On Hermitian Spreads

Ilaria Cardinali   Rocco Trombetti

Abstract

Let $\perp$ be the polarity of $PG(5,q)$ defined by the elliptic quadric $Q^-(5,q)$. A locally Hermitian spread $\mathcal{S}$ of $Q^-(5,q)$, with respect to a line L, is associated in a canonical way with a spread $\mathcal{S}_\Lambda$ of the 3-dimensional projective space $L^\perp = \Lambda$, and conversely. In this paper we give a geometric characterization of the regular spreads of Λ which induce Hermitian spreads of $Q^-(5,q)$.

1 Introduction

Let $Q^-(5,q)$ be the elliptic quadric of $PG(5,q)$. A *spread* $\mathcal{S}$ of $Q^-(5,q)$ is a partition of the pointset of $Q^-(5,q)$ into lines. Let L be a fixed line of $\mathcal{S}$. For each line M of $\mathcal{S}$, the subspace $\langle L, M\rangle$ has dimension 3 and intersects $Q^-(5,q)$ in a non-singular hyperbolic quadric. Let $\mathcal{R}_{L,M}$ be the regulus of $\langle L, M\rangle \cap Q^-(5,q)$ containing the lines L and M. The spread $\mathcal{S}$ is *locally Hermitian* with respect to L if $\mathcal{R}_{L,M}$ is contained in $\mathcal{S}$ for all lines M of $\mathcal{S}$ different from L. If the spread $\mathcal{S}$ is locally Hermitian with respect to all the lines of $\mathcal{S}$, then the spread is called *Hermitian* or *regular* and is unique up to isomorphism (see [8], Section 8.1).

Let $\perp$ be the polarity of $PG(5,q)$ defined by $Q^-(5,q)$. As in [3], it is possible to associate with a locally Hermitian spread with respect to a line L of $Q^-(5,q)$, a spread of the 3-dimensional projective space $L^\perp = \Lambda$ in the following way. Let M be any line of $\mathcal{S}$ different from L; then $m_{L,M} = \langle L, M\rangle^\perp$ is a line of Λ disjoint from $\langle L, M\rangle$.

Then, $\mathcal{S}_\Lambda = \{m_{L,M} \mid M \in \mathcal{S}, M \neq L\} \cup \{L\}$ is a spread of Λ and in [3] it has been shown that for each spread $\mathcal{F}$ of Λ containing L, there is a locally Hermitian spread

Received by the editors July 2002.
Communicated by J. Thas.
1991 *Mathematics Subject Classification* : 51E20.
Key words and phrases : Hermitian spreads, ovoid.

$\mathcal{S}(\mathcal{F}, L)$ of $Q^-(5, q)$ (with respect to L) such that $\mathcal{S}(\mathcal{F}, L)_\Lambda = \mathcal{F}$. This construction is the dual of the one in [8], Section 8.1, Case d (see, e.g., [3]).

By [8], if $\mathcal{S}$ is a Hermitian spread of $Q^-(5, q)$, then the spread $\mathcal{S}_\Lambda$ is regular for each line L of $\mathcal{S}$ but, surprisingly, for $q > 2$, there exists a regular spread $\mathcal{F}$ of Λ containing L such that $\mathcal{S}(\mathcal{F}, L)$ is not Hermitian (see [3], Theorem 6). Note that for $q = 2$ every regular spread induces a Hermitian spread (see [3]). An open problem was the characterization of those regular spreads of Λ inducing Hermitian spreads of $Q^-(5, q)$.

Let $Q^+(5, q^2)$ be a non-singular hyperbolic quadric of $PG(5, q^2)$. Choose $\Sigma \simeq PG(5, q)$ in such a way that $Q^-(5, q) = Q^+(5, q^2) \cap \Sigma$, with Σ the set of fixed points with respect to a Baer involution σ of $PG(5, q^2)$ preserving the hyperbolic quadric $Q^+(5, q^2)$.

