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Idempotent Ideals and Unions of Nets of Prüfer Domains

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0. Introduction

In this paper, all rings considered are assumed to be commutative rings with an identity element. It is known that an integral domain D may contain an idempotent proper ideal A. But when this occurs, A is not finitely generated [21, p. 215], so that D is not Noetherian. Also, it is easy to show that for any positive integer k there exists a ring R which is not a domain and such that R contains an ideal A with the property that $A \supset A^2 \supset \cdots \supset A^k = A^{k+1} = \cdots$. Whether an integral domain R with this property exists is a heretofore open question which we answer affirmatively in §2.

Nakano in [16] has considered the problem of determining when an ideal of D is idempotent, where D is the integral closure of Z, the domain of ordinary integers, in an infinite algebraic number field. In fact, the paper [16] is one of a series of papers which Nakano has written concerning the ideal structure of D. In [18], Ohm has generalized and simplified many of Nakano's results from [16] and [17], showing that as far as the structure of the set of primary ideals of D is concerned, the assumption that D is the integral closure of Z in an algebraic number field is superfluous; the essential requirement on D being that it is a *Prüfer domain* according to the following definition: The integral domain J is a Prüfer domain if for each proper prime ideal P of J, J_P is a valuation ring; equivalently, J is a Prüfer domain if each nonzero finitely generated ideal of J is invertible [10, p. 554].

Following Ohm's example, we show in §3 that most of Nakano's results in [16] carry over to the case when D is the integral closure of a fixed Prüfer domain D_0 in an algebraic extension of the quotient field of D_0.

If J is an integral domain with quotient field K, a domain J_0 between J and K will be called an *overring* of J. In case J_0 is a valuation ring, we call J_0 a *valuation overring* of J. We say that J is an *almost Dedekind* domain if for each maximal ideal M of J, J_M is a rank one discrete valuation ring [5], [1].

1. Preliminary results on Prüfer domains.

We list in this section some results in the theory of Prüfer domains which we shall use frequently in the remainder of this paper.

In [18, pp. 1025–1027] Ohm, generalizing the results of Nakano in [16] [17], proved this result concerning the structure of the set of primary ideals of a Prüfer domain.

THEOREM 1.1. *Suppose P is a prime ideal of the Prüfer domain D, let $\mathcal{S}=\{Q_\alpha\}_{\alpha\in A}$ be the set of P-primary ideals of D, and let Q, Q_1, Q_2 be fixed elements of $\mathcal{S}$.*

(a) $\mathcal{S}$ is closed under multiplication.

(b) If $Q^n=Q^{n+1}$ for some positive integer n, then $Q=Q^2=P$.

(c) If $Q_1\subseteq Q\subset P$, then Q_1 contains a power of Q. Thus $\bigcap_{\alpha\in A}Q_\alpha=\bigcap_{n=1}^{\infty}Q^n=P_0$, and P_0 is a prime ideal. There are no prime ideals of D properly between P_0 and P.

(d) If $P\neq P^2$, then $\mathcal{S}=\{P^i\}_{i=1}^{\infty}$.

(e) If $Q\subset P$, then $Q^2\subset QP$.

(f) If $Q_1\subset Q_2$ and if $Q_1:Q_2=Q_1$, then $Q=P=P^2$.

If J is an integral domain having quotient field K and if $\{V_\alpha\}_{\alpha\in A}$ is the family of valuation overrings of J, an ideal B of J is called a *valuation ideal* of J if there exists an element α of A and an ideal B_α of V_α such that $B=B_\alpha\cap J$; in this case we necessarily have $B=BV_\alpha\cap J$ [22, p. 340]. If N is a J-submodule of K, the *completion* of N, denoted by $\tilde{N}$, is defined to be $\bigcap_{\alpha\in A}NV_\alpha$. If $N=\tilde{N}$, we say that N is *complete*. In case N is an ideal of J, $\tilde{N}$ is an ideal of $\bar{J}$, the integral closure of J. In [7, p. 238], Gilmer and Ohm established this result:

THEOREM 1.2. *In an integral domain D, these conditions are equivalent:*

(a) *D is a Prüfer domain.*

(b) *Each ideal of D is complete.*

(c) *Each ideal of D is an intersection of valuation ideals.*

The final result we state concerns Prüfer domains under integral extensions. (a) was proved by Gilmer in [6, Cor 2]. The "if" part of (b) is due to Prüfer [20, p. 31]. (c) and the "only if" part of (b) are due to Heinzer [8, Thm. 1, Cor. 2]. Butts and Phillips proved (d) in [1, p. 270]. (e) is easily shown and we list it here merely as a matter of convenience.

