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Convergence rate of multinomial goodness-of-fit statistics to chi-square distribution

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Abstract. Let $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_k)'$ be a random vector with multinomial distribution. In this paper we investigate the convergence rate of so-called power divergence family of statistics $\{I^\lambda(\boldsymbol{Y}), \lambda \in \mathbf{R}\}$ introduced by Cressie and Read (1984) to chi-square distribution. It is proved that for every $k \geq 4$

$$\Pr(2nI^\lambda(\boldsymbol{Y}) < c) = G_{k-1}(c) + O(n^{-1+\mu(k-1)}),$$

where $G_r(c)$ is the distribution function of chi-square random variable with r degrees of freedom, $\mu(r) = 6/(7r+4)$ for $3 \leq r \leq 7$, $\mu(r) = 5/(6r+2)$ for $r \geq 8$. This refines Zubov and Ulyanov's result (2008). The proof uses Krätzel-Nowak's theorem (1991) on the number of integer points in a convex body with smooth boundary.

1. Introduction and main result

Let $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_k)'$ be a random vector with the multinomial distribution $M_k(n, \boldsymbol{\pi})$, i.e.,

$$\Pr(Y_1 = n_1, Y_2 = n_2, \dots, Y_k = n_k) = \begin{cases} n! \prod_{j=1}^k \frac{\pi_j^{n_j}}{n_j!} & \sum_{j=1}^k n_j = n \\ 0 & \text{otherwise,} \end{cases}$$

where $n_j = 0, 1, \dots, n$, $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_k)'$, $\pi_j > 0$, $\sum_{j=1}^k \pi_j = 1$. For testing the simple hypothesis $\mathrm{H} : \boldsymbol{\pi} = \boldsymbol{p}$ ($\boldsymbol{p}$ is a fixed vector) against $\mathrm{K} : \boldsymbol{\pi} \neq \boldsymbol{p}$ the power divergence statistics (introduced by Cressie and Read in [2]) can be used:

$$2nI^\lambda = \frac{2}{\lambda(\lambda+1)} \sum_{j=1}^k Y_j \left[\left(\frac{Y_j}{np_j} \right)^\lambda - 1 \right], \qquad \lambda \in \mathbf{R},$$

where $\boldsymbol{p} = (p_1, p_2, \dots, p_k)'$, $p_j > 0$ $(j = 1, 2, \dots, k)$ and $\sum_{j=1}^k p_j = 1$.

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REMARK 1. *When $\lambda = 0$ or $\lambda = -1$, this notation should be understood as a result of passing to the limit.*

Throughout this paper we will use the following notation:

$$\boldsymbol{x} = (x_1, \dots, x_r)',$$

$$\boldsymbol{x}^* = (x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_r)',$$

For any $B \subset \mathbf{R}^r$ and for any $l \in \{1, \dots, r\}$ denote

$$B_l = \{\boldsymbol{x}^* : \boldsymbol{x} \in B\}.$$

DEFINITION 1. A set $B \subset \mathbf{R}^r$ is called an *extended convex set*, if B has the following representation for every $l \in \{1, 2, \dots, r\}$:

$$B = \{\boldsymbol{x} : \lambda_l(\boldsymbol{x}^*) < x_l < \theta_l(\boldsymbol{x}^*), \boldsymbol{x}^* \in B_l\},$$

where λ_l, θ_l are continuous functions on B_l.

It is known (see Cressie, Read [2]), that under the null hypothesis $2nI^{\lambda}$ has the chi-square distribution with $r = k - 1$ degrees of freedom in the limit. Moreover the distribution function of $2nI^{\lambda}$ has the following expansion:

$$\Pr(2nI^{\lambda} < c) = \Pr(\chi_r^2 < c) + J_2 + O(n^{-1}), \tag{1}$$

where

$$J_2 = -\frac{1}{\sqrt{n}} \sum_{l=1}^{r} n^{-(r-l)/2} \sum_{x_{l+1} \in L_{l+1}} \cdots \sum_{x_r \in L_r}$$
$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{B_l^{\lambda}}(\boldsymbol{x}^*)[S_1(\sqrt{n}x_l + p_l n)\phi(\boldsymbol{x})]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} dx_1 \dots dx_{l-1}, \tag{2}$$

$$L_j = \left\{x_j : x_j = \frac{1}{\sqrt{n}}(n_j - np_j), n_j \in \mathbf{Z}\right\}, \tag{3}$$

$$S_1(x) = x - [x] - \frac{1}{2},$$

$$[h(\boldsymbol{x})]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} = h(x_1, \dots, x_{l-1}, \theta_l(\boldsymbol{x}^*), x_{l+1}, \dots, x_r) - h(x_1, \dots, x_{l-1}, \lambda_l(\boldsymbol{x}^*), x_{l+1}, \dots, x_r),$$

$$\phi(\boldsymbol{x}) = (2\pi)^{-r/2} |\Omega|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}'\Omega^{-1}\boldsymbol{x}\right),$$

$$\Omega = \operatorname{diag}(p_1, \dots, p_r) - (p_1, \dots, p_r)'(p_1, \dots, p_r).$$

Here $\chi_A(x)$ is an indicator function, $\theta_l(\boldsymbol{x}^*)$ and $\lambda_l(\boldsymbol{x}^*)$ are continuous functions from Definition 1 for the set

$$B^\lambda = \{\boldsymbol{x} : 2nI^\lambda(\boldsymbol{x}) < c\} \tag{4}$$

with

$$2nI^\lambda(\boldsymbol{x}) = \frac{2}{\lambda(\lambda+1)} \sum_{j=1}^{k} (np_j + \sqrt{n}x_j) \left[\left(1 + \frac{x_j}{\sqrt{np_j}} \right)^\lambda - 1 \right], \tag{5}$$

$$x_k = -(x_1 + \cdots + x_r).$$

In Lemma 2 and Lemma 7 we shall prove that B^λ is a convex set with smooth boundary. Hence B^λ is an extended convex set (see Definition 1) and it follows from Yarnold's result [6] that

$$J_2 = O(n^{-1/2}).$$

This was improved by Zubov and Ulyanov in [7]. They showed that

$$J_2 = O(n^{-1+1/(r+1)}).$$

Our main result is the following

THEOREM 1. *If $2nI^\lambda$ is the power divergence statistic defined above, then*

$$\Pr(2nI^\lambda < c) = \Pr(\chi_r^2 < c) + J_2 + O(n^{-1})$$

and

$$J_2 = O(n^{-1+\mu(r)}), \tag{6}$$

where

$$\mu(r) \stackrel{\text{def}}{=} \begin{cases} \frac{6}{7r+4} & \textit{for } 3 \le r \le 7 \\ \frac{5}{6r+2} & \textit{for } r \ge 8. \end{cases} \tag{7}$$

