neighbourhood since $(\oplus_{n=1}^{\infty} X_n)_p$ is barrelled. Thus, $G \in \mathcal{F}_r$.

Let us show that considering X_j as a subspace of $(\oplus_{n=1}^{\infty} X_n)_p$ there is $j \in \mathbb{N}$ such that $\cup\{F, F \in \mathcal{F}_j\} \supseteq X_j$. Otherwise for each $j \in \mathbb{N}$ there would be some norm one $x_j \in X_j$ verifying that $x_j \notin \cup\{F, F \in \mathcal{F}_j\}$. Defining $\mathbf{x}_j \in (\oplus_{n=1}^{\infty} X_n)_p$ such that $x_{jk} = 0$ if $j \neq k$ while $x_{jj} = x_j$, then $(\mathbf{x}_j)$ is a basic sequence in $(\oplus_{n=1}^{\infty} X_n)_p$ equivalent to the unit vector basis of ℓ_p if $p < \infty$ or c_0 if $p = \infty$. Hence, reasoning as in the previous lemma, we have that the closed linear span L of $(\mathbf{x}_j)$ in $(\oplus_{n=1}^{\infty} \hat{X}_n)_p$, is contained in $(\oplus_{n=1}^{\infty} X_n)_p$. Since $\mathcal{F}$ covers $L \subseteq (\oplus_{n=1}^{\infty} X_n)_p$ and L is a Banach space, it follows that there is some $F \in \mathcal{F}$ so that $\mathbf{x}_j \in F$ for each $j \in \mathbb{N}$. But, as we have seen that $\mathcal{F} = \cup\{\mathcal{F}_n, n \in \mathbb{N}\}$, there is a $k \in \mathbb{N}$ such that $F \in \mathcal{F}_k$. Therefore $\mathbf{x}_k \in \cup\{G : G \in \mathcal{F}_k\}$, which is a contradiction.

Finally, choose a positive integer m such that $\cup\{F : F \in \mathcal{F}_m\} \supseteq X_m$. As X_m is UBL, there is $G \in \mathcal{F}_m$ with $G \supseteq X_m$. This is a contradiction, since $G \in \mathcal{F}_m$ if and only if ($G \in \mathcal{F}$ and) G does not contain X_m. $\square$

Theorem 2 *Let I be a non-empty index set and let $\{X_i, i \in I\}$ be a family of normed spaces. Then $(\oplus_{i\in I} X_i)_p$, with $1 \leq p \leq \infty$, is UBL if and only if X_i is UBL for each $i \in I$.*

Proof. If I is finite, the conclusion is obvious, and if $I = \mathbb{N}$ the result has been proved in the previous theorem. Thus we may assume that card $I > \aleph_0$. If $(\oplus_{i\in I} X_i)_p$ is not UBL there exists a sequence $\{W_n, n \in \mathbb{N}\}$ of closed absolutely convex subsets of $(\oplus_{i\in I} X_i)_p$ covering $(\oplus_{i\in I} X_i)_p$ such that no W_n is a neighbourhood of the origin in $(\oplus_{i\in I} X_i)_p$. Hence there is a sequence $(\mathbf{x}_n)$ in the unit sphere of $(\oplus_{i\in I} X_i)_p$ such that $\mathbf{x}_n \notin \operatorname{span}(W_n)$ for each $n \in \mathbb{N}$. As each $\mathbf{x}_n$ is countably supported $J := \cup\{\operatorname{supp} \mathbf{x}_n, n \in \mathbb{N}\}$ is a countable subset of I. But $(\oplus_{j\in J} X_j)_p$ is UBL as a consequence of the previous theorem, and hence there is some $m \in \mathbb{N}$ such that $W_m \cap (\oplus_{j\in J} X_j)_p$ is a neighbourhood of the origin in $(\oplus_{j\in J} X_j)_p$. Therefore $\mathbf{x}_m \in \operatorname{span}(W_m)$, a contradiction. $\square$

Open problem : Assuming that μ is atomless and X is a normed space, is $L_p(\mu, X), 1 \leq p < \infty$, a TB space?

ACKNOWLEDGMENT. The author is indebted to the referee for his useful comments and suggestions.

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