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Monotonicity of the Modified Likelihood Ratio Test for a Covariance Matrix

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1. Introduction and Summary

In our previous paper [4], we have proved that the modified likelihood ratio test (=modified LR test) for the equality of a covariance matrix Σ to a given one Σ_0 in a p-variate normal distribution is unbiased. The power function of this test depends only on the characteristic roots of $\Sigma\Sigma_0^{-1}$. In this note we prove that this power function is a monotonically increasing (decreasing) function of each of the characteristic roots of $\Sigma\Sigma_0^{-1}$, when it is greater (less) than one, that is, it has the monotonicity property.

2. The monotonicity of the test

Let $p\times 1$ vectors $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$ $(N>p)$ be a random sample from a multivariate normal distribution with unknown mean vector $\boldsymbol{\mu}$ and unknown covariance matrix Σ ($\det \Sigma \neq 0$). We wish to test the hypothesis H: $\Sigma=\Sigma_0$ against the alternatives K: $\Sigma\neq\Sigma_0$ where $\boldsymbol{\mu}$ is unknown and Σ_0 is a given positive definite matrix (p.d. matrix). The LR critical region for this problem is given by, as in Anderson [1],

$$\omega'=\left\{\boldsymbol{S}\,|\,\boldsymbol{S} \textit{ is } \text{p.d.} \textit{ and } |\boldsymbol{S}\Sigma_0^{-1}|^{\frac{N}{2}}\operatorname{etr}\left[-\frac{1}{2}\Sigma_0^{-1}\boldsymbol{S}\right]\leq c_\alpha\right\}, \tag{2.1}$$

where the symbol etr means exptr, $\boldsymbol{S}=\sum_{\alpha=1}^{N}(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})'$ and $\bar{\boldsymbol{X}}=N^{-1}\sum_{\alpha=1}^{N}\boldsymbol{X}_\alpha$. The constant c_α is determined such that the level of this test is α. By replacing $|\boldsymbol{S}\Sigma_0^{-1}|^{N/2}$ to $|\boldsymbol{S}\Sigma_0^{-1}|^{(N-1)/2}$ as in our previous paper [4], we can prove the following theorem.

Theorem 1. *For testing the hypothesis H: $\Sigma=\Sigma_0$ against the alternatives K: $\Sigma\neq\Sigma_0$ for unknown mean $\boldsymbol{\mu}$, the following modified LR critical region given by*

$$\omega=\left\{\boldsymbol{S}\,|\,\boldsymbol{S} \textit{ is } \text{p.d.} \textit{ and } |\boldsymbol{S}\Sigma_0^{-1}|^{\frac{n}{2}}\operatorname{etr}\left[-\frac{1}{2}\Sigma_0^{-1}\boldsymbol{S}\right]\leq c_\alpha\right\} \tag{2.2}$$

has the monotonicity property with respect to each of the p-characteristic roots of $\Sigma\Sigma_0^{-1}$, that is, $\operatorname{ch}(\Sigma\Sigma_0^{-1})=(\delta_1^2, \cdots, \delta_p^2)$, where $\boldsymbol{S}=\sum_{\alpha=1}^{N}(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})'$ and $n=N-1$. More precisely, the power function increases (decreases) with respect *to each of any* δ_i^2 *when* $\delta_i^2 \geq 1$ ($\delta_i^2 \leq 1$) *for fixed* $\delta_1^2, \cdots, \delta_{i-1}^2, \delta_{i+1}^2, \cdots, \delta_p^2$.