2. Properties of B^λ

2.1. Convexity.

LEMMA 1. *Let $2nI^\lambda(\boldsymbol{x})$ be the function defined by* (5)*; then $2nI^\lambda(\boldsymbol{x})$ is a strictly convex function on the domain*

$$Q = \{\boldsymbol{x} : x_j > -\sqrt{n}p_j, x_1 + \cdots + x_r < \sqrt{n}p_k\}. \tag{8}$$

PROOF. The set Q is convex since it is an open r-dimensional pyramid. Calculating second partial derivatives of $2nI^\lambda(\boldsymbol{x})$ we obtain

$$\frac{\partial^2(2nI^\lambda)}{\partial x_i^2} = \frac{2}{p_i}\left(1+\frac{x_i}{\sqrt{n}p_i}\right)^{\lambda-1} + \frac{2}{p_k}\left(1-\frac{x_1+\cdots+x_r}{\sqrt{n}p_k}\right)^{\lambda-1}, \qquad i=\overline{1,r}, \quad (9)$$

$$\frac{\partial^2(2nI^\lambda)}{\partial x_i \partial x_j} = \frac{2}{p_k}\left(1-\frac{x_1+\cdots+x_r}{\sqrt{n}p_k}\right)^{\lambda-1}, \qquad i \neq j. \quad (10)$$

All these derivatives are continuous on Q. Hence, $2nI^\lambda(\boldsymbol{x})$ is twice differentiable on Q. The lemma will be proved if we show that $d^2(2nI^\lambda)$ is a positive definite quadratic form on Q. For this, by Sylvester criterion, it is sufficient to prove that the principle minors Δ_l, $l=\overline{1,r}$ of the matrix $A = \left(\frac{\partial^2(2nI^\lambda)}{\partial x_i \partial x_j}\right)$ are positive on Q. The proof is by induction on l and is left to the reader. □

LEMMA 2. *Let B^λ be the set considered in Introduction; then B^λ is a strictly convex set.*

PROOF. Consider any $\boldsymbol{x_1} \in \partial B^\lambda$, $\boldsymbol{x_2} \in \partial B^\lambda$, $t \in (0,1)$. This means that $2nI^\lambda(\boldsymbol{x_1}) = c$, $2nI^\lambda(\boldsymbol{x_2}) = c$. From Lemma 1 it follows that $2nI^\lambda(\boldsymbol{x})$ is a strictly convex function on Q. Therefore

$$\begin{aligned} 2nI^\lambda(\boldsymbol{x_1} + t(\boldsymbol{x_2} - \boldsymbol{x_1})) &< 2nI^\lambda(\boldsymbol{x_1}) + t(2nI^\lambda(\boldsymbol{x_2}) - 2nI^\lambda(\boldsymbol{x_1})) \\ &= (1-t)2nI^\lambda(\boldsymbol{x_1}) + t2nI^\lambda(\boldsymbol{x_2}) = (1-t)c + tc = c. \end{aligned}$$

This implies that $\boldsymbol{x_1} + t(\boldsymbol{x_2} - \boldsymbol{x_1}) \in B^\lambda$. Consequently B^λ is a strictly convex set. □

2.2. Boundedness.

LEMMA 3. *Let $f_n(x) \to f(x)$ pointwise and the set $A = \{x : f(x) \le c\}$ be bounded. Let the functions $f_n(x)$ be continuous, convex, strictly decreasing for $x \in [\alpha_n, 0)$, $f_n(0) = 0$, strictly increasing for $x \in (0, \beta_n]$ and*

$$f_n(\alpha_n) \to +\infty, \quad f_n(\beta_n) \to +\infty, \qquad n \to +\infty.$$

Then the sets $A_n = \{x : f_n(x) \le c\}$ are uniformly bounded.

PROOF. Assume the converse. Let N be a number such that

$$\min(f_N(\alpha_N), f_N(\beta_N)) > c.$$

Then for any natural $n > N$ the set A_n is a finite segment $[a_n, b_n]$ containing 0, where $a_n < 0$, $b_n > 0$ are solutions of the equation $f_n(x) = c$. By our assumption at least one of the sequences $\{a_n\}$ and $\{b_n\}$ is unbounded. Let it be $\{b_n\}$ (for the case of $\{a_n\}$ the argument is similar). Further we select from $\{b_n\}$ an infinitely large subsequence $\{b_{n_k}\}$. Then

$$[0, b_{n_1}] \subset [0, b_{n_2}] \subset \cdots \subset [0, b_{n_k}] \subset \cdots$$

and, therefore, for any $l > k$

$$f_{n_l}(b_{n_k}) \le c.$$

Passing to the limit at $l \to +\infty$ we obtain

$$f(b_{n_k}) \le c \Rightarrow b_{n_k} \in A,$$

but this contradicts with boundedness of A. $\square$

LEMMA 4. *For any $l \in \{1, \dots, r\}$ the set B_l^λ can be represented as*

$$B_l^\lambda = \{\boldsymbol{y} \in \mathbf{R}^{r-1} : F(\boldsymbol{y}) < c\},$$

where $\boldsymbol{y} = \boldsymbol{x}^*$ and

$$F(\boldsymbol{y}) = \frac{2}{\lambda(\lambda+1)} \sum_{j=1}^{r} (nq_j + \sqrt{n} y_j) \left[\left(1 + \frac{y_j}{\sqrt{n} q_j} \right)^{\lambda} - 1 \right],$$

$$\boldsymbol{q} = (p_1, \dots, p_{l-1}, p_{l+1}, \dots, p_r)', \qquad q_r = p_l + p_k,$$

$$y_r = -(y_1 + \cdots + y_{r-1}).$$

PROOF. The proof is found in [7]. $\square$

It can easily be seen that $2nI^1(\boldsymbol{x})$ is a quadratic form, which doesn't depend on n (see (5)). Therefore, $B^1 = \{2nI^1(\boldsymbol{x}) < c\}$ is an ellipsoid, which doesn't depend on n. However the set $B^\lambda = \{2nI^\lambda(\boldsymbol{x}) < c\}$ depends on n (see (4)).

LEMMA 5. *The set B^λ, $\lambda \ne 1$, is uniformly bounded w.r.t. n.*

PROOF. The proof is by induction on dimension r:

(1) For $r = 1$ the lemma follows from Lemma 3.

