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# Permanence and global asymptotic stability for a generalized nonautonomous Lotka-Volterra competition system

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**Abstract.** We consider nonautonomous $N$-dimensional generalized Lotka-Volterra competition systems. Under certain conditions we show that such systems are weakly permanent or permanent and for two solutions $u$ and $v$ of such systems, the difference $u - v$ tends to zero at the infinity. Our results give generalizations of previous ones.

## 1. Introduction and statements of the main results

In this paper we consider the system of differential equations

$$u_i' = u_i\left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i, u_j)\right], \qquad i = 1, \dots, N,\ N \geq 2, \qquad \text{(GLV)}$$

where the functions $a_i(t)$, $1 \leq i \leq N$, and $b_{ij}(t)$, $1 \leq i, j \leq N$, are assumed to be continuous and bounded on $[c, \infty)$, $c \geq 0$. For a bounded function $g(t)$ on $[c, \infty)$, we put $g_M := \sup_{t \geq c} g(t)$, $g_L := \inf_{t \geq c} g(t)$. We assume that

$$b_{ij}(t) \geq 0, \qquad t \geq c,\ 1 \leq i, j \leq N; \tag{1.1}$$

$$a_{iL} > 0, \qquad b_{iiL} > 0, \qquad 1 \leq i \leq N. \tag{1.2}$$

Furthermore let the functions $f_{ij}(x, y)$, $1 \leq i, j \leq N$, be continuously differentiable on $[0, \infty)^2$, and we impose the following conditions on $f_{ij}$:

$$\begin{cases} f_{ij}(x, y) > 0, \quad (x, y) \in \mathbf{R}_+^2,\ 1 \leq i, j \leq N; \\ D_1 f_{ij}(x, y) \geq 0, \quad (x, y) \in \mathbf{R}_+^2,\ 1 \leq i, j \leq N; \\ D_2 f_{ij}(x, y) > 0, \quad (x, y) \in \mathbf{R}_+^2,\ 1 \leq i, j \leq N; \\ f_{ij}(0, 0) = 0, \quad 1 \leq i, j \leq N; \\ \lim\limits_{x \to \infty} f_{ii}(x, x) = \infty, \quad 1 \leq i \leq N, \end{cases} \tag{1.3}$$

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where $\mathbf{R}_+ = (0, \infty)$ and $D_i$, $i = 1, 2$, denotes the differentiation with respect to the $i$th variable.

REMARK 1. *In* (GLV), *we note that for* $i, j = 1, \dots, N$, *the variables* $x$ *and* $y$ *of* $f_{ij}$ *correspond to* $u_i$ *and* $u_j$, *respectively.*

Throughout the paper we make use of the well-known fact (see Lemma 1, [1], [3]–[7], [9]–[11]) that if $u = (u_1, \dots, u_N)$ is a local solution of (GLV) with $u(t_0) \in \mathbf{R}_+^N$, then $u$ can be extended to the interval $[t_0, \infty)$ and $u(t) \in \mathbf{R}_+^N$ for $t \in [t_0, \infty)$. Therefore in the sequel we may assume that all solutions of (GLV) exist near $\infty$ and are positive there.

System (GLV) is a generalization of the following nonautonomous $N$-dimensional Lotka-Volterra competition system that S. Ahmad and A. C. Lazer [1] considered:

$$u_i' = u_i \left[ a_i(t) - \sum_{j=1}^{N} b_{ij}(t) u_j \right], \qquad i = 1, \dots, N, \ N \geq 2. \tag{LV}$$

To state the work of S. Ahmad and A. C. Lazer [1] we introduce the symbols: For a continuous and bounded function $g$ on $[c, \infty)$, we set

$$A[g, t_1, t_2] = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(s) ds,$$

where $c \leq t_1 < t_2$. We define the *upper average* $M[g]$ and the *lower average* $m[g]$, respectively, by

$$M[g] = \lim_{s \to \infty} \sup\{A[g, t_1, t_2] \mid t_2 - t_1 \geq s\},$$

$$m[g] = \lim_{s \to \infty} \inf\{A[g, t_1, t_2] \mid t_2 - t_1 \geq s\}.$$

For system (LV), S. Ahmad and A. C. Lazer [1] supposed conditions (1.1), (1.2) and the *average conditions* such that

$$m[a_i] > \sum_{j \neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \qquad 1 \leq i \leq N. \tag{A}$$

Under these conditions they have shown the following: *Let* $u = (u_1, \dots, u_N)$ *and* $v = (v_1, \dots, v_N)$ *be solutions of* (LV) *on* $[t_0, \infty)$, $t_0 \geq c$. *Then*

(I) $0 < \liminf_{t \to \infty} u_i(t) \leq \limsup_{t \to \infty} u_i(t) < \infty$ *for* $1 \leq i \leq N$;

(II) $\lim_{t \to \infty} (u_i(t) - v_i(t)) = 0$ *for* $1 \leq i \leq N$.

Remark 2. (LV) *and* (GLV) *are called permanent if there are positive constants* $\beta$, $\lambda$, *independent of the initial conditions, such that*

$$\beta \le \liminf_{t\to\infty} u_i(t) \le \limsup_{t\to\infty} u_i(t) \le \lambda, \qquad 1 \le i \le N$$

*for any solution of* (LV) *and* (GLV) *(see* [7]–[11]*). In this paper, following* [7], *we introduce the definition:* (LV) *and* (GLV) *are called weakly permanent if* (I) *hold for any solution of the equation.*

K. Gopalsamy [4, 5] and C. Alvarez and A. Tineo [6] have shown (I) and (II) for solutions of (LV) under the conditions

$$a_{iL} > \sum_{j\ne i} \frac{b_{ijM} a_{jM}}{b_{jjL}}, \qquad 1 \le i \le N, \tag{GAT}$$

that are stronger conditions than (A).

Our main aim is to show that (I) and (II) are still valid for solutions of (GLV). To state the results we introduce the notation and the symbols: For $i = 1, \dots, N$, we put

$$\tilde{f}_i(x) = f_{ii}(x, x), \qquad x \in \mathbf{R}_+.$$

By assumption (1.3), $\tilde{f}_i$, $i = 1, \dots, N$, have the inverse functions $\tilde{f}_i^{-1} : \mathbf{R}_+ \to \mathbf{R}_+$. For $R > 0$ and $\delta \ge 0$, we define two constants $C^*(\delta, R)$ and $C_*(\delta, R)$, respectively, by

$$C^*(\delta, R) = \max\{D_k f_{ij}(x, y) \mid 1 \le i, j \le N, k = 1, 2, (x, y) \in [\delta, R]^2\},$$

$$C_*(\delta, R) = \min\{D_2 f_{ij}(x, y) \mid 1 \le i, j \le N, (x, y) \in [\delta, R]^2\}.$$

The number $C_*(0, R)$ will be employed only when $D_2 f_{ij}(0, 0) > 0$, $1 \le i, j \le N$.