Denote by Λ^* the 3-dimensional subspace of $PG(5, q^2)$ defined by Λ. Recall that each regular spread of Λ has exactly two transversal lines over $GF(q^2)$ (see, e.g., [1]). Note that Λ^* intersects $Q^+(5, q^2)$ in two conjugate planes with respect to σ meeting Λ in exactly the line L.

In this paper we prove that the only regular spreads (containing L) of Λ which give Hermitian spreads of $Q^-(5, q)$ are those whose transversals lie on the two planes of $Q^+(5, q^2)$ through L.

2 Regular Spreads of $PG(3, q)$

Let L_1, L_2, L_3 be any three disjoint lines of $PG(3, q)$ and denote by $\mathcal{R}(L_1, L_2, L_3)$ the regulus of $PG(3, q)$ containing L_1, L_2 and L_3 (see e.g. [2]). A spread $\mathcal{F}$ of $PG(3, q)$ is *regular* if $\mathcal{R}(L_1, L_2, L_3)$ is contained in $\mathcal{F}$ for all distinct L_1, L_2, $L_3 \in \mathcal{F}$. Put $\bar{\Lambda} = PG(3, q^2)$ and let τ be a Baer involution of $\bar{\Lambda}$. The set Λ of the points of $\bar{\Lambda}$ fixed by τ is a 3-dimensional projective space $PG(3, q)$. A line of $\bar{\Lambda}$ contains exactly $0, 1$ or $q + 1$ points of Λ, and a point x of $\bar{\Lambda} \setminus \Lambda$ is incident with the extension of exactly one line of Λ. If m is an imaginary line of $\bar{\Lambda}$ (i.e., a line disjoint from Λ), then also m^τ is imaginary. For each point x of m, let $L(x)$ be the line joining the points $x \in m$ and $x^\tau \in m^\tau$. Then $L(x)$ intersects Λ in exactly $q + 1$ points and $l(x) = L(x) \cap \Lambda$ is a line of Λ. The lineset $\mathcal{F}_m = \{l(x) : x \in m\}$ is a regular spread of Λ and for each regular spread $\mathcal{F}$ of Λ there are exactly two imaginary lines m and m^τ such that $\mathcal{F}_m = \mathcal{F}_{m^\tau} = \mathcal{F}$ (see, e.g., [2]). The set $\mathcal{R}$ is a regulus of $\mathcal{F}_m$ if and only if $\{x \in m : l(x) \in \mathcal{R}\}$ is a Baer subline of m (see, e.g., [1]), i.e. a subline isomorphic to $PG(1, q)$. The lines m and m^τ are called *transversals* of $\mathcal{F}_m$.

3 Hermitian spreads of $Q^-(5, q)$

Let $\mathcal{H} = H(3, q^2)$ be the Hermitian variety of $PG(3, q^2)$ with equation $X_0X_3^q + X_1X_2^q + X_2X_1^q + X_3X_0^q = 0$ (for more details see, e.g., [4]) . An *ovoid* of $\mathcal{H}$ is a set of $q^3 + 1$ points of $\mathcal{H}$ such that no two of them are collinear on $\mathcal{H}$ (see e.g. [6]). The incidence structure formed by all points and lines on $\mathcal{H}$ is the dual of the incidence structure formed by all points and lines of the elliptic quadric $Q^-(5, q)$ (see [5]); call this duality ρ. A locally Hermitian spread $\mathcal{S}$ of $Q^-(5, q)$ with respect to a line L is the image under ρ of an ovoid $\mathcal{O}$ of $\mathcal{H}$ whose points lie on q^2 Baer sublines on $\mathcal{O}$

through the point P, with $P^\rho = L$. The spread $\mathcal{S} = \mathcal{O}^\rho$ is Hermitian if and only if $\mathcal{O}$ is classical (see [8]), i.e., is a non-singular Hermitian curve.