(2) Let it be true for the dimension $(r-1)$. From Lemma 4 it follows that projection B_r^λ of the set B^λ on subspace

$$x_r = 0$$

is an $(r-1)$-dimensional set, that has the same form as B^λ. By induction assumption B_r^λ is uniformly bounded w.r.t. n. Thus, for all $\boldsymbol{x} \in B^\lambda$

$$|x_i| \le C_1, \qquad i = \overline{1, r-1}$$

By the same argument we see that B_1^λ is uniformly bounded w.r.t. n. Hence for all $\boldsymbol{x} \in B^\lambda$

$$|x_r| \le C_2. \qquad \square$$

2.3. Smoothness. In the rest of this paper we will use such concepts from differential geometry as *manifold*, *smooth manifold*, *manifold of class* C^∞, *surface*, etc. Definitions of these concepts can be found e.g., in [5].

LEMMA 6. *Let* $f : \mathbf{R}^n \to \mathbf{R}^1$ *be a function of class* C^∞, $M_c = \{\boldsymbol{x} : f(\boldsymbol{x}) = c\}$. *If the gradient of* f *is nonzero throughout* M_c, *then* M_c *is a smooth* $(n-1)$-*dimensional manifold of class* C^∞.

PROOF. The proof is found in [5], Ch. 3, §2, Theorem 1. □

REMARK 2. *The lemma is true if* f *is defined on a set* $Q \in \mathbf{R}^n$ *such that* $Q \supset M_c$.

LEMMA 7. *The surface*

$$\partial B^\lambda = \{\boldsymbol{x} : 2nI^\lambda(\boldsymbol{x}) = c\} \tag{11}$$

is an $(r-1)$-*dimensional surface of class* C^∞.

PROOF. The domain of the function $2nI^\lambda(\boldsymbol{x})$ is the set Q defined by (8). Q increases infinitely with the growth of n. Since the set B^λ is bounded (see Lemma 5) there exists the number N such that Q contains the surface ∂B^λ for all $n \geq N$. The function $2nI^\lambda(\boldsymbol{x})$ is indefinitely differentiable as a superposition of indefinitely differentiable functions. By direct computations we see that

$$\frac{\partial(2nI^\lambda)}{\partial x_j} = \frac{2\sqrt{n}}{\lambda}\left(1 + \frac{x_j}{\sqrt{n}p_j}\right)^\lambda - \frac{2\sqrt{n}}{\lambda}\left(1 - \frac{x_1 + \cdots + x_r}{\sqrt{n}p_k}\right)^\lambda, \qquad j = \overline{1,r}. \tag{12}$$

Let us show that the gradient of $2nI^\lambda(\boldsymbol{x})$ is nonzero throughout ∂B^λ. Assume the converse. Then there exists $\boldsymbol{x}^{\mathbf{0}} \in \partial B^\lambda$ such that

$$\operatorname{grad}[2nI^\lambda(\boldsymbol{x}^0)] = 0 \Rightarrow \frac{\partial(2nI^\lambda)}{\partial x_j}(\boldsymbol{x}^{\mathbf{0}}) = 0, \qquad j = \overline{1,r}$$

$$\Leftrightarrow \frac{x_j^0}{\sqrt{n}p_j} = -\frac{x_1^0 + \cdots + x_r^0}{\sqrt{n}p_k}, \qquad j = \overline{1,r}.$$

We can write these last r equations in the matrix form:

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt{n}p_1} + \frac{1}{\sqrt{n}p_k} & \frac{1}{\sqrt{n}p_k} & \cdots & \frac{1}{\sqrt{n}p_k} \\ \frac{1}{\sqrt{n}p_k} & \frac{1}{\sqrt{n}p_2} + \frac{1}{\sqrt{n}p_k} & \cdots & \frac{1}{\sqrt{n}p_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}p_k} & \frac{1}{\sqrt{n}p_k} & \cdots & \frac{1}{\sqrt{n}p_r} + \frac{1}{\sqrt{n}p_k} \end{pmatrix}}_{C} \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_r^0 \end{pmatrix} = 0.$$

The matrix C is nondegenerate (the proof is left to the reader). Hence $\boldsymbol{x}^{\mathbf{0}} = (0, \dots, 0)'$, and $2nI^{\lambda}(\boldsymbol{x}^{\mathbf{0}}) = 2nI^{\lambda}(0, \dots, 0) = 0 < c$. But this contradicts with the choice of $\boldsymbol{x}^{\mathbf{0}} \in \partial B^{\lambda}$. This contradiction proves that

$$\mathrm{grad}[2nI^{\lambda}(\boldsymbol{x})] \neq 0$$

throughout the surface ∂B^{λ}. Applying Lemma 6 to the mapping $2nI^{\lambda}$ proves the lemma. $\square$

2.4. Parameterization. Consider the function

$$U_n(\rho, \boldsymbol{t}) = 2nI^{\lambda}(\boldsymbol{x}(\rho, \boldsymbol{t})) - c, \tag{13}$$

where the mapping $\boldsymbol{x}(\rho, \boldsymbol{t}) = (x_1(\rho, \boldsymbol{t}), \dots, x_r(\rho, \boldsymbol{t}))'$ is defined by the equalities

$$\begin{cases} x_1(\rho, \boldsymbol{t}) = \rho \sin t_1 \dots \sin t_{r-1}, \\ x_j(\rho, \boldsymbol{t}) = \rho \cos t_{j-1} \prod_{l=j}^{r-1} \sin t_l \quad \text{for } j = \overline{2, r-1}, \\ x_r(\rho, \boldsymbol{t}) = \rho \cos t_{r-1}, \end{cases}$$

on the set

$$S = \Bigg\{ (\rho, \boldsymbol{t}) : \rho \in [0, +\infty), t_1 \in [0, 2\pi], t_l \in [0, \pi], l = \overline{2, r-1},$$
$$x_j(\rho, \boldsymbol{t}) > -\sqrt{n} p_j, j = \overline{1, r}, \sum_{j=1}^{r} x_j(\rho, \boldsymbol{t}) < \sqrt{n} p_k \Bigg\}. \tag{14}$$

Remark 3. *Notice that the function U defined by* (13) *depends on n for $\lambda \neq 1$ (denote it by U_n) and doesn't depend on n for $\lambda = 1$ (denote it by U).*

Let $\boldsymbol{y}(\boldsymbol{t}) = (y_1(\boldsymbol{t}), \dots, y_r(\boldsymbol{t}))'$ be a mapping defined by the equalities

$$\begin{cases} y_1(\boldsymbol{t}) = \sin t_1 \dots \sin t_{r-1}, \\ y_j(\boldsymbol{t}) = \cos t_{j-1} \prod_{l=j}^{r-1} \sin t_l \quad \text{for } j = \overline{2, r-1}, \\ y_r(\boldsymbol{t}) = \cos t_{r-1}. \end{cases} \tag{15}$$

Then $\boldsymbol{x}(\boldsymbol{t}) = \rho \boldsymbol{y}(\boldsymbol{t})$ and

$$\|\boldsymbol{y}(\boldsymbol{t})\| = 1. \tag{16}$$

Lemma 8. *For sufficiently large n we have*

$$\frac{\partial U_n}{\partial \rho} > 0$$

on $S \backslash \{\rho = 0\}$.