Let $R > 0$ and $\delta \ge 0$. For system (GLV) we introduce the conditions

$$m[a_i] > \frac{C^*(\delta, 2R)}{C_*(\delta, 2R)} \sum_{j\ne i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \qquad 1 \le i \le N \tag{GA}$$

provided that $C_*(\delta, 2R) > 0$. As seen from below (GA) can be regarded as a generalization of (A).

Remark 3. *If* $f_{ij}(x, y) = y$, $1 \le i, j \le N$, *we have* $C^*(\delta, R) = C_*(\delta, R) = 1$ *for* $R > 0$ *and* $\delta \ge 0$. *Since* $C^*(\delta, R)/C_*(\delta, R) = 1$, *conditions* (GA) *reduce to conditions* (A).

Theorem 1. *Let conditions* (1.1), (1.2) *and* (1.3) *hold, and* $D_2 f_{ij}(0, 0) > 0$, $1 \le i, j \le N$. *Suppose that* (GA) *hold for* $\delta = 0$ *and for some* $R$ *satisfying*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \le i \le N\},$$

*that is,*

$$m[a_i] > \frac{C^*(0,2R)}{C_*(0,2R)} \sum_{j \neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \qquad 1 \leq i \leq N. \tag{1.4}$$

*Then* (I) *and* (II) *hold for solutions* $u = (u_1, \dots, u_N)$ *and* $v = (v_1, \dots, v_N)$ *of* (GLV) *on* $[t_0, \infty)$, $t_0 \geq c$.

When $D_2 f_{ij}(0,0)$ may vanish for some $i, j \in \{1, \dots, N\}$, we give a variant of Theorem 1:

THEOREM 2. *Let conditions* (1.1), (1.2) *and* (1.3) *hold and let* $t_0 \geq c$. *Suppose that* (GA) *hold for some* $\delta > 0$ *and* $R$ *satisfying*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \leq i \leq N\},$$

*that is,*

$$m[a_i] > \frac{C^*(\delta,2R)}{C_*(\delta,2R)} \sum_{j \neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \qquad 1 \leq i \leq N,$$

*and*

$$\liminf_{t \to \infty} \frac{a_i(t) - \sum_{j \neq i} f_{ij}(R,R) b_{ij}(t)}{b_{ii}(t)} > \max_{1 \leq j \leq N} \{f_{jj}(\delta,\delta)\}, \qquad 1 \leq i \leq N \tag{1.5}$$

*hold. Then* (I) *and* (II) *hold for solutions* $u = (u_1, \dots, u_N)$ *and* $v = (v_1, \dots, v_N)$ *of* (GLV) *on* $[t_0, \infty)$, $t_0 \geq c$.

REMARK 4. *Conditions in Theorem* 2 *imply that* (GLV) *is permanent (see Proposition* 4*).*

We give examples of systems (GLV) for which above conditions hold.

EXAMPLE 1.1. We consider the following system for two species

$$u_1' = u_1\left[(\cos t + 7) - (\sin t + 7) \cdot \left(\frac{u_1}{2} + 1\right) u_1 - (\sin t + 1) \cdot \left(\frac{u_1^2}{2} + u_2\right)\right]$$

$$u_2' = u_2\left[(\cos t + 9) - (\sin t + 1) \cdot (u_2 + 1) u_1 - (\sin t + 9) \cdot \left(\frac{u_2}{3} + 1\right) u_2\right],$$

where $f_{11}(x,y) = (y/2+1)y$, $f_{12}(x,y) = (x^2/2 + y)$, $f_{21}(x,y) = (x+1)y$ and $f_{22}(x,y) = (y/3+1)y$.

First we note that $D_2 f_{ij}(0,0) > 0$, $i, j = 1, 2$, hold. Since $a_i(t)$, $i = 1, 2$, are periodic functions, we have $M[a_1] = m[a_1] = 7$, $M[a_2] = m[a_2] = 9$.

Since $(a_1/b_{11})_M \le 8/6$, $(a_2/b_{22})_M \le 10/8$, we have

$$\tilde{f}_1^{-1}\left(\frac{8}{6}\right) = -1 + \sqrt{1 + 2 \cdot \frac{8}{6}} = -1 + \sqrt{\frac{11}{3}} < 1,$$

$$\tilde{f}_2^{-1}\left(\frac{10}{8}\right) = \frac{-1 + \sqrt{1 + (4/3) \cdot (10/8)}}{2/3} = \frac{-3 + \sqrt{24}}{2} < 1.$$

Therefore we can put $R = 1$. Then, since

$$C^*(0, 2R) = 3, \qquad C_*(0, 2R) = 1,$$

we have

$$m[a_1] - \frac{C^*(0, 2R) b_{12M} M[a_2]}{C_*(0, 2R) b_{22L}} = 7 - \frac{3 \cdot 2 \cdot 9}{1 \cdot 8} = \frac{1}{4} > 0,$$

$$m[a_2] - \frac{C^*(0, 2R) b_{21M} M[a_1]}{C_*(0, 2R) b_{11L}} = 9 - \frac{3 \cdot 2 \cdot 7}{1 \cdot 6} = 2 > 0.$$

Hence conditions (GA) hold.

Example 1.2. We consider the following system for two species

$$u_1' = u_1\left[(\cos t + 7) - (\sin t + 7) \cdot \left(\frac{u_1}{2} + 1\right)u_1 - \left\{\frac{1}{10}(\sin t + 1)\right\} \cdot u_1 u_2\right]$$

$$u_2' = u_2\left[(\cos t + 9) - \left\{\frac{1}{6}(\sin t + 1)\right\} \cdot \frac{3u_2 u_1}{u_2 + 1} - (\sin t + 9) \cdot \left(\frac{u_2}{3} + 1\right)u_2\right],$$

where $f_{11}(x, y) = (y/2 + 1)y$, $f_{12}(x, y) = xy$, $f_{21}(x, y) = 3xy/(x + 1)$ and $f_{22}(x, y) = (y/3 + 1)y$.