PROOF. The statistic $\boldsymbol{S}$ is distributed according to the Wishart distribution $\boldsymbol{W}(\Sigma, n)$, so the power function of the test $\boldsymbol{\omega}$ is given by

$$P_K(\omega \mid \Sigma) = c_{p,n}\int_{S\in\omega} |\boldsymbol{S}|^{\frac{1}{2}(n-p-1)} |\Sigma^{-1}|^{\frac{n}{2}} \operatorname{etr}\left[-\frac{1}{2}\Sigma^{-1}\boldsymbol{S}\right] d\boldsymbol{S} \tag{2.3}$$

where $c_{p,n}^{-1} = \pi^{p(p-1)/4} 2^{np/2} \Pi_{i=1}^{p} \Gamma[(n-i+1)/2]$. Put $\boldsymbol{A} = \Sigma_0^{-1/2}\boldsymbol{S}\Sigma_0^{-1/2}$, then the matrix $\boldsymbol{A}$ is also p.d. and the Jacobian is given by $|\partial \boldsymbol{A}/\partial \boldsymbol{S}| = |\Sigma_0^{-1}|^{(p+1)/2}$. Therefore we have

$$P_K(\omega_1 \mid \Sigma) = c_{p,n}\int_{A\in\omega_1} |\boldsymbol{A}|^{\frac{1}{2}(n-p-1)} |\Sigma_0^{-1}\Sigma|^{-\frac{n}{2}} \operatorname{etr}\left[-\frac{1}{2}\Sigma^{-1}\Sigma_0^{\frac{1}{2}}\boldsymbol{A}\Sigma_0^{\frac{1}{2}}\right] d\boldsymbol{A} \tag{2.4}$$

where $\omega_1 = \left\{\boldsymbol{A} \mid \boldsymbol{A} \textit{ is } \text{p.d.} \textit{ and } |\boldsymbol{A}|^{n/2} \operatorname{etr}\left[-\frac{1}{2}\boldsymbol{A}\right] \leq c_\alpha\right\}$. Let $\boldsymbol{T}$ be an orthogonal matrix such that $\boldsymbol{T}'\Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2}\boldsymbol{T} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} = \operatorname{diag}(\delta_1^2, \cdots, \delta_p^2)$. Put $\boldsymbol{B} = \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}$, then $\boldsymbol{B}$ is also p.d. and the Jacobian is given by $|\partial \boldsymbol{B}/\partial \boldsymbol{A}| = 1$. On the other hand, $\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} \in \omega_1$ is equivalent to $\boldsymbol{A} \in \omega_1$ for any orthogonal matrix $\boldsymbol{T}$. Thus we have

$$P_K(\omega_1 \mid \Sigma) = c_{p,n}\int_{B\in\omega_1} |\boldsymbol{B}|^{\frac{1}{2}(n-p-1)} |\boldsymbol{\Lambda}|^{-\frac{n}{2}} \operatorname{etr}\left[-\frac{1}{2}\boldsymbol{\Lambda}^{-1}\boldsymbol{B}\right] d\boldsymbol{B}. \tag{2.5}$$

So the power function depends only on $\boldsymbol{\Lambda}$, namely, $P_K(\omega_1 \mid \Sigma) = P_K(\omega_1 \mid \boldsymbol{\Lambda})$. Put $\boldsymbol{B} = \boldsymbol{D}^{1/2}\boldsymbol{R}\boldsymbol{D}^{1/2\dagger}$ where $\boldsymbol{D}^{1/2} = \operatorname{diag}(b_{11}^{1/2}, \cdots, b_{pp}^{1/2})$ and b_{ii} means i-th diagonal element of $\boldsymbol{B}$, then the Jacobian $|\partial \boldsymbol{B}/\partial(\boldsymbol{R}, \boldsymbol{D})| = |\boldsymbol{D}|^{(p-1)/2}$. Put $\omega_{\boldsymbol{R}} = \{\boldsymbol{D} \mid \boldsymbol{B} = \boldsymbol{D}^{1/2}\boldsymbol{R}\boldsymbol{D}^{1/2} \in \omega_1\}$, then we can write

$$\begin{aligned} P_K(\omega_1 \mid \boldsymbol{\Lambda}) &= c_{p,n}\int_{R>0} |\boldsymbol{R}|^{\frac{1}{2}(n-p-1)} d\boldsymbol{R} \int_{\omega_R} |\boldsymbol{D}|^{\frac{n}{2}-1} |\boldsymbol{\Lambda}|^{-\frac{n}{2}} \operatorname{etr}\left[-\frac{1}{2}\boldsymbol{\Lambda}^{-1}\boldsymbol{D}\right] d\boldsymbol{D} \\ &= c_{p,n}\int_{R>0} |\boldsymbol{R}|^{\frac{1}{2}(n-p-1)} \beta(\delta_1^2, \cdots, \delta_p^2 \mid \boldsymbol{R}) d\boldsymbol{R}, \end{aligned} \tag{2.6}$$