Proof. Calculating partial derivative of $U_n(\rho, \boldsymbol{t})$ w.r.t. ρ we have

$$
\begin{aligned}
\frac{\partial U_n}{\partial \rho} &= \sum_{j=1}^{k} \frac{\partial (2nI^{\lambda})}{\partial x_j} \frac{\partial x_j}{\partial \rho} \\
&= 2\sqrt{n} \sum_{j=1}^{r} \frac{y_j(\boldsymbol{t})}{\lambda} \left(1 + \frac{\rho y_j(\boldsymbol{t})}{\sqrt{n} p_j}\right)^{\lambda} \\
&\quad - 2\sqrt{n} \frac{\sum_{j=1}^{r} y_j(\boldsymbol{t})}{\lambda} \left(1 - \frac{\rho \sum_{j=1}^{r} y_j(\boldsymbol{t})}{\sqrt{n} p_k}\right)^{\lambda}. \qquad (17)
\end{aligned}
$$

Consider the function

$$
f(x) = \frac{x}{\lambda}(1+x)^{\lambda} - \frac{x}{\lambda}, \qquad x \in \mathbf{R}.
$$

It can easily be seen that in sufficiently small neighborhood of $x = 0$

$$
f(x) \geq 0, \qquad f(x) = 0 \Leftrightarrow x = 0, \tag{18}
$$

Using $f(x)$ we can rewrite (17) in the following way

$$
\frac{1}{2\sqrt{n}} \frac{\partial U_n}{\partial \rho} = \sum_{j=1}^{r} \frac{1}{s_j(\rho)} f(s_j(\rho) y_j(\boldsymbol{t})) + \frac{1}{s_k(\rho)} f\left(-s_k(\rho) \sum_{j=1}^{r} y_j(\boldsymbol{t})\right) \geq 0, \tag{19}
$$

where $s_j(\rho) = \frac{\rho}{\sqrt{n p_j}} > 0$. Assume that $\frac{\partial U_n}{\partial \rho}(\rho^0, \boldsymbol{t}^0) = 0$. Then by non-negativity of every term in (19) we obtain

$$
\begin{cases} f(s_j(\rho^0) y_j(\boldsymbol{t}^0)) = 0, \ j = \overline{1, r} \\ f(-s_k(\rho^0) \sum_{j=1}^{r} y_j(\boldsymbol{t}^0)) = 0. \end{cases} \Rightarrow \{\text{from } (18)\} \Rightarrow y_j(\boldsymbol{t}^0) = 0, \ j = \overline{1, r}
$$

$$
\Rightarrow \boldsymbol{y}(\boldsymbol{t}^0) = \mathbf{0}.
$$

But this contradicts (16). Therefore, $\frac{\partial U_n}{\partial \rho} > 0$ for all $(\rho, \boldsymbol{t}) \in S$ and $\rho \neq 0$. □

Lemma 8 and implicit function theorem give us the following

Lemma 9. *Let $U_n(\rho^0, \boldsymbol{t}^0) = 0$, where U_n is defined by* (13). *Then for any sufficiently small $\varepsilon > 0$ there exists a neighborhood $V(\boldsymbol{t}^0)$ of $\boldsymbol{t}^0$ and a unique function $\rho_n(\boldsymbol{t})$ such that $|\rho_n(\boldsymbol{t}) - \rho^0| < \varepsilon$ and*

$$
U_n(\rho_n(\boldsymbol{t}), \boldsymbol{t}) = 0.
$$

for all $\boldsymbol{t} \in V(\boldsymbol{t}^0)$ and the function $\rho_n(\boldsymbol{t})$ is continuous and infinitely differentiable on $V(\boldsymbol{t}^0)$.

Put by definition

$$T = [0, 2\pi] \times \underbrace{[0, \pi] \times [0, \pi] \times \cdots \times [0, \pi]}_{(r-2)\ \text{times}}.$$

Lemma 10. *The surface ∂B^λ has infinitely differentiable parameterization*

$$\boldsymbol{x}^n(\boldsymbol{t}) = \rho_n(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t}), \qquad \boldsymbol{t} \in \mathbf{R}^{r-1},$$

where $\boldsymbol{y}(\boldsymbol{t})$ is defined by (15).

Proof. From Lemma 2 it follows that the set $B^\lambda = \{\boldsymbol{x} : 2nI^\lambda(\boldsymbol{x}) < c\}$ is convex. Therefore its boundary $\partial B^\lambda = \{\boldsymbol{x} : 2nI^\lambda(\boldsymbol{x}) = c\}$ is a convex surface containing the origin since $2nI^\lambda(0, \dots, 0) = 0 < c$. Hence for all $\boldsymbol{t^0} \in T$ the ray that starts at the origin in the direction of the vector $\boldsymbol{y}(\boldsymbol{t}^0)$ intersects ∂B^λ at the unique point $\boldsymbol{x}^0$. Using the transformation

$$\boldsymbol{x} = \rho \boldsymbol{y}(\boldsymbol{t})$$

we turn to spherical coordinate system. Then the point $\boldsymbol{x}^0$ turns to the point $(\rho^0, \boldsymbol{t}^0)$, where $\rho^0 = \|\boldsymbol{x}^0\|$. By construction $\boldsymbol{x}^0$ lies on the surface ∂B^λ, thus

$$U_n(\rho^0, \boldsymbol{t}^0) = 2nI^\lambda(\boldsymbol{x}^0) - c = 0.$$

Therefore from Lemma 9 it follows that there exists a neighborhood $V(\boldsymbol{t}^0)$ of $\boldsymbol{t}^0$ and a unique function $\rho_n(\boldsymbol{t})$ such that $U_n(\rho_n(\boldsymbol{t}), \boldsymbol{t}) = 0$ and $\rho_n(\boldsymbol{t})$ is continuous and infinitely differentiable on $V(\boldsymbol{t}^0)$. Put

$$\boldsymbol{x}^n(\boldsymbol{t}) = \rho_n(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t}).$$

Then

$$2nI^\lambda(\boldsymbol{x}^n(\boldsymbol{t})) = U_n(\rho_n(\boldsymbol{t}), \boldsymbol{t}) + c = c, \qquad \boldsymbol{t} \in V(\boldsymbol{t}^0),$$

and the functions $x_j^n(\boldsymbol{t})$, $j = \overline{1, r}$ are continuous and infinitely differentiable on $V(\boldsymbol{t}^0)$.