First we note that $D_2 f_{12}(0, 0) = 0$. From Example 1.1, we put $R = 1$ and $\delta = 9/20$. Then we have

$$\begin{aligned} a_1(t) - f_{12}(R, R) b_{12}(t) &= (\cos t + 7) - 1 \cdot 1 \cdot \frac{1}{10}(\sin t + 1) \ge \cos t + \frac{34}{5} \\ &\ge \frac{29}{5} = \frac{29}{40} \cdot 8 \ge \frac{29}{40} \cdot (\sin t + 7) = \frac{29}{40} \cdot b_{11}(t) \\ a_2(t) - f_{21}(R, R) b_{21}(t) &= (\cos t + 9) - \frac{1}{1 + 1} \cdot 3 \cdot 1 \cdot \frac{1}{6}(\sin t + 1) \ge \cos t + \frac{17}{2} \\ &\ge \frac{15}{2} = \frac{3}{4} \cdot 10 \ge \frac{3}{4} \cdot (\sin t + 9) = \frac{3}{4} \cdot b_{22}(t). \end{aligned}$$

Then, since $f_{22}(\delta,\delta) \le f_{11}(\delta,\delta) = 441/800 < 29/40$, (1.5) hold. Moreover, since

$$C^*(\delta, 2R) = 3, \qquad C_*(\delta, 2R) = 9/20,$$

we have

$$m[a_1] - \frac{C^*(\delta,2R)b_{12M}M[a_2]}{C_*(\delta,2R)b_{22L}} = 7 - \frac{3\cdot(1/5)\cdot 9}{(9/20)\cdot 8} = \frac{11}{2} > 0,$$

$$m[a_2] - \frac{C^*(\delta,2R)b_{21M}M[a_1]}{C_*(\delta,2R)b_{11L}} = 9 - \frac{3\cdot(1/3)\cdot 7}{(9/20)\cdot 6} = \frac{173}{27} > 0.$$

Hence conditions (GA) hold.

The rest of this paper is organized as follows. In Section 2 we give important propositions that are employed in proving Theorems 1 and 2. The proofs of Theorems 1 and 2 are given in Sections 3 and 4, separately.

## 2. Preliminary propositions

To prove Theorems 1 and 2, we prove important propositions that are generalizations of [1, Lemmas 3.1 and 3.2].

PROPOSITION 1. *Let* $u = (u_1, \dots, u_N)$ *and* $v = (v_1, \dots, v_N)$ *be solutions of* (GLV).

(i) *Let* $D_2 f_{ij}(0,0) > 0$, $1 \le i, j \le N$. *Suppose that there exist constants* $A, B > 0$ *and* $T = T_{u,v} \ge t_0$ *such that for* $j = 1, \dots, N$ *and* $t \ge T$,

$$A \le u_j(t), v_j(t) \le B. \tag{2.1}$$

*Suppose moreover that for system* (GLV) *there exist positive constants* $\alpha_1, \dots, \alpha_N$ *such that for* $j = 1, \dots, N$,

$$\liminf_{t\to\infty}\left[\alpha_j b_{jj}(t) - \frac{C^*(0,2B)}{C_*(0,2B)}\sum_{i\ne j}\alpha_i b_{ij}(t)\right] > 0. \tag{2.2}$$

*Then there exist some constants* $\tilde{T} \ge T$, $C = C_{A,B} > 0$ *and* $\gamma = \gamma_{A,B} > 0$ *such that for* $t \ge \tilde{T}$,

$$\sum_{i=1}^N |u_i(t) - v_i(t)| \le \left(\sum_{i=1}^N |u_i(\tilde{T}) - v_i(\tilde{T})|\right) C e^{-\gamma(t-\tilde{T})}. \tag{2.3}$$

(ii) *Suppose that there exist constants* $A, B > 0$ *and* $T = T_{u,v} \ge t_0$ *satisfying* (2.1) *for* $j = 1, \dots, N$ *and* $t \ge T$. *Suppose moreover that for* *system* (GLV) *there exist positive constants* $\alpha_1, \dots, \alpha_N$ *such that for* $j = 1, \dots, N$,

$$\liminf_{t\to\infty}\left[\alpha_j b_{jj}(t) - \frac{C^*(A,2B)}{C_*(A,2B)}\sum_{i\neq j}\alpha_i b_{ij}(t)\right] > 0. \tag{2.4}$$

*Then the same conclusion as in* (i) *holds.*

When (GA) hold, we can reduce condition (2.2) to a simpler one.

Proposition 2. *Suppose that conditions* (GA) *hold for some* $\delta \geq 0$ *and* $R > 0$. *Then there exist some positive constants* $\alpha_1, \dots, \alpha_N$ *such that for* $j = 1, \dots, N$,

$$\alpha_j b_{jjL} - \frac{C^*(\delta,2R)}{C_*(\delta,2R)}\sum_{i\neq j}\alpha_i b_{ijM} > 0. \tag{2.5}$$

Note that (2.5) with $\delta = 0$ implies (2.2) with $B = R$, and (2.5) with $\delta > 0$ implies (2.4) with $A = \delta$ and $B = R$. So (GA) automatically imply (2.2) or (2.4) according as $\delta = 0$ or $\delta > 0$.

Remark 5. *By Proposition* 1, *in order to prove Theorem* 1, *it is sufficient to prove*

$$0 < \liminf_{t\to\infty}\ u_i(t) \leq \limsup_{t\to\infty}\ u_i(t) < R, \qquad 1 \leq i \leq N, \tag{2.6}$$

*where* $u = (u_1, \dots, u_N)$ *is a solution of* (GLV) *defined near* $\infty$ *and* $R$ *is the constant indicated in Theorem* 1. *Similarly, to prove Theorem* 2, *it is sufficient to prove*

$$\delta < \liminf_{t\to\infty}\ u_i(t) \leq \limsup_{t\to\infty}\ u_i(t) < R, \qquad 1 \leq i \leq N. \tag{2.7}$$

*In Sections* 3 *and* 4 *we shall prove* (2.6) *and* (2.7) *(Propositions* 3 *and* 4*).*

**2.1. Proofs of Propositions 1 and 2.** We shall prove (ii) of Proposition 1; (i) of Proposition 1 can be proved similarly.