where $\beta(\delta_1^2, \cdots, \delta_p^2 \mid \boldsymbol{R}) = \int_{\omega_R} \Pi_{i=1}^{p} b_{ii}^{(n/2)-1} (\delta_i^2)^{-n/2} \exp[-b_{ii}/2\delta_i^2] db_{ii}$, and the region $\boldsymbol{R} > 0$ means the set of all p.d. matrices such that all diagonal elements are one. We can show that if $\delta_i^{*2} \geq \delta_i^2 \geq 1$ or $\delta_i^{*2} \leq \delta_i^2 \leq 1$, then

$$\beta(\delta_1^2, \cdots, \delta_{i-1}^2, \delta_i^{*2}, \delta_{i+1}^2, \cdots, \delta_p^2 \mid \boldsymbol{R}) \geq \beta(\delta_1^2, \cdots, \delta_i^2, \cdots, \delta_p^2 \mid \boldsymbol{R}). \tag{2.7}$$

For instance, the range of the integration with respect to variable b_{11} for fixed $b_{22}, \cdots, b_{pp}$ is written as $b_{11}^{n/2} \exp[-b_{11}/2] \leq c_\alpha |\boldsymbol{R}|^{-n/2} \Pi_{i=2}^{p} b_{ii}^{-n/2} \exp[b_{ii}/2]$. By the following lemma which assures the monotonicity of the power function of the test (2.2) in case of $p=1$, we have $\beta(\delta_1^{*2}, \delta_2^2, \cdots, \delta_p^2 \mid \boldsymbol{R}) \geq \beta(\delta_1^2, \cdots, \delta_p^2 \mid \boldsymbol{R})$

† This transformation was used by Gleser [2] to prove the unbiasedness of the sphericity test.

for $\delta_1^{*2} \geq \delta_1^2 \geq 1$ or $\delta_1^{*2} \leq \delta_1^2 \leq 1$. By integrating the inequality multiplied by $|\boldsymbol{R}|^{(n-p-1)/2}$ on both sides of (2.7) with respect to $\boldsymbol{R}$, we have the desired conclusion.

Now it is enough to prove Theorem 1 in the univariate case, which was stated by Ramachandran [3]. He, however, did not mention its proof explicitly. So we write the lemma with the sketch of the proof. We use σ^2 and σ_0^2 instead of covariance matrix $\sum$ and $\sum_0$.

Lemma. *For testing the hypothesis H: $\sigma=\sigma_0$ against the alternatives K: $\sigma \neq \sigma_0$ for unknown mean μ, the modified LR critical region given by*

$$\omega=\left\{S \mid S>0 \text{ and } (S\sigma_0^{-2})^{\frac{n}{2}} \exp\left[-\frac{1}{2} S\sigma_0^{-2}\right] \leq c_\alpha\right\} \tag{2.8}$$

has the monotonicity property with respect to $\delta^2=\sigma^2/\sigma_0^2$, where $S=\sum_{\alpha=1}^N (X_\alpha-\bar{X})^2$, $\bar{X}=N^{-1}\sum_{\alpha=1}^N X_\alpha$ and $n=N-1$.

Proof. Under the alternative K, the statistic S/σ^2 is distributed according to χ^2 distribution with n degrees of freedom, so the power function of the test ω is given by

$$P(\omega \mid \sigma^2)=c_{1,n}\int_{S\in\omega} S^{\frac{n}{2}-1}(\sigma^2)^{-\frac{n}{2}} \exp[-S/2\sigma^2]\, dS \tag{2.9}$$

where $c_{1,n}^{-1}=2^{\frac{n}{2}}\Gamma[n/2]$. Putting $z=\sigma_0^{-2}S$, we can write the power function (2.9) as

