ON γ-P-REGULAR SPACES

SABIR HUSSAIN AND BASHIR AHMAD

Abstract. We define and explore γ-P-regularity which is a generalization of P-regularity [16] and γ-regularity [3]. We also define and discuss strongly γ-semi-continuous functions.

1. Introduction

Signal and Arya [21] defined a new separation axiom called almost regularity which is weaker than regularity. It has been shown [21] that for Hausdorff spaces, this axiom occupies a position between Urysohn's separation axiom and T_3-axiom. Maheswari and Prasad [17] have defined another axiom called s-regularity which is weaker than regularity (without T_2). C. Dorsett [7] defined and investigated a new separation axiom called semi-regularity. It is shown that s-regularity is weaker than semi-regularity. Semi-regularity due to C. Dorsett [7] and almost regularity due to Signal and Arya [20] are both weaker than regularity, but both are independent of each other. M. Khan and B. Ahmad [16] defined another form of regularity called P-regularity which implies semi-regularity [7] as well as almost regularity [21]. S. Kasahara [12] defined an operation α on topological spaces. H. Ogata [13] introduced the concept of γ-open sets and investigated the related topological properties. B. Ahmad and F. U. Rehman [1, 20] introduced the notions of γ-exterior and γ-boundary points in topological spaces. They also continued studying the properties and characterizations of (γ, β)-continuous mappings introduced by H. Ogata [13]. B. Ahmad and S. Hussain further studied the properties of γ-operations in topological spaces [2, 3]. They introduced and explored the notion of γ-semi-open sets in topological spaces [5, 10]. The concept of γ-s-closed spaces and γ-s-closed subspaces have been studied [4, 8]. It is known [8] that the concept of γ-s-closed space is a generalization of s-closed space [14].

In this paper, we define and explore γ-P-regularity which is a generalization of P-regularity [16] and γ-regularity [3]. We also define and discuss strongly γ-semi-continuous functions in Section 4.

First, we recall preliminaries used in the sequel. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Definition 2.1. **[12]** *Let X be a space. An operation $\gamma\colon \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ.*

Definition 2.2. **[13]** *Let X be a space and $A \subseteq X$. A point $x \in A$ is said to be γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus,*

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\}.$$

Note that A is γ-open [13] if and only if $A = int_\gamma(A)$. A set A is called γ-closed [20] if and only if $X - A$ is γ-open.

Definition 2.3. **[13]** *A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called the γ-closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.*

Definition 2.4. **[13]** *An operation γ on τ is said be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.*

Definition 2.5. **[13]** *An operation γ on τ is said to be open, if for every nbd U of each $x \in X$, there exists a γ-open set B such that $x \in B$ and $U^\gamma \subseteq B$.*

Definition 2.6. **[3]** *An operation γ on τ is said to be γ-open (resp. closed), if U^γ is γ-open (resp. closed) for every open (resp. closed) set U in X.*

Definition 2.7. **[3]** *A space X is said to be γ-regular space, if for any γ-closed set A and $x \notin A$, there exist γ-open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.*

Definition 2.8. **[10]** *A subset A of a space X is said to be a γ-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed if and only if $X - A$ is γ-semi-open in X. Note that A is γ-semi-closed if and only if $int_\gamma cl_\gamma(A) \subseteq A$.*

Definition 2.9. **[5]** *Let A be a subset of a space X. The intersection of all γ-semi-closed sets containing A is called the γ-semi-closure of A and is denoted by $scl_\gamma(A)$. Note that A is γ-semi-closed if and only if $scl_\gamma(A) = A$.*

Definition 2.10. **[5]** *Let A be a subset of a space X. The union of γ-semi-open subsets of A is called the γ-semi-interior of A and is denoted by $sint_\gamma(A)$.*

Definition 2.11. **[5]** *A subset A of a space X is said to be γ-semi-regular, if it is both γ-semi-open and γ-semi-closed. The class of all γ-semi-regular sets of X is denoted by $SR_\gamma(A)$. Note that if γ is a regular operation, then the union of γ-semi-regular sets is γ-semi-regular.*

Definition 2.12. **[16]** *A space X is said to be P-regular, if for each semi-closed set F and $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.*