Proof of (ii) of proposition 1. Firstly, from (2.4), there exist some $\tilde{T} \geq T$ and $\varepsilon > 0$ such that for $j = 1, \dots, N$ and $t \geq \tilde{T}$,

$$\alpha_j b_{jj}(t) - \frac{C^*(A,2B)}{C_*(A,2B)}\sum_{i\neq j}\alpha_i b_{ij}(t) \geq \varepsilon. \tag{2.8}$$

Next let

$$\theta(t) = \sum_{i=1}^{N} \alpha_i \left| \log\left(\frac{u_i(t)}{v_i(t)}\right) \right|, \qquad t \geq t_0.$$

Here we note that $\log(u_i(t)/v_i(t))$, $1 \leq i \leq N$, are Lipschitz continuous on every finite interval $I \subset [t_0, \infty)$ and

$$\left| \left|\log\left(\frac{u_i(t_1)}{v_i(t_1)}\right)\right| - \left|\log\left(\frac{u_i(t_2)}{v_i(t_2)}\right)\right| \right| \leq \left| \log\left(\frac{u_i(t_1)}{v_i(t_1)}\right) - \log\left(\frac{u_i(t_2)}{v_i(t_2)}\right) \right|,$$

where $t_1, t_2 \in I$. Therefore, since $|\log(u_i(t)/v_i(t))|$, $1 \leq i \leq N$, are absolutely continuous on $I$, $\theta(t)$ is differentiable for almost all $t \geq t_0$.

From (GLV), for almost all $t \geq T$,

$$\begin{aligned}
\theta'(t) &= \sum_{i=1}^{N} \alpha_i \left[\frac{u_i'}{u_i} - \frac{v_i'}{v_i}\right] \operatorname{sgn}(u_i - v_i) \\
&= \sum_{i=1}^{N} \alpha_i \left[ -\sum_{j=1}^{N} b_{ij}(t)(f_{ij}(u_i, u_j) - f_{ij}(v_i, v_j)) \right] \operatorname{sgn}(u_i - v_i) \\
&= \sum_{j=1}^{N} \sum_{i=1}^{N} [-\alpha_i b_{ij}(t)(f_{ij}(u_i, u_j) - f_{ij}(v_i, v_j)) \operatorname{sgn}(u_i - v_i)] \\
&= \sum_{j=1}^{N} \Bigg[ -\alpha_j b_{jj}(t)(\tilde{f}_j(u_j) - \tilde{f}_j(v_j)) \operatorname{sgn}(u_j - v_j) \\
&\qquad - \sum_{i \neq j} \alpha_i b_{ij}(t)(f_{ij}(u_i, u_j) - f_{ij}(v_i, v_j)) \operatorname{sgn}(u_i - v_i) \Bigg],
\end{aligned}$$

where $u = u(t)$, $v = v(t)$. Here, by the mean value theorem, we note that there exist $0 < w_{ij} < 1$, $1 \leq i, j \leq N$, such that for $i, j = 1, \ldots, N$,

$$\begin{aligned}
f_{ij}(u_i, u_j) - f_{ij}(v_i, v_j) &= (u_i - v_i) D_1 f_{ij}(w_{ij}u_i + z_{ij}v_i, w_{ij}u_j + z_{ij}v_j) \\
&\quad + (u_j - v_j) D_2 f_{ij}(w_{ij}u_i + z_{ij}v_i, w_{ij}u_j + z_{ij}v_j), \qquad (2.9)
\end{aligned}$$

where $z_{ij} = 1 - w_{ij}$, $1 \leq i, j \leq N$. Thus we have

$$\begin{aligned}
\theta'(t) = \sum_{j=1}^{N} \Bigg[ &-\alpha_j b_{jj}(t)(D_1 f_{jj}(w_{jj}u_j + z_{jj}v_j, w_{jj}u_j + z_{jj}v_j) \\
&+ D_2 f_{jj}(w_{jj}u_j + z_{jj}v_j, w_{jj}u_j + z_{jj}v_j))(u_j - v_j) \operatorname{sgn}(u_j - v_j)
\end{aligned}$$

$$- \sum_{i \neq j} \alpha_i b_{ij}(t) D_1 f_{ij}(w_{ij} u_i + z_{ij} v_i, w_{ij} u_j + z_{ij} v_j)(u_i - v_i) \operatorname{sgn}(u_i - v_i)$$

$$\left. - \sum_{i \neq j} \alpha_i b_{ij}(t) D_2 f_{ij}(w_{ij} u_i + z_{ij} v_i, w_{ij} u_j + z_{ij} v_j)(u_j - v_j) \operatorname{sgn}(u_i - v_i) \right]$$

$$\leq \sum_{j=1}^{N} \left[ -C_*(A, 2B) \alpha_j b_{jj}(t) |u_j - v_j| + \sum_{i \neq j} C^*(A, 2B) \alpha_i b_{ij}(t) |u_j - v_j| \right].$$

By (2.8), we have

$$\theta'(t) \leq -\varepsilon C_*(A, 2B) \sum_{j=1}^{N} |u_j - v_j| \qquad \text{a.e. } t \geq \tilde{T}.$$

By (2.1) and the mean value theorem, we note that for $i = 1, \dots, N$ and $t \geq T$,

$$\frac{1}{B} |u_i(t) - v_i(t)| \leq \left| \log\left( \frac{u_i(t)}{v_i(t)} \right) \right| \leq \frac{1}{A} |u_i(t) - v_i(t)|. \tag{2.10}$$

Therefore, for almost all $t \geq \tilde{T}$, we have

$$\theta'(t) \leq -\frac{\varepsilon A C_*(A, 2B)}{\alpha^*} \sum_{i=1}^{N} \alpha_i \left| \log\left( \frac{u_i(t)}{v_i(t)} \right) \right| = -\frac{\varepsilon A C_*(A, 2B)}{\alpha^*} \theta(t),$$

where $\alpha^* = \max_{1 \leq i \leq N} \{\alpha_i\}$. Thus, since $\theta(t)$ is absolutely continuous on $[t_0, \infty)$, for $t \geq \tilde{T}$,

$$\theta(t) \leq \theta(\tilde{T}) \exp\left( -\frac{\varepsilon A C_*(A, 2B)}{\alpha^*} (t - \tilde{T}) \right). \tag{2.11}$$

Let us put $\alpha_* = \min_{1 \leq i \leq N} \{\alpha_i\}$. By (2.1), we have

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq \frac{B}{\alpha_*} \sum_{i=1}^{N} \alpha_i \left| \log\left( \frac{u_i(t)}{v_i(t)} \right) \right| = \frac{B}{\alpha_*} \theta(t) \tag{2.12}$$

for $t \geq T$ and

$$\theta(\tilde{T}) \leq \frac{\alpha^*}{A} \sum_{i=1}^{N} |u_i(\tilde{T}) - v_i(\tilde{T})|. \tag{2.13}$$

Therefore it follows from (2.11)–(2.13) that

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq \frac{\alpha^* B}{\alpha_* A} \sum_{i=1}^{N} |u_i(\tilde{T}) - v_i(\tilde{T})| \exp\left( -\frac{\varepsilon A C_*(A, 2B)}{\alpha^*} (t - \tilde{T}) \right).$$

Hence, putting $C = \alpha^* B/\alpha_* A > 0$ and $\gamma = \varepsilon A C_*(A, 2B)/\alpha_* > 0$, we can obtain (2.3). $\square$

Next we shall prove Proposition 2.