Let $P = (1,0,0,0) \in \mathcal{H}$ and note that the polar plane of P with respect to $\mathcal{H}$ is $\pi_1 : X_3 = 0$. In order to parametrize the ovoid $\mathcal{O}$, one can fix any plane π_0 of $PG(3,q^2)$ not through P and consider the set $\mathcal{P}$ of the points of π_0 obtained as intersection with the q^2 Baer sublines on $\mathcal{O}$ through P. Since $\mathcal{O}$ is an ovoid, the points of $\mathcal{P}$ have the property that the line joining any two of them intersects π_1 in a non-singular point.

Let π_0 have equation $X_0 = 0$. The points of π_0 not contained in π_1 have coordinates $\{(0,\alpha,\beta,1) : \alpha,\beta \in GF(q^2)\}$. As the line joining $(0,\alpha,\beta,1)$ and $(0,\alpha,\beta',1)$, with $\beta \neq \beta'$, intersects π_1 in a singular point, it follows that for any α there is exactly one β such that $(0,\alpha,\beta,1) \in \mathcal{P}$. Hence, put $\beta = f(\alpha)$, f a function from $GF(q^2)$ to itself. Then $\mathcal{P}$ is the set of points of π_0 with coordinates $(0,\alpha,f(\alpha),1)$, $\alpha \in GF(q^2)$. Since the line joining any two points of $\mathcal{P}$ intersects π_1 in a non-singular point, it is easy to verify that:

$$(*) \quad tr((f(\alpha)-f(\alpha'))^q(\alpha-\alpha')) \neq 0, \quad \forall \alpha,\alpha' \in GF(q^2),\ \alpha \neq \alpha',$$

where tr is the trace function from $GF(q^2)$ to $GF(q)$.

The ovoid $\mathcal{O}$ is the set of points of $\mathcal{H}$ lying on the q^2 lines joining P with any point of $\mathcal{P}$. So $(c,\alpha,f(\alpha),1) \in \mathcal{O}$ if and only if $tr(c+\alpha^q f(\alpha)) = 0$, $\forall \alpha, c \in GF(q^2)$, i.e. if and only if $c = \gamma - \alpha^q f(\alpha)$, $\gamma \in GF(q^2)$ with $tr(\gamma) = 0$. Hence we can write

$$\mathcal{O} = \{P_{\gamma,\alpha} \mid \gamma,\alpha \in GF(q^2) \text{ with } tr(\gamma)=0\} \cup \{P\}$$

where $P_{\gamma,\alpha} = (\gamma - \alpha^q f(\alpha), \alpha, f(\alpha), 1)$. Apply the Klein correspondence ϕ from the lineset of $PG(3,q^2)$ onto the pointset of the hyperbolic quadric $\mathcal{K} = Q^+(5,q^2)$ with equation $X_0X_5 + X_1X_4 + X_2X_3 = 0$. The correspondence ϕ associates to the line of $PG(3,q^2)$ spanned by the points $u = (u_0,u_1,u_2,u_3)$ and $v = (v_0,v_1,v_2,v_3)$, the point of $PG(5,q^2)$ with Plücker coordinates $(p_{01},p_{02},p_{03},p_{12},-p_{13},p_{23})$ where $p_{ij} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}$, $i,j = 0,...,3$, $i<j$. To each point $P_{\gamma,\alpha}$ of $\mathcal{O}$ there corresponds the line $L_{\gamma,\alpha} = (\Im(P_{\gamma,\alpha}))^\phi$ of $\mathcal{K}$ where $\Im(P_{\gamma,\alpha})$ is the set of lines of $\mathcal{H}$ through $P_{\gamma,\alpha}$. Note that the lines $l_1 = \langle (f(\alpha)^q,-1,0,0), (\gamma-\alpha^q f(\alpha),\alpha,f(\alpha),1)\rangle$ and $l_2 = \langle (\alpha^q,0,-1,0), (\gamma-\alpha^q f(\alpha),\alpha,f(\alpha),1)\rangle$ belong to $\Im(P_{\gamma,\alpha})$. Hence, the ovoid $\mathcal{O}$ can be associated with the partial line-spread $\mathcal{S}^+ = \mathcal{O}^\phi$ of $\mathcal{K}$:

$$\mathcal{S}^+ = \{L_{\gamma,\alpha} : \gamma,\alpha \in GF(q^2)\} \cup \{L^*\}$$

where $L^* = P^\phi = \langle (1,0,0,0,0,0), (0,1,0,0,0,0)\rangle$ and
$L_{\gamma,\alpha} = (\Im(P_{\gamma,\alpha}))^\phi = \langle l_1^\phi, l_2{}^\phi\rangle = \langle (\gamma - \alpha^q f(\alpha) + \alpha f(\alpha)^q, f(\alpha)^{q+1}, f(\alpha)^q, -f(\alpha), 1, 0),$
$(\alpha^{q+1}, \gamma, \alpha^q, \alpha, 0, -1)\rangle$.

The Hermitian variety $\mathcal{H}$ is the dual of the elliptic quadric $Q^-(5,q) = \mathcal{K} \cap \Sigma$ where $\Sigma = \{(x_0, x-x^q, y, y^q, z-z^q, x_5) : x_0,x_5 \in GF(q), x,y,z \in GF(q^2)\}$ is the subgeometry ($\simeq PG(5,q)$) of the points of $PG(5,q^2)$ fixed by the involutorial collineation:

$$(x_0,x_1,x_2,x_3,x_4,x_5)^\sigma = (x_0^q, -x_1^q, x_3^q, x_2^q, -x_4^q, x_5^q)$$

(see, e.g., [4], §15.2). Using this duality, the ovoid $\mathcal{O}$ corresponds to the locally Hermitian spread $\mathcal{S}^- = \mathcal{S}^+ \cap \Sigma$ of $Q^-(5,q)$ with respect to the line $L = L^* \cap \Sigma$.

Lemma 3.1. *$\mathcal{S}^-$ is a Hermitian spread of $Q^-(5,q)$ if and only if $f(\alpha) = a\alpha + b$, with $a, b \in GF(q^2)$ and $tr(a) \neq 0$.*

Proof. $\mathcal{S}^-$ is Hermitian if and only if $\mathcal{O}$ is a classical ovoid of $\mathcal{H}$, i.e. if and only if it is contained in a non-singular plane. This implies that $f(\alpha) = a\alpha + b$, with $a, b \in GF(q^2)$ and condition $(*)$ forces $tr(a) \neq 0$. Conversely, assuming $f(\alpha) = a\alpha + b$ and $tr(a) \neq 0$, the ovoid $\mathcal{O}$ is contained in the non-singular plane $aX_1 - X_2 + bX_3 = 0$, hence $\mathcal{O}$ is a classical ovoid and $\mathcal{S}^-$ is Hermitian. ■

In the sequel we use the symbol $\perp$ to refer both to the polarity of $PG(5,q^2)$ induced by $\mathcal{K}$ and to the polarity of $PG(5,q)$ induced by $Q^-(5,q)$. The 3-dimensional projective space $\Lambda^* = L^{*\perp}$ has equations $X_4 = X_5 = 0$. Using the construction of [3], the following partial spread $\mathcal{S}^+$ of $\mathcal{K}$ induces the partial spread of Λ^*:

$$\mathcal{S}_{\Lambda^*} = \{L_\alpha : \alpha \in GF(q^2)\} \cup \{L^*\}$$

where $L_\alpha = \langle L^*, L_{\gamma,\alpha}\rangle^\perp = \{(\alpha X_2 + \alpha^q X_3, f(\alpha)X_2 - f(\alpha)^q X_3, X_2, X_3, 0, 0) : X_2, X_3 \in GF(q^2)\}$. If we intersect $\mathcal{S}_{\Lambda^*}$ with Σ, we obtain a spread $\mathcal{S}_\Lambda$ of the 3-dimensional projective space $L^\perp = \Lambda = \Lambda^* \cap \Sigma$.