Definition 2.13. **[3]** *A space X is said to be γ_o-compact, if every cover $\{V_i : i \in I\}$ of X by γ-open sets of X, there exists a finite subset I_o of I such that $X = \bigcup_{i \in I_o} cl_\gamma(V_i)$.*

Definition 2.14. **[9]** *A space X is said to be almost γ-regular, if for any γ-semi-closed set A, and $x \notin A$, there exist disjoint γ-open sets V and W such that $x \in V$ and $A \subseteq W$.*

3. γ-P-Regular Spaces

Definition 3.1. *A space X is said to be γ-P-regular, if for each γ-semi-closed set F and $x \notin F$, there exist disjoint γ-open sets U, V such that $x \in U$ and $F \subseteq V$.*

The following implications follow from [3, 7, 8, 16, 21].

$$P-regularity \Rightarrow regularity \Rightarrow \gamma-regularity$$
$$P-regularity \Rightarrow regularity \Rightarrow almost-regularity$$
$$P-regularity \Rightarrow semi-regularity \Rightarrow \gamma-semi-regularity$$

We note that each regularity and P-regularity [16] implies γ-P-regularity, and γ-regularity [3], however, the converse is not true in general as shown in the following example.

Example 3.2. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $a \in X$, define the operation $\gamma\colon \tau \to P(X)$ by*

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } a \in A \\ int(cl(A)), & \text{if } a \notin A \end{cases}$$

Clearly $\{a, c\}, \{b\}, X, \phi$ are γ-open and γ-semi-open sets. Also, $\{a\}$, $\{b\}$, $\{a, c\}$, X ,ϕ are semi-open sets in X. Here, X is γ-P-regular and γ-regular, but not P-regular and regular.

Theorem 3.3. *Let X be a space. If γ is an open operation, then the following are equivalent.*

(1) *X is a γ-P-regular space.*

(2) *For every $x \in X$ and every γ-semi-open set V containing x, there exists a γ-open set U such that $x \in U \subseteq cl_\gamma(U) \subseteq V$.*
(3) *For every γ-semi-closed set A, the intersection of all the γ-closed nbds of A is A.*
(4) *For every set A and a γ-semi-open set B such that $A \cap B \neq \phi$, there exists a γ-open set F such that $A \cap F \neq \phi$ and $cl_\gamma(F) \subseteq B$.*
(5) *For every nonempty set A and γ-semi-closed B such that $A \cap B = \phi$, there exist disjoint γ-open sets L and M such that $A \cap L \neq \phi$ and $B \subseteq M$.*

Proof. (1) $\Rightarrow$ (2). Let V be a γ-semi-open set containing x. Then $X - V$ is γ-semi-closed and $x \notin X - V$. Since X is γ-P-regular, there exist disjoint γ-open sets U_1, U_2 such that $x \in U_1$ and $X - V \subseteq U_2$. Thus, we obtain $x \in U_1 \subseteq cl_\gamma(U_1) \subseteq V$. This proves (2).

(2) $\Rightarrow$ (3). Let A be a γ-semi-closed set and $x \notin A$. Then $X - A$ is γ-semi-open and $x \in X - A$. By (2), there exists a γ-open set U such that $x \in U \subseteq cl_\gamma(U) \subseteq X - A$. Thus, $A \subseteq X - cl_\gamma(U) \subseteq X - U$. Consequently, $X - U$ is a γ-closed nbd of A [20] and $x \notin X - U$. This proves (3).

(3) $\Rightarrow$ (4). Let $A \cap B \neq \phi$ and B be a γ-semi-open set. Let $x \in A \cap B$. Then $X - B$ is γ-semi-closed and $x \notin X - B$. Then by (3), there exists a γ-closed nbd V of $X - B$ and $x \notin V$. Let $X - B \subseteq U \subseteq V$, where U is γ-open. Then $F = X - V$ is γ-open such that $x \in F$ and $A \cap F \neq \phi$. Also, $X - U$ is γ-closed and $cl_\gamma(F) = cl_\gamma(X - V) \subseteq X - U \subseteq B$. This shows that $cl_\gamma(F) \subseteq B$.