Proof of proposition 2. First let $C = [c_{ij}]$ be an $N \times N$ matrix defined by the following:

$$c_{ij} = \frac{C^*(\delta, 2R) b_{ijM}}{C_*(\delta, 2R) b_{jjL}} \quad (i \neq j), \qquad c_{ii} = 0.$$

Furthermore we put $w = (M[a_1], \dots, M[a_N])^{\mathrm{T}}$, where $(M[a_1], \dots, M[a_N])^{\mathrm{T}}$ denotes the transpose of $(M[a_1], \dots, M[a_N])$. Here, from conditions (GA), we note that for $i = 1, \dots, N$,

$$\sum_{j \neq i} \frac{C^*(\delta, 2R) b_{ijM}}{C_*(\delta, 2R) b_{jjL}} M[a_j] < m[a_i] \leq M[a_i].$$

Therefore we have

$$Cw < w, \tag{2.14}$$

which means that for $i = 1, \dots, N$, the $i$th entry of the vector $Cw$ is strictly less than the $i$th entry of $w$. By (2.14), there exists an $N \times N$ matrix $P = [p_{ij}]$ such that $c_{ij} < p_{ij}$, $1 \leq i, j \leq N$, and

$$Pw < w. \tag{2.15}$$

Since $P$ is a strictly positive matrix, it follows from the Perron-Frobenius theorem [2] that there exists a $v = (\alpha_1, \dots, \alpha_N)^{\mathrm{T}}$ and $\lambda > 0$ such that $\alpha_j > 0$, $1 \leq j \leq N$, and

$$P^{\mathrm{T}} v = \lambda v. \tag{2.16}$$

We can rewrite (2.16) as

$$\lambda v^{\mathrm{T}} = v^{\mathrm{T}} P. \tag{2.17}$$

From (2.15) and (2.17), we have

$$\lambda v^{\mathrm{T}} w = v^{\mathrm{T}} P w < v^{\mathrm{T}} w.$$

Therefore $\lambda < 1$. By (2.16), we have

$$C^{\mathrm{T}} v < P^{\mathrm{T}} v = \lambda v < v. \tag{2.18}$$

Hence the $j$th of (2.18) implies

$$\sum_{i \neq j} \frac{C^*(\delta, 2R) b_{ijM}}{C_*(\delta, 2R) b_{jjL}} \alpha_i < \alpha_j;$$

that is (2.5). $\square$

## 3. Proof of Theorem 1

From Remark 5, in order to prove Theorem 1, it is sufficient to prove the following proposition.

Proposition 3. *Let conditions* (1.1), (1.2) *and* (1.3) *hold and let* $t_0 \geq c$. *Let* $R > 0$ *be a number such that*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \leq i \leq N\}.$$

*Suppose that* $D_2 f_{ij}(0,0) > 0$, $1 \leq i, j \leq N$ *and conditions* (1.4) *hold. Then for any solution* $u = (u_1, \dots, u_N)$ *of* (GLV) *on* $[t_0, \infty)$,

$$0 < \liminf_{t\to\infty} u_i(t) \leq \limsup_{t\to\infty} u_i(t) \leq R, \qquad 1 \leq i \leq N. \tag{3.1}$$

Before proving Proposition 3, we give several lemmas that are employed in the proof of Proposition 3.

Lemma 1. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Let* $t_0 \geq c$ *and let* $u = (u_1, \dots, u_N)$ *be a solution of* (GLV) *on* $[t_0, \infty)$. *Then the following statements* (i) *and* (ii) *hold:*

(i) *For* $i = 1, \dots, N$ *and* $t \geq t_0$,

$$u_i(t) \leq \max\{u_i(t_0), \tilde{f}_i^{-1}((a_i/b_{ii})_M)\}.$$

(ii) *Let* $R > 0$ *be a number such that*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \leq i \leq N\}.$$

*Then*

$$\limsup_{t\to\infty} u_i(t) \leq R, \qquad 1 \leq i \leq N. \tag{3.2}$$

Proof. (i) First we prove the following claim:

Claim. *If there exist some* $T$ *and* $i \in \{1, \dots, N\}$ *such that* $\tilde{f}_i(u_i(T)) \geq (a_i/b_{ii})_M$, *then* $u_i'(T) \leq 0$.

In fact from the assumption, we have

$$\begin{aligned} u_i'(T) &= u_i(T)\left[a_i(T) - \sum_{j\neq i} b_{ij}(T) f_{ij}(u_i(T), u_j(T)) - b_{ii}(T)\tilde{f}_i(u_i(T))\right] \\ &\leq u_i(T)[a_i(T) - b_{ii}(T)\tilde{f}_i(u_i(T))] \leq 0. \end{aligned}$$

By the above claim, we can prove (i).

(ii) From (i), it suffices to prove the existence of some $T \geq t_0$ satisfying $u_i(T) \leq R$, $1 \leq i \leq N$. The proof is divided into three cases.

Case 1. The case where $u_i(t) \le R$ near $\infty$.
Case 2. The case where $u_i(t) \ge R$ near $\infty$.
Case 3. Cases 1 and 2 do not hold.
In Case 1, there is nothing to prove.
In Case 2, by the proof of (i), we have

$$u_i'(t) \le u_i(t)[a_i(t) - b_{ii}(t)\tilde{f}_i((u_i(t)))] \le [(a_i/b_{ii})_M - \tilde{f}_i(R)]u_i(t_0)b_{ii}(t)$$

for $t \ge t_0$. Therefore we obtain

$$u_i(t) \le [(a_i/b_{ii})_M - \tilde{f}_i(R)]u_i(t_0)\int_{t_0}^{t} b_{ii}(s)ds + u_i(t_0).$$

Hence, by $[(a_i/b_{ii})_M - \tilde{f}_i(R)] < 0$ and (1.2), it follows that $u_i(t) \to -\infty$ as $t \to \infty$, but this is a contradiction.