$$P(\omega_1 \mid \delta^2)=c_{1,n}\int_{z\in\omega_1} z^{\frac{n}{2}-1}(\delta^2)^{-\frac{n}{2}} \exp[-z/2\delta^2]\, dz, \tag{2.10}$$

where $\omega_1=\{z \mid z>0 \text{ and } z^{\frac{n}{2}} \exp[-z/2] \leq c_\alpha\}$. Since the equation $z^{\frac{n}{2}} \exp[-z/2] =c_\alpha$ has exactly two solutions $z=c_1$ and $c_2 (c_1<c_2)$, we obtain

$$\begin{aligned} \frac{dp(\omega_1 \mid \delta^2)}{d\delta^2} &= c_{1,n}(\delta^2)^{-\left(\frac{n}{2}\right)-1}\left\{c_2^{\frac{n}{2}} \exp[-c_2/2\delta^2]-c_1^{\frac{n}{2}} \exp[-c_1/2\delta^2]\right\} \\ &= c_{1,n}(\delta^2)^{-\left(\frac{n}{2}\right)-1} c_1^{\frac{n}{2}} \exp[-c_2/2\delta^2]\{\exp[(c_2-c_1)/2] \\ &\qquad -\exp[(c_2-c_1)/2\delta^2]\}. \end{aligned} \tag{2.11}$$

Thus if $\delta^2>1$ then $dP(\omega_1 \mid \delta^2)/d\delta^2>0$ and if $\delta^2<1$ then $dP(\omega_1 \mid \delta^2)/d\delta^2<0$. Therefore the lemma is proved.

By the analogous argument as in the proof of Theorem 1, we have the following theorem.

Theorem 2. *For testing the hypothesis H': $\sum=\sum_0$, $\boldsymbol{\mu}=\boldsymbol{\mu}_0$ against the alternatives K': $\sum \neq \sum_0$, $\boldsymbol{\mu}=\boldsymbol{\mu}_0$ where $\sum_0$ and $\boldsymbol{\mu}_0$ are known, the LR critical region given by*

$$\omega^* = \left\{ S^* \mid S^* \text{ is p.d. and } |S^* \Sigma_0^{-1}|^{\frac{N}{2}} \operatorname{etr}\left[-\frac{1}{2} \Sigma_0^{-1} S^* \right] \leq c_\alpha \right\}, \tag{2.12}$$

where $S^* = \sum_{\alpha=1}^{N} (X_\alpha - \mu_0)(X_\alpha - \mu_0)'$, has the monotonicity property with respect to each of $\mathrm{ch}(\Sigma \Sigma_0^{-1})$.

We can also generalize Theorem 1 to the k sample case. Let $p \times 1$ vectors $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ $(N_i > p)$ be a random sample from p-variate normal distribution with mean μ_i and covariance matrix Σ_i $(i = 1, \cdots, k)$. Put $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)(X_{j\alpha} - \bar{X}_j)'$ and $n_j = N_j - 1$. Then we have the following theorem by the sama argument as in the proof of Theorem 1.

THEOREM 3. *For testing the hypothesis* H'': $\Sigma_j = \Sigma_{0j}$ $(j = 1, 2, \cdots, k)$ *against the alternatives* K'': $\Sigma_i \neq \Sigma_{0i}$ *for some* i, *where the mean* μ_j *is unspecified and* Σ_{0j} *is a given* p.d. *matrix* $(j = 1, 2, \cdots, k)$, *the modified LR critical region given by*

$$\omega'' = \left\{ (S_1, S_2, \cdots, S_k) \mid S_j \text{ is p.d. } (j = 1, 2, \cdots, k) \text{ and } \right. \tag{2.13}$$

$$\left. \Pi_{j=1}^{k} \left[|S_j \Sigma_{0j}^{-1}|^{\frac{n_j}{2}} \operatorname{etr}\left(-\frac{1}{2} \Sigma_{0j}^{-1} S_j \right) \right] \leq c_\alpha \right\}$$

has the monotonicity property with respect to each of $\mathrm{ch}(\Sigma_j \Sigma_{0j}^{-1})$ $(j = 1, 2, \cdots, k)$.

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