THEOREM 1.3. *Let D be an integral domain with quotient field K and let J be a domain integral over D such that J has quotient field L.*

(a) *If D is Prüfer and if B is an ideal of D, there is an ideal C of J such that $C\cap D=B$. In particular, $B=BJ\cap D$.*

If J is the integral closure of D in L, then

(b) *J is Prüfer if and only if D is Prüfer.*

(c) *If J is almost Dedekind, D is almost Dedekind.*

(d) *If D is almost Dedekind and $[L:K]<\infty$, then J is almost Dedekind.*

(e) *If D is Prüfer, if P is a prime ideal of D, and if M is a prime ideal of J, then $M\cap D=P$ if and only if $J_M\cap K=D_P$.*

2. Idempotent ideals of an integral domain

In [16], Nakano determines conditions under which a fixed ideal A of the integral closure Z' of Z in an infinite algebraic number field is idempotent. Nakano's major results in this area are contained in his Sätze 9–11. We first show in Theorems 2.1, 2.3 that the results of Nakano are valid in any Prüfer domain. Then we turn to a study of idempotent ideals of an arbitrary integral domain. In particular, we show that for any positive integer k there is an integral domain D_k and a maximal ideal M_k of D_k such that $M_k\supset M_k^2\supset \cdots\supset M_k^k=M_k^{k+1}=\cdots$.

Before proving Theorem 2.1, we introduce some terminology due originally to Krull. If A is an ideal of the ring R and if S is a multiplicative system in R, the ideal $A_S=\{x\in R\mid xs\in A \text{ for some } s\in S\}$ is called the *isolated component ideal* (i.K.I) of A with respect to S. Hence if "e" and "c" denote extension and contraction of ideals of R with respect to the ring R_S (see [21, pp. 218–227]), then $A_S=A^{ec}$. In case S is the complement of a prime ideal P in R, we use the notation A_P instead of A_{R-P}. If P is a minimal prime of A, then A^e has radical P^e in R_P, and P^e is maximal in R_P. Hence A_P is P-primary in this case, $A\subseteq A_P$, and each P-primary ideal containing A contains A_P. We call A_P the isolated primary P-component of A [18, p. 1024].

THEOREM 2.1. *Suppose D is a Prüfer domain and A is an idempotent ideal of D. If P is a prime ideal of D containing A, then A_P is an idempotent prime ideal. In particular, each minimal prime of A is an isolated primary component of A and is idempotent.*

PROOF. In the proof, we use strongly the result, established in [4, p. 248], that an idempotent ideal of a valuation ring is prime.

Thus, since $A=A^2$, $AD_P=[AD_P]^2$ and D_P is a valuation ring since D is a Prüfer domain. Consequently, AD_P is prime in D_P: $AD_P=QD_P$ for some prime ideal Q of D containing A. We have $Q^2D_P=A^2D_P=AD_P=QD_P$, and by Theorem 1.1, Q^2 is Q-primary. Hence $A_P=AD_P\cap D=Q=QD_P\cap D=Q^2=Q^2D_P\cap D$, and A_P is an idempotent prime ideal, as we wished to show. Q.E.D.

LEMMA 2.2. *Suppose V is a valuation ring and that P is a proper idempotent prime ideal of V. If A is an ideal of V with radical P and if $A\neq P$, there is a P-primary ideal Q such that $A\subseteq Q\subset P$.*

PROOF. By [4, Proposition 1.10, p. 249], $P^2=P$ is generated by $\{p^2\mid p\in P\}$. Hence there is an element x of P such that $x^2\notin A$. Therefore, $A\subset(x^2)\subset P$. If Q is the i.K.I. of $(x^2)$with respect to P, it follows that Q is P-primary and that $A \subset (x^2) \subseteq Q \subset (x) \subset P$. Q.E.D.

THEOREM 2.3. *If A is an ideal of the Prüfer domain D such that each isolated primary component of A is idempotent, then A is idempotent.*

PROOF. To show that $A = A^2$, it suffices to show that $AD_M = A^2D_M$ for each maximal ideal M of D containing A [22, p. 94].

Thus, if P is the minimal prime of A contained in M, then PD_M is the radical of AD_M. Since A_P is P-primary and is idempotent, Theorem 1.1 shows that $A_P = P$. Hence PD_M is the only PD_M-primary ideal of D_M containing AD_M. Since PD_M is idempotent, Lemma 2.2 shows that $AD_M = PD_M$. Moreover, $A^2D_M = (PD_M)^2 = PD_M = AD_M$ and our proof is complete. Q.E.D.

THEOREM 2.4. *Suppose A is a finitely generated ideal of the Prüfer domain D, that $\{P_\alpha\}$ is the set of minimal primes of A and for each α, $N(P_\alpha)$ is the intersection of the set of P_α-primary ideals. Then $\bigcap_{n=1}^{\infty} A^n = \bigcap_\alpha N(P_\alpha)$*

PROOF. We first observe that since each P_α is a minimal prime of the finitely generated ideal A, $N(P_\alpha) \subset P_\alpha$ for each α [7, Theorem 4.3].