In Case 3, there exists some $\tilde{t} \ge t_0$ such that $u_i(\tilde{t}) = R$. By (i), we have $u_i(t) \le u_i(\tilde{t}) = R$ for $t \ge \tilde{t}$, but this is a contradiction. □

Henceforth let $R$ be a number such that

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \,|\, 1 \le i \le N\}.$$

Then, from Lemma 1, there exists some $T = T_u \ge t_0$ such that for $i = 1, \dots, N$,

$$u_i(t) \le R, \qquad t \ge T, \tag{3.3}$$

where $t_0 \ge c$ and $u$ is a solution of (GLV) on $[t_0, \infty)$. Furthermore, by (1.3) and (2.9), it follows that for $i, j = 1, \dots, N$,

$$f_{ij}(u_i(t), u_j(t)) \le C^*(0, R)(u_i(t) + u_j(t)), \qquad t \ge T. \tag{3.4}$$

Then the following lemma holds.

LEMMA 2. *Let conditions* (1.1), (1.2) *and* (1.3) *hold and let* $t_0 \ge c$. *Let* $u = (u_1, \dots, u_N)$ *be a solution of* (GLV) *on* $[t_0, \infty)$. *Then there exists some* $\alpha > 0$ *such that*

$$\sum_{i=1}^{N} u_i(t) \ge \alpha, \qquad t \ge T.$$

PROOF. We set $a_* = \min_{1 \le i \le N}\{a_{iL}\}$ and $b^* = \max_{1 \le i, j \le N}\{b_{ijM}\}$. Moreover let

$$V(t) = \sum_{i=1}^{N} u_i(t), \qquad t \ge T.$$

Then, by (GLV) and (3.4), the following claim holds:

Claim. *If there exists some* $\tilde{t} \geq T$ *such that* $0 < V(\tilde{t}) \leq a_*/(N+1)b^*C^*(0,R)$, *then* $V'(\tilde{t}) \geq 0$.

In fact, from the assumption of the above claim, we have

$$\begin{aligned} V'(\tilde{t}) &= \sum_{i=1}^{N} u_i(\tilde{t}) \left[ a_i(\tilde{t}) - \sum_{j=1}^{N} b_{ij}(\tilde{t}) f_{ij}(u_i(\tilde{t}), u_j(\tilde{t})) \right] \\ &\geq \sum_{i=1}^{N} u_i(\tilde{t}) \left[ a_* - b^* C^*(0,R) \left( N u_i(\tilde{t}) + \sum_{j=1}^{N} u_j(\tilde{t}) \right) \right] \\ &\geq V(\tilde{t})[a_* - (N+1) b^* C^*(0,R) V(\tilde{t})] \geq 0. \end{aligned}$$

By the above claim, we can obtain $V(t) \geq \min\{V(T), a_*/(N+1)b^*C^*(0,R)\} =: \alpha$ for $t \geq T$. □

The proofs of Lemmas 3–5 are based on [1].

Lemma 3 (see [1, Lemma 2.3]). *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Assume that* (3.1) *do not hold. Then there exist some* $t_0 \geq c$, *some solution* $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_N)$ *of* (GLV) *on* $[t_0, \infty)$ *and a maximal nonempty subset* $J$ *of* $\{1, \dots, N\}$ *such that* $J \neq \{1, \dots, N\}$ *and*

$$\inf_{t \geq t_0} \max\{\tilde{u}_j(t) \mid j \in J\} = 0. \tag{3.5}$$

Henceforth let $\tilde{u}$ and $J$ denote the solution and the subset of $\{1, \dots, N\}$ given in Lemma 3.

Lemma 4 (see [1, Lemma 2.4]). *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Assume that* (3.1) *do not hold. Let* $\xi > 0$ *be a number such that*

$$\xi = \min\{\tilde{u}_j(T) \mid j \in J\}.$$

*Then there exist some sequences* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *such that for* $n \geq 1$,

$$T \leq s_n < t_n; \tag{3.6}$$

$$t_n - s_n > n; \tag{3.7}$$

$$\max\{\tilde{u}_j(t) \mid j \in J, s_n \leq t \leq t_n\} = \xi/n; \tag{3.8}$$

*and there exists* $j_n \in J$ *such that*

$$\tilde{u}_{j_n}(s_n) = \xi/n. \tag{3.9}$$

PROOF. First let

$$p_i(t) = a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i, u_j), \qquad t \geq t_0,\ 1 \leq i \leq N.$$

By (1.2), (3.3) and (3.4), there exists some number $r > 0$ such that for $i = 1, \dots, N$,

$$p_i(t) \geq -r, \qquad t \geq T. \tag{3.10}$$

By the definition of the set $J$ and (3.10), we can prove Lemma 4 as in the proof of [1, Lemma 2.4]. □

Henceforth let $j_n \in J$, $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ denote the sequences given in Lemma 4.

LEMMA 5 (see [1, Lemma 2.5]). *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Assume that* (3.1) *do not hold. Let $K$ be the complement of $J$*: $K = \{1, 2, \dots, N\} - J$. *Then there exists some number $\eta > 0$ such that for $n \geq 1$ and $k \in K$,*

$$\tilde{u}_k(s_n) \geq \eta, \qquad \tilde{u}_k(t_n) \geq \eta. \tag{3.11}$$

Henceforth let $\eta > 0$ denotes the number given in Lemma 5.

Since $J$ is a finite set, there exist some integer $j_* \in J$ and some increasing sequence $\{n_q\}_{q=1}^{\infty}$ such that

$$j_{n_q} = j_*, \qquad q \geq 1. \tag{3.12}$$

For simplicity, we put $c_q = s_{n_q}$ and $d_q = t_{n_q}$ for $q \geq 1$. By (3.7) and (3.8), we have

$$d_q - c_q \geq n_q, \qquad q \geq 1; \tag{3.13}$$

$$\lim_{q \to \infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_j(t)dt = 0, \qquad j \in J. \tag{3.14}$$

Since $\tilde{u}_{j_*}(c_q) \geq \tilde{u}_{j_*}(d_q)$ for $q \geq 1$ from (3.8) and (3.9), we have

$$\log\left(\frac{\tilde{u}_{j_*}(d_q)}{\tilde{u}_{j_*}(c_q)}\right) \leq 0, \qquad q \geq 1. \tag{3.15}$$

Then the following lemma holds.