(4) $\Rightarrow$ (5). Suppose $A \cap B = \phi$, where $A \neq \phi$ and B is γ-semi-closed. Then $X - B$ is γ-semi-open and $A \cap (X - B) \neq \phi$. By (4), there exists a γ-open set L such that $A \cap L \neq \phi$ and $L \subseteq cl_\gamma(L) \subseteq X - B$. Let $M = X - cl_\gamma(L)$. Then $B \subseteq M$. L and M are γ-open sets such that $M = X - cl_\gamma(L) \subseteq (X - L)$. This proves (5).

(5) $\Rightarrow$ (1). This is straightforward. This completes the proof. $\square$

Let $B \subseteq X$ and $\gamma\colon \tau \to P(X)$ be an operation. We define $\gamma_B\colon \tau_B \to P(X)$ by $\gamma_B(U \cap B) = \gamma(U) \cap B$. Then γ_B is an operation and satisfies $cl_{\gamma_B}(U \cap B) \subseteq cl_\gamma(U \cap B) \subseteq cl_\gamma(U) \cap cl_\gamma(B)$.

We know that a subset A of (B, τ_B) is γ_B-open, if for each $x \in A$ there exists a τ_B-open set U such that $x \in U$ and $\gamma_B(U) \subseteq A$.

Observe that if W is γ-open in X and $U = W \cap B$, then U is a γ_B-open in B. In the case that γ is open, we obtain that any γ_B-open set U in B can be written as $U = H \cap B$, where H is a γ-open set in X. And the proof is as follows.

Let U be a γ_B-open set in B. Then for each $x \in U$ there exists a τ_B-open set W_x in B such that $W_x \subseteq W_x^\gamma \subseteq U$. Since $W_x = H_x \cap B$ where H_x is an open sets in X, we obtain that $U = \bigcup_{x \in U} H_x \cap B = \bigcup_{x \in U} H_x^\gamma \cap B$.

Using the fact that γ is open, we obtain that for each $y \in H_x$ there exists a γ-open set D_y such that $H_x^\gamma \subseteq D_y$. If we take $D = \bigcup_{y \in U} D_y$, D is γ-open in X and $U = D \cap B$.

The complement of a γ-semi-open set is called γ-semi-closed set. Observe that A is a γ-semi-closed if there exists a γ-closed set U in X such that $int_\gamma(U) \subseteq A \subseteq U$. The collection of all γ-semi-closed sets in X is denoted by $SC_\gamma(X)$.

From this we obtain that if $A \subseteq B \subseteq X$ and A is γ_B-semi-open in B, then there exists a γ_B-open set O such that $O \subseteq A \subseteq cl_{\gamma_B}(O)$.

Now using this fact, we can prove the following theorem.

Theorem 3.4. *Let X be a space, γ be an open operator, and B a γ-semi-closed set in X containing a subset A of X. If A is a γ_B-semi-closed in the subspace B, then A is a γ-semi-closed in X.*

Proof. Let A be a γ_B-semi-closed in the subspace B. Then there exists a γ_B-closed set U in B such that $int_{\gamma_B}(U) \subseteq A \subseteq U$. U is a γ_B-closed set in B. Using the above argument, there exists a γ-closed set V in X such that $U = V \cap B$. Thus we have $int_{\gamma_B}(V \cap B) \subseteq V \cap B$. $V \cap B$ is γ-semi-closed set in X and $int_{\gamma_B}(U) \subseteq A \subseteq U$. We obtain that $int_\gamma(U) \subseteq int_{\gamma_B}(U) \subseteq A \subseteq U$. We conclude that A is γ-semi-closed in X. □

Theorem 3.5. *Every γ_Y-semi-closed subspace of a γ-P-regular space X is γ_Y-P-regular, where γ is open.*

Proof. Let F be any γ_Y-semi-closed subset of Y and $x \in Y - F$. Then by Theorem 3.4, $F \in SC_\gamma(X)$. X is γ-P-regular. Therefore, there exist γ-open sets U_X and V_X such that $x \in U_X$, $F \subseteq V_X$ and $U_X \cap V_X = \phi$. Let $U = U_X \cap Y$ and $V = V_X \cap Y$. Then clearly U and V are disjoint γ_Y-open subsets of Y containing x and F, respectively. This proves that Y is γ_Y-P-regular. □

Definition 3.6. *A space X is said to be locally γ-indiscrete if every γ-open set is γ-closed.*

Theorem 3.7. *Every almost γ-regular (resp. γ-semi-regular) locally γ-indiscrete space X is γ-P-regular, where γ is open.*