Since $\boldsymbol{t}^0$ is taken arbitrarily, the lemma is proved. □

Throughout this paper we denote uniform convergence of $f_n(x)$ to $f(x)$ as $n \to \infty$ by

$$f_n(x) \rightrightarrows f(x), \qquad n \to \infty$$

Lemma 11. *$2nI^\lambda(\boldsymbol{x}) \rightrightarrows 2nI^1(\boldsymbol{x})$, $n \to \infty$, on any bounded set. Moreover $\forall m \in \mathbf{N}$ and $\forall e_1, \dots, e_r \in \mathbf{N}_0 : e_1 + \cdots + e_r = m$*

$$\frac{\partial^m (2nI^\lambda)}{\partial x_1^{e_1} \dots \partial x_r^{e_r}}(\boldsymbol{x}) \rightrightarrows \frac{\partial^m (2nI^1)}{\partial x_1^{e_1} \dots \partial x_r^{e_r}}(\boldsymbol{x}), \qquad n \to \infty,$$

on any bounded set.

Proof. The uniform convergence $2nI^{\lambda}(\boldsymbol{x}) \rightrightarrows 2nI^{1}(\boldsymbol{x})$ follows from Taylor expansion of $2nI^{\lambda}(\boldsymbol{x})$ w.r.t. n. From (12) it follows that

$$\frac{\partial(2nI^{\lambda})}{\partial x_i} \rightrightarrows \frac{2}{p_i}x_i + \frac{2}{p_k}(x_1 + \cdots + x_r) = \frac{\partial(2nI^{1})}{\partial x_i}, \qquad i = \overline{1,r}.$$

From (9) and (10) it follows that

$$\frac{\partial^2(2nI^{\lambda})}{\partial x_i^2} \rightrightarrows \frac{2}{p_i} + \frac{2}{p_k} = \frac{\partial^2(2nI^{1})}{\partial x_i^2}, \qquad i = \overline{1,r},$$

$$\frac{\partial^2(2nI^{\lambda})}{\partial x_i x_j} \rightrightarrows \frac{2}{p_k} = \frac{\partial^2(2nI^{1})}{\partial x_i x_j}, \qquad i \neq j,$$

and $\forall m \geq 3,\ \forall e_1, \dots, e_r \in \mathbf{N}_0 : e_1 + \cdots + e_r = m$

$$\frac{\partial^m(2nI^{\lambda})}{\partial x_1^{e_1} \dots \partial x_r^{e_r}}(\boldsymbol{x}) \rightrightarrows 0 = \frac{\partial^m(2nI^{1})}{\partial x_1^{e_1} \dots \partial x_r^{e_r}}(\boldsymbol{x}). \qquad \square$$

Let $\rho_n(\boldsymbol{t})$ be the function constructed in Lemma 10 for the surface ∂B^{λ}. Then from Lemma 5 it follows that $|\rho_n(\boldsymbol{t})| \leq R$, where $R = \sup_n\{\max_{\boldsymbol{t} \in T} \rho_n(\boldsymbol{t})\} < \infty$.

Lemma 12. *$\forall m \in \mathbf{N}$ and $\forall e_1, \dots, e_r \in \mathbf{N}_0 : e_1 + \cdots + e_r = m$*

$$\frac{\partial^m U_n}{\partial \rho^{e_r} \partial t_1^{e_1} \partial t_2^{e_2} \dots \partial t_{r-1}^{e_{r-1}}} \overset{[0,R]\times T}{\rightrightarrows} \frac{\partial^m U}{\partial \rho^{e_r} \partial t_1^{e_1} \partial t_2^{e_2} \dots \partial t_{r-1}^{e_{r-1}}}.$$

Proof. By induction on m and direct calculation we can show that

$$\frac{\partial^m U_n}{\partial \rho^{e_r} \partial t_1^{e_1} \partial t_2^{e_2} \dots \partial t_{r-1}^{e_{r-1}}} = \sum_{l=1}^{m} \sum_{\substack{e_1+\cdots+e_r=l \\ e_j \in \mathbf{N}_0}} \frac{\partial^l(2nI^{\lambda})}{\partial x_1^{e_1} \dots x_r^{e_r}} P_{l,\boldsymbol{e}}(\rho, \boldsymbol{t}),$$

where $P_{l,\boldsymbol{e}}(\rho, \boldsymbol{t})$ doesn't depend on n. Now the lemma follows from Lemma 11. $\square$

Lemma 13. *Let $\rho_n(\boldsymbol{t})$ and $\rho(\boldsymbol{t})$ be the functions constructed in Lemma* 10 *for the surfaces ∂B^{λ} $(\lambda \neq 1)$ and ∂B^{1} respectively. Then*

$$|\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})| \leq Cn^{-1/2}.$$

Moreover, $\forall m \in \mathbf{N}$ and $\forall e_1, \dots, e_{r-1} \in \mathbf{N}_0 : e_1 + \cdots + e_{r-1} = m$

$$\frac{\partial^m \rho_n}{\partial t_1^{e_1} \dots \partial t_{r-1}^{e_{r-1}}}(\boldsymbol{t}) \overset{T}{\rightrightarrows} \frac{\partial^m \rho}{\partial t_1^{e_1} \dots \partial t_{r-1}^{e_{r-1}}}(\boldsymbol{t}).$$

PROOF. First let us show that $\rho_n(\boldsymbol{t}) \overset{T}{\rightrightarrows} \rho(\boldsymbol{t})$. Assume the converse. Then there exists a number $\varepsilon > 0$ and a sequence $\{\boldsymbol{t}^n\}$ such that

$$|\rho_n(\boldsymbol{t}^n) - \rho(\boldsymbol{t}^n)| \geq \varepsilon. \tag{20}$$