LEMMA 6. *Let conditions* (1.1), (1.2) *and* (1.3) *hold. Assume that* (3.1) *do not hold and $D_2 f_{ij}(0,0) > 0$, $1 \leq i, j \leq N$. Then for each $k \in K$ there exists some $x_k \geq 0$ such that*

$$M[a_k] \geq \frac{C_*(0,R)}{C^*(0,R)} \sum_{l \in K} b_{klL} x_l, \tag{3.16}$$

$$m[a_{j_*}] \leq \sum_{l \in K} b_{j_* lM} x_l. \tag{3.17}$$

Proof. Firstly, by (3.3), we note that for $k \in K$ and $q \geq 1$,

$$0 \leq \frac{1}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_i(t) dt \leq R.$$

Therefore, by considering subsequences of $\{c_q\}_{q=1}^{\infty}$ and $\{d_q\}_{q=1}^{\infty}$ if necessary, we can assume that for $k \in K$,

$$\lim_{q \to \infty} \frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_k(t) dt =: x_k \geq 0 \tag{3.18}$$

exists.

Similarly, since $a_{kL} \leq a_k(t) \leq a_{kM}$ for $t \geq c$, by considering subsequences of $\{c_q\}_{q=1}^{\infty}$ and $\{d_q\}_{q=1}^{\infty}$ if necessary, we can assume that for $k \in K$,

$$\lim_{q \to \infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_k(t) dt$$

exists. Furthermore, since (3.13) implies $d_q - c_q \to \infty$ as $q \to \infty$, it follows that for $k \in K$,

$$\lim_{q \to \infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_k(t) dt \leq M[a_k] \tag{3.19}$$

Since, from (3.3) and Lemma 5, $\eta \leq \tilde{u}_k(c_q)$, $\tilde{u}_k(d_q) \leq R$ for $k \in K$ and $q \geq 1$, we have

$$\lim_{q \to \infty} \frac{1}{d_q - c_q} \log\left(\frac{\tilde{u}_k(d_q)}{\tilde{u}_k(c_q)}\right) = 0, \qquad k \in K. \tag{3.20}$$

Since, by (1.3) and (2.9),

$$f_{ij}(\tilde{u}_i(t), \tilde{u}_j(t)) \geq C_*(0,R) \tilde{u}_j(t), \qquad 1 \leq i, j \leq N,\ t \geq T,$$

it follows from (GLV) that for $k \in K$ and $t \geq T$,

$$a_k(t) = \frac{\tilde{u}_k'(t)}{\tilde{u}_k(t)} + \sum_{j=1}^{N} b_{kj}(t) f_{kj}(\tilde{u}_k(t), \tilde{u}_j(t)) \geq \frac{\tilde{u}_k'(t)}{\tilde{u}_k(t)} + \sum_{l \in K} C_*(0,R) b_{klL} \tilde{u}_l(t).$$

Therefore, from (3.18)–(3.20), we have

$$
\begin{aligned}
M[a_k] &\geq \lim_{q\to\infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_k(t)dt \\
&\geq \lim_{q\to\infty} \frac{1}{d_q - c_q} \log\left(\frac{\tilde{u}_k(d_q)}{\tilde{u}_k(c_q)}\right) \\
&\quad + \lim_{q\to\infty} \sum_{l\in K} \frac{C_*(0,R)b_{klL}}{C^*(0,R)} \left(\frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_l(t)dt\right) \\
&= \frac{C_*(0,R)}{C^*(0,R)} \sum_{l\in K} b_{klL} x_l
\end{aligned}
$$

for $k \in K$; that is (3.16).

Next, similarly as in the proof of (3.16) by considering subsequences of $\{c_q\}_{q=1}^{\infty}$ and $\{d_q\}_{q=1}^{\infty}$ if necessary, we have

$$
m[a_{j_*}] \leq \lim_{q\to\infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_{j_*}(t)dt. \tag{3.21}
$$

Moreover it follows from (GLV) and (3.4) that for $t \geq T$,

$$
\begin{aligned}
a_{j_*}(t) &= \frac{\tilde{u}_{j_*}'(t)}{\tilde{u}_{j_*}(t)} + \sum_{l=1}^{N} b_{j_*l}(t) f_{j_*l}(\tilde{u}_{j_*}(t), \tilde{u}_l(t)) \\
&\leq \frac{\tilde{u}_{j_*}'(t)}{\tilde{u}_{j_*}(t)} + \sum_{l=1}^{N} b_{j_*lM} C^*(0,R)(\tilde{u}_{j_*}(t) + \tilde{u}_l(t)).
\end{aligned}
$$

Therefore, by (3.15), we have

$$
\begin{aligned}
\frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_{j_*}(t)dt &\leq \sum_{l=1}^{N} b_{j_*lM} \left(\frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_{j_*}(t) + \tilde{u}_l(t)dt\right) \\
&\leq N b_{j_*lM} \frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_{j_*}(t)dt + \sum_{l\in J} b_{j_*lM} \frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_l(t)dt \\
&\quad + \sum_{l\in K} b_{j_*lM} \frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_l(t)dt.
\end{aligned}
$$

Hence, by $j_* \in J$, (3.14) and (3.21), we have

$$
\begin{aligned}
m[a_{j_*}] &\le \lim_{q\to\infty} \frac{1}{d_q - c_q} \int_{c_q}^{d_q} a_{j_*}(t)\,dt \\
&\le \lim_{q\to\infty} \sum_{l\in K} b_{j_* lM} \frac{C^*(0,R)}{d_q - c_q} \int_{c_q}^{d_q} \tilde{u}_l(t)\,dt = \sum_{l\in K} b_{j_* lM} x_l;
\end{aligned}
$$

that is (3.17). □

By employing Lemma 6, we prove Proposition 3.

Proof of proposition 3. We assume to the contrary that conditions (1.4) hold and (3.1) do not hold. Then by the above lemmas, there exist a proper subset $K$ of $\{1,\dots,N\}$, an integer $j_* \in \{1,\dots,N\}\backslash K$, and some numbers $x_k \ge 0$, $k\in K$, satisfying (3.16) and (3.17).

Since, by (3.16),

$$
\frac{C_*(0,R) b_{kkL} x_k}{C^*(0,R)} \le \sum_{l\in K} \frac{C_*(0,R) b_{klL} x_l}{C^*(0,R)} \le M[a_k]
$$

for $k\in K$, we have

$$
x_k \le \frac{C^*(0,R) M[a_k]}{C_*(0,R) b_{kkL}} \le \frac{C^*(0,2R) M[a_k]}{C_*(0,2R) b_{kkL}}, \qquad k\in K.
$$

Therefore it follows from (3.17) that

$$
m[a_{j_*}] \le \sum_{k\in K} b_{j_* kM} x_k \le \frac{C^*(0,2R)}{C_*(0,2R)} \sum_{k\in K} \frac{b_{j_* kM} M[a_k]}{b_{kkL}};
$$

but this is a contradiction to an inequality derived from (1.4):

$$
m[a_{j_*}] > \frac{C^*(0,2R)}{C_*(0,2R)} \sum_{k\neq j_*} \frac{b_{j_* kM} M[a_k]}{b_{kkL}} \ge \frac{C^*(0,2R)}{C_*(0,2R)} \sum_{k\in K} \frac{b_{j_* kM} M[a_k]}{b_{kkL}}.
$$

□

By Propositions 1–3, we can prove Theorem 1.