Proof. Let V be a γ-semi-closed subset of X and $x \notin V$. Since X is locally γ-indiscrete, V is γ-regularly-closed [4]. Since X is almost γ-regular [9], there exist disjoint γ-open sets U_1 and U_2 such that $x \in U_1$ and $V \subseteq U_2$. This proves that X is γ-P-regular. This completes the proof. □

Corollary 3.8. *Every γ-regular, locally γ-indiscrete space is γ-P-regular, where γ is open.*

Definition 3.9. [9] *Two sets A and B are γ-strongly separated if there exist disjoint γ-open (closed) sets U, V such that $A \subset U$ and $B \subset V$.*

Theorem 3.10. *In a γ-P-regular space X, every pair consisting of a γ_o-compact set A and a disjoint γ-semi-closed set B can be γ-strongly separated, where γ is regular and open.*

Proof. Since X is γ-P-regular, for each $x \in A$ there exist disjoint γ-open sets U_x and V_{B_x} such that $x \in U_x$ and $B \subseteq V_{B_X}$. Clearly $\{U_x \cap A : x \in A\}$ is relatively γ-open covering of the γ_o-compact set A. This gives that there exists a finite subfamily $\{U_{x_i} \cap A : i = 1, 2, \dots, n\}$ such that $A \subseteq \bigcup_{i \in I_o} cl_\gamma(U_{x_i} \cap A) = U$. Let $V = \bigcap cl_\gamma\{V_{Bx_i} : i = 1, 2, \dots, n\}$. Then U and V are the required disjoint γ-closed sets, since γ is open [13]. This completes the proof. □

Corollary 3.11. *If X is γ-P-regular, A is a γ_o-compact subset of X, and B is a γ-semi-open set containing A. Then there exists a γ-semi-regular set V such that $A \subseteq V \subseteq B$, where γ is regular and open.*

Proof. Let B be γ-semi-open set. Then $X - B$ is a γ-semi-closed set such that $(X - B) \cap A = \phi$. Since A is γ_o-compact, by Theorem 3.10, there exist γ-open sets U_1 and U_2 such that $A \subseteq U_1$, $X - B \subseteq U_2$ and $U_1 \cap U_2 = \phi$ [3]. Then clearly, $A \subseteq U_1 \subseteq scl_\gamma(U_1) \subseteq B$. Let $V = scl_\gamma(U_1)$. Then $A \subseteq V \subseteq B$, where V is a γ-semi-regular set. This completes the proof. □

Corollary 3.12. *If X is a γ-P-regular space and A, B are subsets of X such that A is γ_o-compact and B is γ-semi-closed with $A \cap B = \phi$, then there exist γ-semi-regular sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$, where γ is regular and open.*

Proof. Since $X - B$ is a γ-semi-open set containing A and X is γ-P-regular, by Corollary 3.11, there exists a γ-semi-regular set V_1 such that $A \subseteq V_1 \subseteq X - B$. Again since V_1 is a γ-semi-open set containing A, there exists a γ-semi-regular set U such that $A \subseteq U \subseteq V_1$. Let $V = X - V_1$. Then $A \subseteq U$, $B \subseteq V$, where U and V are disjoint γ-semi-regular sets. This completes the proof. □

4. Strongly γ-Semi-Continuous Functions

Definition 4.1. [6] *A function $f : X \to Y$ is said to be γ-semi-continuous if for any γ-open set B of Y, $f^{-1}(B)$ is γ-semi-open in X.*

Definition 4.2. [6] *A function $f : X \to Y$ is said to be γ-semi-open (resp. closed) if for each γ-open (resp. closed) set U in X, $f(U)$ is γ-semi-open (resp. closed) in Y.*

Definition 4.3. **[13]** *A function $f\colon X \to Y$ is said to be (γ, β)-continuous if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^\beta$, where $\gamma\colon \tau \to P(X)$ and $\beta\colon \delta \to P(Y)$ are operations on τ and δ, respectively.*

Definition 4.4. **[1]** *A function $f\colon X \to Y$ is said to be (γ, β)-closed (resp. open) if for any γ-closed (resp. open) set A of X, $f(A)$ is β-closed (resp. open) in Y.*

Next we give the following definition.