By compactness of T we select a converging subsequence $\{\boldsymbol{t}^{n_k}\} : \boldsymbol{t}^{n_k} \to \boldsymbol{t}^0$. Thus by continuity of $\rho(\boldsymbol{t})$ we have

$$\rho(\boldsymbol{t}^{n_k}) \to \rho(\boldsymbol{t}^0).$$

Now by compactness of $[0, R]$ we select a converging subsequence $\{\rho_{n_{k_l}}(\boldsymbol{t}^{n_{k_l}})\}$: $\rho_{n_{k_l}}(\boldsymbol{t}^{n_{k_l}}) \to \rho_1$, and from (20) we have

$$\rho_1 \neq \rho(\boldsymbol{t}^0). \tag{21}$$

Further

$$\begin{aligned} U_{n_{k_l}}(\rho_{n_{k_l}}(\boldsymbol{t}_{n_{k_l}}), \boldsymbol{t}_{n_{k_l}}) = 0 &\Rightarrow \{U_n \rightrightarrows U\} \\ &\Rightarrow U(\rho_{n_{k_l}}(\boldsymbol{t}_{n_{k_l}}), \boldsymbol{t}_{n_{k_l}}) + \alpha_l = 0,\ \alpha_l = o(1) \\ &\Rightarrow \{l \to \infty\} \Rightarrow U(\rho^1, \boldsymbol{t}^0) = 0. \end{aligned}$$

Combining this with (21) we get contradiction with the identity

$$U(\rho(\boldsymbol{t}), \boldsymbol{t}) = 0, \qquad \forall \boldsymbol{t} \in T,$$

which follows from the proof of Lemma 10. Hence

$$\rho_n(\boldsymbol{t}) \overset{T}{\rightrightarrows} \rho(\boldsymbol{t}).$$

Now from Taylor expansion of $U_n(\rho, \boldsymbol{t})$ and boundedness of B^λ we get

$$|U_n(\rho, \boldsymbol{t}) - U(\rho, \boldsymbol{t})| \leq C_1 n^{-1/2}.$$

From Lemma 8 and its proof it follows that $\frac{\partial U}{\partial \rho}(\rho, \boldsymbol{t}) > 0$ for $\rho > 0$ and $\frac{\partial U}{\partial \rho}(0, \boldsymbol{t}) = 0$. By direct computation we obtain $\frac{\partial^2 U}{\partial \rho^2}(\rho, \boldsymbol{t}) > 0$ throughout domain. Thus $\frac{\partial U}{\partial \rho}$ is strictly increasing function for $\rho \geq 0$ and vanishing at $\rho = 0$. The set B^1 is an ellipsoid, thus its radius $\rho(\boldsymbol{t}) \geq K > 0$ and we have

$$\begin{aligned} C_2|\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})| &\leq \left|\frac{\partial U}{\partial \rho}(\xi_n(\boldsymbol{t}), \boldsymbol{t})\right| |\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})| = |U(\rho_n(\boldsymbol{t}), \boldsymbol{t}) - U(\rho(\boldsymbol{t}), \boldsymbol{t})| \\ &= |U(\rho_n(\boldsymbol{t}), \boldsymbol{t}) - U_n(\rho_n(\boldsymbol{t}), \boldsymbol{t})| \leq C_1 n^{-1/2}, \end{aligned}$$

where $\xi_n(\boldsymbol{t}) = \rho(\boldsymbol{t}) + \underbrace{\theta_n(\boldsymbol{t})}_{\in (0,1)}[\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})] \overset{T}{\rightrightarrows} \rho(\boldsymbol{t}) \geq K > 0.$

The second part of the lemma follows from Lemma 12 and the formulae for derivatives of implicit function. □

2.5. Support function properties. Remember that $2nI^1(\boldsymbol{x})$ is a quadratic form, which doesn't depend on n (see (5)). Therefore, $B^1 = \{2nI^1(\boldsymbol{x}) < c\}$ is an ellipsoid, which doesn't depend on n. Let $H_n(\boldsymbol{u})$ be a support function of B^λ for any $\lambda \neq 1$ and $H(\boldsymbol{u})$ be a support function of B^1. Throughout we assume that $\boldsymbol{u}$ takes values from bounded set.

LEMMA 14. $\forall m \in \mathbf{N},\ \forall e_1 \in \mathbf{N}_0, \dots, e_r \in \mathbf{N}_0 : e_1 + \cdots + e_r = m$

$$\frac{\partial^m H_n}{\partial u_1^{e_1} \dots \partial u_r^{e_r}}(\boldsymbol{u}) \rightrightarrows \frac{\partial^m H}{\partial u_1^{e_1} \dots \partial u_r^{e_r}}(\boldsymbol{u}), \qquad n \to \infty.$$

PROOF. Let $\boldsymbol{x}_n(\boldsymbol{t})$ and $\boldsymbol{x}(\boldsymbol{t})$ be parameterizations from Lemma 10 for the sets B^λ ($\lambda \neq 1$) and B^1 respectively. In order to simplify the computation we will prove the statement for $r = 2$. In this case

$$\begin{aligned} H_n(u,v) &= \sup_{(x_1,x_2) \in \partial B^\lambda} \{ux_1 + vx_2\} = \max_{t \in [0,2\pi]} \{u\rho_n(t) \cos t + v\rho_n(t) \sin t\} \\ &= u\rho_n(t_n^*(u,v)) \cos t_n^*(u,v) + v\rho_n(t_n^*(u,v)) \sin t_n^*(u,v), \end{aligned}$$

where $t_n^*(u,v)$ is a maximum point of smooth and periodic function

$$f_n(t,u,v) = u\rho_n(t) \cos t + v\rho_n(t) \sin t$$

w.r.t. t. From Lemma 13 it follows that

$$\frac{\partial f_n}{\partial u} \rightrightarrows \frac{\partial f}{\partial u},$$

where

$$f(t,u,v) = u\rho(t) \cos t + v\rho(t) \sin t.$$

Let $t^*(u,v)$ be the maximum point of $f(t,u,v)$ w.r.t. $t \in [0,2\pi]$. In a similar fashion as we proved the convergence of radii in Lemma 13 we can show that $t_n^*(u,v) \rightrightarrows t^*(u,v)$. Thus we have

$$\begin{aligned} \frac{\partial H_n}{\partial u} &= \frac{\partial}{\partial u}(f_n(t_n^*(u,v),u,v)) \\ &= \underbrace{\frac{\partial f_n}{\partial t}(t_n^*(u,v),u,v)}_{=0} \frac{\partial t_n^*}{u}(u,v) + \frac{\partial f_n}{\partial u}(t_n^*(u,v),u,v) \\ &\rightrightarrows \frac{\partial f}{\partial u}(t^*(u,v),u,v) = \frac{\partial H}{\partial u}. \end{aligned}$$