Remark 6. *Proposition* 3 *can be proved even if conditions* (1.4) *are slightly weakened to the following:*

$$
m[a_i] > \frac{C^*(0,R)}{C_*(0,R)} \sum_{j\neq i} \frac{b_{ijM} M[a_j]}{b_{jjL}}, \qquad 1 \le i \le N.
$$

## 4. Proof of Theorem 2

From Remark 5, in order to prove Theorem 2, it is sufficient to prove the following proposition.

PROPOSITION 4 (see [3, Lemmas 3.1 and 4.1]). *Let conditions* (1.1), (1.2) *and* (1.3) *hold and let* $t_0 \geq c$. *Let* $R > 0$ *be a number such that*

$$R > \max\{\tilde{f}_i^{-1}((a_i/b_{ii})_M) \mid 1 \leq i \leq N\}.$$

*Suppose that for system* (GLV) *there exists some number* $\delta > 0$ *such that* (1.5) *hold. Then for any solution* $u = (u_1, \dots, u_N)$ *of* (GLV) *on* $[t_0, \infty)$,

$$\delta \leq \liminf_{t\to\infty} u_i(t) \leq \limsup_{t\to\infty} u_i(t) \leq R, \qquad 1 \leq i \leq N. \tag{4.1}$$

PROOF. Step 1. Firstly, similarly as in Section 3, there exists $T = T_u \geq t_0$ such that (3.3) hold. Furthermore, from (1.5), there exist some numbers $\delta' > \max_{1\leq j\leq N}\{\tilde{f}_j(\delta)\}$ and $\tilde{T} \geq T$ such that for $i = 1, \dots, N$ and $t \geq \tilde{T}$,

$$a_i(t) - \sum_{j\neq i} f_{ij}(R,R) b_{ij}(\tilde{t}) \geq \delta' b_{ii}(t).$$

Then we can claim the following:

Claim 1. *If there exist some* $\tilde{t} \geq \tilde{T}$ *and* $i \in \{1, \dots, N\}$ *such that* $0 < \tilde{f}_i(u_i(\tilde{t})) \leq \tilde{f}_i(\delta)$, *then* $u_i'(\tilde{t}) \geq 0$.

In fact, from the assumption of Claim 1, (3.3) and (1.3), we have

$$\begin{aligned} u_i'(\tilde{t}) &= u_i(\tilde{t})\left[a_i(\tilde{t}) - \sum_{j\neq i} b_{ij}(\tilde{t}) f_{ij}(u_i(\tilde{t}), u_j(\tilde{t})) - b_{ii}(\tilde{t})\tilde{f}_i(u_i(\tilde{t}))\right] \\ &\geq u_i(\tilde{t})\left[a_i(\tilde{t}) - \sum_{j\neq i} b_{ij}(\tilde{t}) f_{ij}(R,R) - b_{ii}(\tilde{t})\tilde{f}_i(\delta)\right] \\ &\geq u_i(\tilde{t})(\delta' b_{ii}(\tilde{t}) - b_{ii}(\tilde{t})\tilde{f}_i(\delta)) \geq 0. \end{aligned}$$

Step 2. By Claim 1, we can claim the following:

Claim 2. For $i = 1, \dots, N$,

$$\tilde{f}_i(u_i(t)) \geq \min\{\tilde{f}_i(u_i(\tilde{T})), \tilde{f}_i(\delta)\}, \qquad t \geq \tilde{T}.$$

From Claim 2, in order to prove Lemma 4, it is sufficient to prove the following: For $i = 1, \dots, N$, there exists some $\hat{t}_i \geq \tilde{T}$ such that

$$\tilde{f}_i(u_i(\hat{t}_i)) \geq \tilde{f}_i(\delta).$$

We assume to the contrary that there exists some number $i \in \{1, \dots, N\}$ such that $\tilde{f}_i(u_i(t)) < \tilde{f}_i(\delta) < \delta'$ for $t \geq \tilde{T}$. Then it follows from Step 1 that

$$u_i'(t) \geq (\delta' - \tilde{f}_i(\delta)) u_i(t) b_{ii}(t) \geq 0, \qquad t \geq \tilde{T}.$$

Therefore, since

$$u_i'(t) \geq (\delta' - \tilde{f}_i(\delta))u_i(\tilde{T})b_{ii}(t),$$

we have

$$u_i(t) \geq (\delta' - \tilde{f}_i(\delta))u_i(\tilde{T}) \int_{\tilde{T}}^{t} b_{ii}(s)ds + u_i(\tilde{T}), \qquad t \geq \tilde{T}.$$

Thus, by noting that $\delta' - \tilde{f}_i(\delta) > 0$, it follows from (1.2) that $u_i(t) \to \infty$ as $t \to \infty$ but this is a contradiction. Hence we can prove Lemma 4 by setting $\hat{t} = \max_{1 \leq i \leq N}\{\hat{t}_i\}$. □

By Propositions 1–4, we can prove Theorem 2.

Remark 7. *Proposition* 4 *can be proved even if conditions* (1.3) *are slightly weakened to the following:*

$$\begin{cases} f_{ij}(x, y) > 0, \quad (x, y) \in \mathbf{R}_+^2, \ 1 \leq i, j \leq N; \\ D_1 f_{ij}(x, y) \geq 0, \quad (x, y) \in \mathbf{R}_+^2, \ 1 \leq i, j \leq N; \\ D_2 f_{ij}(x, y) \geq 0, \quad (x, y) \in \mathbf{R}_+^2, \ 1 \leq i, j \leq N; \\ (D_1 f_{ii} + D_2 f_{ii})(x, x) > 0, \quad x \in \mathbf{R}_+, \ 1 \leq i \leq N; \\ f_{ij}(0, 0) = 0, \quad 1 \leq i, j \leq N; \\ \lim_{x \to \infty} f_{ii}(x, x) = \infty, \quad 1 \leq i \leq N. \end{cases}$$

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