Definition 4.5. *A function $f\colon X \to Y$ is said to be strongly (γ, β)-continuous if $f^{-1}(U)$ is γ-open in X, for every β-clopen set U in Y.*

Definition 4.6. *A function $f\colon X \to Y$ is said to be strongly γ-semi-continuous if $f^{-1}(U)$ is γ-open (resp. closed) in X, for every γ-semi-open (resp. closed) set U in Y.*

Definition 4.7. **[4]** *A subset A of X is called γ-regular-open if $A = int_\gamma(cl_\gamma(A))$.*

Note that A is γ-regular closed if and only if $X - A$ is γ-regular open.

Definition 4.8. *A function $f\colon X \to Y$ is said to be γ-s-open (resp. closed) if $f(U)$ is γ-open (resp. closed) in X, for every γ-semi-open (resp. closed) set U in Y.*

Theorem 4.9. *Let $f\colon X \to Y$ be a (γ, β)-clopen and γ-semi-continuous mapping with γ_o-compact point inverses. If X is γ-P-regular, then Y is γ-P-regular, where β is β-open and γ is a regular and γ-open operation.*

Proof. Let C be a γ-semi-closed subset of Y and $y \notin C$. Then $f^{-1}(C)$ is γ-semi-closed in X [6]. Moreover $f^{-1}(y)$ and $f^{-1}(C)$ are disjoint in X. Then by Theorem 3.9, there exist disjoint γ-closed sets F and G such that $f^{-1}(y) \in F$ and $f^{-1}(C) \subseteq G$. Since f is (γ, β)-closed, there exist β-closed sets V and W in Y containing y and C, respectively, such that $f^{-1}(V) \subseteq F$ and $f^{-1}(W) \subseteq G$ [20]. Since $F \cap G = \phi$, it follows that $V \cap W = \phi$. This proves that Y is γ-P-regular. This completes the proof. $\square$

Theorem 4.10. *If $f\colon X \to Y$ be a function and f is (γ, β)-continuous, then for each β-open set V of Y, $f^{-1}(V)$ is γ-open in X.*

Proof. Let $x \in f^{-1}(V)$, then $f(x) \in V$. Since V is β-open, there exists an open set W in Y such that $f(x) \in W$ and $W^\beta \subseteq V$. Since f is (γ, β)-continuous, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq W^\beta \subseteq V$. Consequently, $x \in U$ and $U^\gamma \subseteq f^{-1}(V)$ which implies $f^{-1}(V)$ is γ-open in X. $\square$

Theorem 4.11. *The inverse image of a β-P-regular space under a (γ, β)-continuous, γ-semi-closed preserving injection is γ-P-regular, where β is open.*

Proof. Let C be a γ-semi-closed subset of X and $x \in X - C$. Since f is a γ-semi-closed preserving injection, $f(C)$ is γ-semi-closed in Y and $f(x) \notin f(C)$. Since Y is β-P-regular, there exist disjoint β-open sets U and V in Y such that $f(x) \in U$ and $f(C) \subseteq V$. This gives that $x \in f^{-1}(U)$ and $C \subseteq f^{-1}(V)$. Since f is (γ, β)-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are γ-open subsets of X [20], such that $f^{-1}(U) \cap f^{-1}(V) = \phi$. This proves that X is γ-P-regular. This completes the proof. □

Theorem 4.12. *Let $f : X \to Y$ be strongly γ-semi-continuous and (γ, β)-clopen surjection. If X is γ-regular, then Y is γ-P-regular.*

Proof. Let C be any γ-semi-closed set of Y and $y \in Y - C$. Let $x \in f^{-1}(y)$. Since f is strongly γ-semi-continuous, $f^{-1}(C)$ is γ-closed in X and $x \in f^{-1}(C)$. Since X is γ-regular [3], there exist disjoint γ-open sets U and V such that $x \in U$, $f^{-1}(C) \subseteq V$. Since f is (γ, β)-closed [1], there exists a β-open set W of Y such that $C \subseteq W$ and $f^{-1}(W) \subseteq V$. This gives that $U \cap f^{-1}(W) = \phi$ or $f(U) \cap W = \phi$. Since f is (γ, β)-open [1], $f(U)$ is β-open in Y and $y \in f(U)$. This proves that Y is γ-P-regular. This completes the proof. □