Arguing as above we see that

$$\frac{\partial H_n}{\partial v} \rightrightarrows \frac{\partial H}{\partial v}.$$

Uniform convergence of the derivatives of higher order is proved in the similar way. □

2.6. Finiteness and non-vanishing of Gaussian curvature.

LEMMA 15. *Gaussian curvature is finite and nonzero throughout ∂B^λ for sufficiently large n.*

PROOF. If $\boldsymbol{u}$ is any point of unit sphere $\{\boldsymbol{x} : \|\boldsymbol{x}\| = 1\}$, and $\boldsymbol{x} = \mathscr{M}(\boldsymbol{u})$ its image under the canonical map, then the Gaussian curvature κ of ∂B^λ $(\lambda \neq 1)$ in $\boldsymbol{x}$ is equal to $(\lambda_1, \ldots, \lambda_{r-1})^{-1}$ where $\{0, \lambda_1, \ldots, \lambda_{r-1}\}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 H_n}{\partial u_i \partial u_j}(\boldsymbol{u})\right)$ (see [1], p. 61f). Notice that Gaussian curvature of the ellipsoid ∂B^1 is finite and nonzero. Now the lemma follows from Viéte's formulas and from Lemma 14. □

3. Preliminary transformation of J_2

By definition put

$$L = \left\{ \boldsymbol{x} : x_j = \frac{1}{\sqrt{n}}(m_j - np_j),\ m_j \in \mathbf{Z},\ j = \overline{1, r} \right\},$$

This means that L is an r-dimensional lattice in $\mathbf{R}^r$ and lattice spacing of L is $\frac{1}{\sqrt{n}}$. Let N^λ be the number of lattice points in B^λ, i.e., $N^\lambda = \#(L \cap B^\lambda)$. Let V^λ be the volume of B^λ.

PROPOSITION 1. *Let J_2 be the term defined by* (2)*; then*

$$J_2 = dn^{-r/2}(N^\lambda - n^{r/2} V^\lambda) + O(n^{-1}), \tag{22}$$

where $d = const > 0$.

PROOF. Let $\boldsymbol{x}^n(\boldsymbol{t}) = \rho_n(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t})$ be the parameterization of ∂B^λ $(\lambda \neq 1)$ constructed in Lemma 10 and $\boldsymbol{x}(\boldsymbol{t}) = \rho(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t})$ be the parameterization of ∂B^1 from the same Lemma. The set B^λ is a strictly convex set (see, Lemma 2). Thus it is an extended convex set. Let us fix arbitrary $\boldsymbol{x}^* \in B_l^\lambda$. Then from Definition 1 it follows that

$$(x_1, \ldots, x_{l-1}, \theta_l(\boldsymbol{x}^*), x_{l+1}, \ldots, x_r) \in \partial B^\lambda,$$

$$(x_1, \ldots, x_{l-1}, \lambda_l(\boldsymbol{x}^*), x_{l+1}, \ldots, x_r) \in \partial B^\lambda.$$

Since $\boldsymbol{x}^n(\boldsymbol{t})$ is the parameterization of ∂B^λ, there exist values $\boldsymbol{t}_l$, $\boldsymbol{u}_l$ such that

$$(x_1, \ldots, x_{l-1}, \theta_l(\boldsymbol{x}^*), x_{l+1}, \ldots, x_r) = \boldsymbol{x}^n(\boldsymbol{t}_l),$$
$$(x_1, \ldots, x_{l-1}, \lambda_l(\boldsymbol{x}^*), x_{l+1}, \ldots, x_r) = \boldsymbol{x}^n(\boldsymbol{u}_l).$$

Denote

$$W_l(x) = S_1(\sqrt{n}x + p_l n).$$

Then

$$\begin{aligned}
&\chi_{B_l^\lambda}(\boldsymbol{x^*})[S_1(\sqrt{n}x_l + p_l n)\phi(\boldsymbol{x})]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} \\
&\quad = W_l(x_l^n(\boldsymbol{t}_l))\phi(\boldsymbol{x}^n(\boldsymbol{t}_l)) - W_l(x_l^n(\boldsymbol{u}_l))\phi(\boldsymbol{x}^n(\boldsymbol{u}_l)) \\
&\quad = \underbrace{W_l(x_l^n(\boldsymbol{t}_l))[\phi(\boldsymbol{x})]_{\boldsymbol{x}(\boldsymbol{t}_l)}^{\boldsymbol{x}^n(\boldsymbol{t}_l)}}_{A} + \underbrace{W_l(x_l^n(\boldsymbol{u}_l))[\phi(\boldsymbol{x})]_{\boldsymbol{x}^n(\boldsymbol{u}_l)}^{\boldsymbol{x}(\boldsymbol{u}_l)}}_{B} + d[W_l(x)]_{x_l^n(\boldsymbol{u}_l)}^{x_l^n(\boldsymbol{t}_l)}, \qquad (23)
\end{aligned}$$

where $d = \phi(\boldsymbol{x}(\boldsymbol{t}_l)) = \phi(\boldsymbol{x}(\boldsymbol{u}_l))$, since $\boldsymbol{x}(\boldsymbol{t}_l), \boldsymbol{x}(\boldsymbol{u}_l) \in \partial B^1$ and

$$\partial B^1 = \{\boldsymbol{x} : 2nI^1(\boldsymbol{x}) = c\} = \{\boldsymbol{x} : \boldsymbol{x}'\Omega^{-1}\boldsymbol{x} = c\}.$$

Further for all $\boldsymbol{t} \in [0, 2\pi] \times [0, \pi] \times \cdots \times [0, \pi]$

$$\begin{aligned}
|\phi(\boldsymbol{x}^n(\boldsymbol{t})) - \phi(\boldsymbol{x}(\boldsymbol{t}))| &= |\phi(\rho_n(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t})) - \phi(\rho(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{t}))| \\
&= |(\phi(\boldsymbol{y}(\boldsymbol{t})))^{\rho_n^2(\boldsymbol{t})} - (\phi(\boldsymbol{y}(\boldsymbol{t})))^{\rho^2(\boldsymbol{t})}| \\
&= |\phi(\boldsymbol{y}(\boldsymbol{t}))^{\eta_n(\boldsymbol{t})} \log(\phi(\boldsymbol{y}(\boldsymbol{t}))| \, |\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})| \, |\rho_n(\boldsymbol{t}) + \rho(\boldsymbol{t})| \\
&\le Cn^{-1/2} \qquad (24)
\end{aligned}$$