Observe that the singular planes of $\mathcal{K}$ through L are exactly $\pi : \; X_2 = X_4 = X_5 = 0$ and $\pi^\sigma : \; X_3 = X_4 = X_5 = 0$.

Theorem 3.2. *The spread $\mathcal{S}^-$ is Hermitian if and only if $\mathcal{S}_\Lambda$ is a regular spread whose transversals lie on the two planes of $\mathcal{K}$ through L.*

Proof. Suppose $\mathcal{S}^-$ is Hermitian. By Lemma 3.1, $f(\alpha) = a\alpha + b$ with $tr(a) \neq 0$. Intersecting π with all lines $L_\alpha \in \mathcal{S}_{\Lambda^*}$ we get $\{(\alpha^q, -f(\alpha)^q, 0, 1, 0, 0), \alpha \in GF(q^2)\}$, which is a line, say m. Similarly, intersecting π^σ with all $L_\alpha \in \mathcal{S}_{\Lambda^*}$, we get a line, say m', with $m' = m^\sigma$. Note that the lines of $\mathcal{S}_\Lambda$ are precisely those joining each point on m with its conjugate on m^σ. Hence $\mathcal{S}_\Lambda$ is regular. Conversely, if $\mathcal{S}_\Lambda$ is a regular spread whose transversals lie in π and π^σ, then $\pi \cap L_\alpha = \{(\alpha^q, -f(\alpha)^q, 0, 1, 0, 0), \alpha \in GF(q^2)\}$ and $\pi^\sigma \cap L_\alpha = \{(\alpha, f(\alpha), 1, 0, 0, 0), \alpha \in GF(q^2)\}$ are lines. This happens only when $f(\alpha) = a\alpha + b$. Finally, condition $(*)$ implies $tr(a) \neq 0$ and, by Lemma 3.1, $\mathcal{S}^-$ is Hermitian. ■

References

[1] R.H. Bruck: Construction problems in finite projective spaces, Finite Geometric Structures and their Application, C.I.M.E., Bressanone 18-27 Giugno 1972, Edizioni Cremonese, 105-188.

[2] A.A. Bruen: Spreads and a conjecture of Bruck and Bose, J. Algebra, 23 (1972), 519-537.

[3] I. Cardinali, G. Lunardon, O. Polverino, R. Trombetti: Spreads in $H(q)$ and 1-systems of $Q(6,q)$, European J. Combin.,23 (2002), 367-376.

[4] J.W.P. Hirschfeld: *Finite projective spaces of three dimensions*, Oxford University Press, Oxford,1985.

[5] S.E. Payne and J.A. Thas: *Finite Generalized Quadrangles*, Research Notes in Mathematics, volume 110, Pitman, Boston,1984.

[6] S.E. Payne and J.A. Thas: Spreads and Ovoids in Finite Generalized Quadrangles, Geom. Dedicata, 52 (1984), 227-253.

[7] J.A. Thas: Ovoids and spreads of finite classical polar spaces, Geom. Dedicata, 10 (1981), 135-144.

[8] J.A. Thas: Semi-partial geometries and spreads of classical polar spaces, J. Combin. Theory Ser. A, 35 (1983), 58-66.

Ilaria Cardinali
Dipartimento Scienze Matematiche e Informatiche "R. Magari"
Universita' degli Studi di Siena
Pian dei Mantellini
53100 Siena, Italy
e-mail: cardinal@unina.it

Rocco Trombetti
Dipartimento di Matematica e Applicazioni
Universita' degli Studi di Napoli "Federico II"
Complesso di Monte S. Angelo, Edificio T,
Via Cintia,
I 80126 Napoli, Italy
e-mail: rtrombet@unina.it