Theorem 4.13. *Let $f : X \to Y$ be strongly γ-semi-continuous and (γ, β)-closed surjection with γ_o-compact point inverses. If X is γ-regular, then Y is γ-P-regular.*

Proof. Let C be a γ-semi-closed set of Y and $y \in Y - C$. Since f is strongly γ-semi-continuous, $f^{-1}(C)$ is γ-closed in X. Moreover, the γ_o-compact sets $f^{-1}(y)$ and $f^{-1}(C)$ are disjoint. Since X is γ-regular, there exist disjoint γ-open sets F and G such that $f^{-1}(y) \in F$ and $f^{-1}(C) \subseteq G$. Since f is (γ, β)-closed, there exist β-open sets V and W in Y containing y and C, respectively, such that $f^{-1}(V) \subseteq F$ and $f^{-1}(W) \subseteq G$. Clearly $V \cap W = \phi$. This proves that Y is γ-P-regular. This completes the proof. □

Theorem 4.14. *The inverse image of a γ-semi-regular space Y under γ-semi-closed preserving and strong γ-semi-continuous injection is γ-P-regular.*

Proof. Let C be any arbitrary γ-closed subset of X such that $x \in X - C$. Since f is γ-semi-closed preserving injection, $f(C)$ is γ-semi-closed in Y and $f(x) \notin f(C)$. Since Y is γ-semi-regular [8], there exist disjoint γ-semi-open sets U and V such that $f(x) \in U$ and $f(C) \subseteq V$. Since f is strong γ-semi-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are γ-open subsets of X containing x and

C such that $f^{-1}(U) \cap f^{-1}(V) = \phi$. This proves that X is γ-P-regular. This completes the proof. $\square$

Theorem 4.15. *The inverse image of a β-regular space under γ-s-closed and (γ, β)-continuous injection is γ-P-regular.*

Proof. Let C be any γ-semi-closed subset of X and $x \in X - C$. Since f is γ-s-closed, $f(C)$ is β-closed in Y and $f(x) \notin f(C)$. Since Y is β-regular, there exist disjoint β-open sets U and V of Y such that $f(x) \in U$ and $f(C) \subseteq V$. Since f is (γ, β)-continuous, using Theorem 4.10, $f^{-1}(U)$ and $f^{-1}(V)$ are γ-open subsets of X. Now $x \in f(f^{-1}(x)) \subseteq f^{-1}(U)$ and $C \subseteq f(f^{-1}(C)) \subseteq f^{-1}(V)$ or $x \in f^{-1}(U)$, $C \subseteq f^{-1}(V)$ and clearly $f^{-1}(U) \cap f^{-1}(V) = \phi$. This proves that X is γ-P-regular. This completes the proof. $\square$

Theorem 4.16. *The inverse image of a β-P-regular space Y under (γ, β)-continuous and γ-semi-closed preserving injection is γ-P-regular.*

Proof. Let C be any arbitrary γ-semi-closed subset of X and $x \in X - C$. Since f is γ-semi-closed preserving injection, $f(C)$ is β-semi-closed in Y and $f(x) \notin f(C)$. Since Y is β-P-regular, there exist disjoint β-open sets U and V of Y such that $f(x) \in U$ and $f(C) \subseteq V$. Since f is (γ, β)-continuous, using Theorem 4.10, $f^{-1}(U)$ and $f^{-1}(V)$ are γ-open subsets of X. Now $x \in f(f^{-1}(x)) \subseteq f^{-1}(U)$ and $C \subseteq f(f^{-1}(C)) \subseteq f^{-1}(V)$ or $x \in f^{-1}(U)$, $C \subseteq f^{-1}(V)$ and clearly $f^{-1}(U) \cap f^{-1}(V) = \phi$. This proves that X is γ-P-regular. This completes the proof. $\square$

5. Acknowledgement

The authors are thankful to the referee for his valuable comments and suggestions towards the improvement of the paper.

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MSC2000: 54A05, 54A10, 54D10

Department of Mathematics, Islamia University, Bahawalpur, Pakistan
E-mail address: sabiriub@yahoo.com

Department of Mathematics, King Abdul Aziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: drbashir9@gmail.com

Current address (S. Hussain): Department of Mathematics, Yanbu University, P. O. Box 31387, Yanbu Alsinaiyah, Saudi Arabia