where $\eta_n(\boldsymbol{t}) = \rho_n^2(\boldsymbol{t}) + \underbrace{\lambda_n(\boldsymbol{t})}_{\in (0,1)}[\rho_n^2(\boldsymbol{t}) - \rho^2(\boldsymbol{t})] \overset{T}{\rightrightarrows} \rho^2(\boldsymbol{t})$. The last estimate in (24) follows from boundedness of B^λ (Lemma 5) and the uniform estimate

$$|\rho_n(\boldsymbol{t}) - \rho(\boldsymbol{t})| \le C_1 n^{-1/2},$$

(cf. Lemma 13). Using (24) for estimation of A and B, and taking into account boundedness of $W_l(x)$, we get

$$A = O(n^{-1/2}), \qquad B = O(n^{-1/2}). \qquad (25)$$

From (23) and (25) it follows that

$$\begin{aligned}
&\chi_{B_l^\lambda}(\boldsymbol{x^*})[S_1(\sqrt{n}x_l + p_l n)\phi(\boldsymbol{x})]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} \\
&\quad = \chi_{B_l^\lambda}(\boldsymbol{x^*})d[S_1(\sqrt{n}x_l + p_l n)]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} + O(n^{-1/2}). \qquad (26)
\end{aligned}$$

Inserting (26) into (2), taking into account boundedness of B^λ and the fact that lattice spacing of L_l, $l = \overline{1,r}$ is equal to $n^{-1/2}$, we obtain

$$J_2 = -\frac{d}{\sqrt{n}} \sum_{l=1}^{r} n^{-(r-l)/2} \sum_{x_{l+1} \in L_{l+1}} \cdots \sum_{x_r \in L_r}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{B_l^\lambda}(\boldsymbol{x}^*) [S_1(\sqrt{n} x_l + p_l n)]_{\lambda_l(\boldsymbol{x}^*)}^{\theta_l(\boldsymbol{x}^*)} dx_1 \ldots dx_{l-1} + O(n^{-1}). \tag{27}$$

Further, arguing as in (Yarnold, [6]), we prove the lemma. □

4. Application of Krätzel-Nowak's theorem

Theorem 2 (Krätzel and Nowak). *Let $\mathscr{B}$ denote a compact convex subset of $\mathbf{R}^r$, $r \geq 3$, which contains the origin as an inner point. Suppose that the boundary $\partial\mathscr{B}$ of $\mathscr{B}$ is an $(r-1)$-dimensional surface of class C^∞ with finite nonzero Gaussian curvature throughout. For $n > 0$ define $A(n)$ as the number of points of the lattice $\mathbf{Z}^r$ in the "blown up" domain $\sqrt{n}\mathscr{B}$, i.e. $A(n) = \#(\sqrt{n}\mathscr{B} \cap \mathbf{Z}^r)$. Then there exists a number C such that*

$$|A(n) - \mathrm{vol}(\mathscr{B}) n^{r/2}| \leq C n^{r/2-1+\mu(r)}, \tag{28}$$

where $\mu(r)$ is defined by (7) *and the number C may depend on $\mathscr{B}$.*

Proof. The proof is found in [4], Section 3. □

Proposition 2. *Let N^λ be the number of lattice points in B^λ and V^λ be the volume of B^λ; then*

$$N^\lambda - n^{r/2} V^\lambda = O(n^{r/2-1+\mu(r)}), \tag{29}$$

where $\mu(r)$ is defined by (7) *and the constant implied by O in* (29) *doesn't depend on n.*

Proof. Since $2nI^\lambda(\mathbf{0}) = 0 < c$, from Lemmas 2, 7, 15 it follows that B^λ satisfies the conditions of Krätzel-Nowak's theorem. Notice that B^λ depends on n for all $\lambda \neq 1$. Thus if we apply Krätzel-Nowak's theorem directly to B^λ we obtain

$$\exists C(n) : |N^\lambda - n^{r/2} V^\lambda| \leq C(n) n^{r/2-1+\mu(r)} \tag{30}$$

with $\mu(r)$ defined by (7). Now in order to replace $C(n)$ by an absolute constant in the estimate (30) we need to check the boundedness of all constants implied in the symbols O, $\ll$, $\asymp$ in the proof of Krätzel-Nowak's theorem. Instead of Lemma 1 in [4] it is sufficient to use Satz 5 from [3]. The boundedness of constants in Satz 5, [3] was shown in [7]. We omit trivial estimates and focus on the part of the proof that requires considerable modification. This part (see pp. 67–68 in [4]) is concerned with constructing a finite covering $\{\mathscr{K}_j\}_{j=1}^J$ of

$$\mathscr{K}_0 \stackrel{\text{def}}{=} \{\boldsymbol{u} \in \mathbf{R}^r : 1 < \|\boldsymbol{u}\| \le 2\}$$

such that

$$\inf_{1 \le j \le J} \inf_{\boldsymbol{u}' \in \mathscr{K}_j} |H^{(m)}(\boldsymbol{u}'; \boldsymbol{v}^{(j)})| > 0, \tag{31}$$

where $v^{(j)}$ are certain integer vectors that depend only on B^1 and

$$H^{(m)}(\boldsymbol{a}; \boldsymbol{u}) \stackrel{\text{def}}{=} \frac{d^m}{d\tau^m} H(\boldsymbol{a} + \tau \boldsymbol{u}) \bigg|_{\tau=0}.$$

Using the well-known identity

$$H_n^{(m)}(\boldsymbol{a}; \boldsymbol{v}) = \sum_{\substack{e_1 + \cdots + e_r = m \\ e_j \in \mathbf{N}_0}} \frac{m!}{e_1! \dots e_r!} \frac{\partial^m H_n}{\partial u_1^{e_1} \dots \partial u_r^{e_r}}(\boldsymbol{a}) v_1^{e_1} \dots v_r^{e_r},$$

and Lemma 14 we obtain

$$H_n^{(m)}(\boldsymbol{a}; \boldsymbol{v}) \stackrel{\boldsymbol{a}}{\rightrightarrows} H^{(m)}(\boldsymbol{a}; \boldsymbol{v}), \qquad n \to \infty. \tag{32}$$

Combining (31) and (32) we have for some $N > 0$

$$\inf_{n \ge N} \inf_{1 \le j \le J} \inf_{\boldsymbol{u}' \in \mathscr{K}_j} |H_n^{(k)}(\boldsymbol{u}'; \boldsymbol{v}^{(j)})| > 0. \tag{33}$$

Using the estimate (33) instead of (31) further in the proof of Krätzel-Nowak's theorem we obtain (29). □

5. Proof of the Theorem 1

The statement of the Theorem 1 follows from (1), Proposition 1 and Proposition 2.

Theorem is proved.

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