We choose an element x of $\bigcap_\alpha N(P_\alpha)$. To show $x \in A^n$ for a given positive integer n, it suffices to show that $x \in A^nD_M$ for an arbitrary maximal ideal M of D containing A^n. Hence, let P_α be the minimal prime of A contained in M. We complete our proof by observing that $x \in N(P_\alpha)D_M \subseteq A^nD_M$. The containment $N(P_\alpha)D_M \subseteq A^nD_M$ follows in this case since A^nD_M has radical PD_M, so that $A^nD_M \not\subseteq N(P_\alpha)D_M \subset P_\alpha D_M$. We conclude that $\bigcap_\alpha N(P_\alpha) \subseteq \bigcap_{n=1}^{\infty} A^n$.

Conversely, if $y \in \bigcap_{n=1}^{\infty} A^n$, then for any α, $y \in (\bigcap_{n=1}^{\infty} A^{2n} D_{P_\alpha}) \cap D$. However, $A^2D_{P_\alpha}$ is a $P_\alpha D_{P_\alpha}$-primary ideal distinct from $P_\alpha D_{P_\alpha}$ so that $\bigcap_{n=1}^{\infty}(A^2D_{P_\alpha})^n$ is the intersection of the set of $P_\alpha D_{P_\alpha}$-primary ideals of D_{P_α} [4, Theorem 1.7]. That is, $\bigcap_1^\infty A^{2n}D_P = N(P_\alpha)D_{P_\alpha}$. It then follows that $y \in N(P_\alpha)D_{P_\alpha} \cap D = N(P_\alpha)$, so that $\bigcap_{n=1}^{\infty} A^n = \bigcap_\alpha N(P_\alpha)$ as we wished to show. Q.E.D.

REMARK 2.5. Theorem 2.4. was proved by Ohm [19, Corollary 1.5] in case A is a principal ideal. Our notation in Theorem 2.4 is that of Ohm, and our method of proof is not essentially different.

REMARK 2.6. In Theorem 2.4, the hypothesis that A is finitely generated is necessary. For example, if A is the maximal ideal of a rank one non-discrete valuation ring, $\bigcap_{n=1}^{\infty} A^n = A$, but the intersection of the set of A-primary ideals is (0). However, it is true that for any ideal A of a Prüfer domain $\bigcap_{n=1}^{\infty} A^n$ is an intersection of prime ideals (In the terminology of Krull [12], an ideal C of a commutative ring T is *semi-prime* if $C = \sqrt{C}$; equivalently, C is semi-prime if C may be expressed as an intersection of prime ideals of T.) This statement follows from the fact that the radical of an ideal B of a ring R is the intersection of the set of prime ideals of R which contain B, [21, p. 151], and from Theorem 2.7.

THEOREM 2.7. *If A is an ideal of the Prüfer domain D and if $B=\bigcap_{n=1}^{\infty} A^n$, then $B=\sqrt{B}$.*

PROOF. Let $u \in \sqrt{B} : u^k \in B$. We show, for n a positive integer, that $u \in A^n$. Hence, if M is a maximal ideal of D containing A, $u^k \in A^{nk}$ implies $u^k \in A^{nk} D_M = (A^n D_M)^k$. Since D_M is a valuation ring, it follows that $u \in A^n D_M$ [7. Lemma 2.8]. Consequently, $u \in A^n$, and $u \in \bigcap_{n=1}^{\infty} A^n = B$. Q.E.D.

We turn now to a consideration of idempotent ideals of an integral domain J which is not assumed to be Prüfer.

THEOREM 2.8. *Suppose A is an idempotent ideal of the domain J. The completion $\tilde{A}$ of A is a semi-prime ideal of $\bar{J}$, the integral closure of J.*

PROOF. Let $\{V_\alpha\}$ be the family of valuation overrings of J. By definition, $\tilde{A} = \bigcap_\alpha A V_\alpha = \bigcap_\alpha (A V_\alpha \cap \bar{J})$. For any α, $A V_\alpha$ is idempotent in V_α, so that $A V_\alpha$ is prime in V_α. Consequently, $\tilde{A} = \bigcap_\alpha (A V_\alpha \cap \bar{J})$ is semi-prime in $\bar{J}$.

Q.E.D.

COROLLARY 2.9. *If A is an idempotent ideal of the domain J such that A is an intersection of valuation ideals of J, then A is semi-prime.*

PROOF. By Theorem 2.8 $\tilde{A}$, the completion of A, is semi-prime in $\bar{J}$, the integral closure of J. But since A is an intersection of valuation ideals of J, $A = \tilde{A} \cap J$. It then follows that A is semi-prime in J. Q.E.D.

COROLLARY 2.10. *Suppose A is an ideal of a domain J such that $A^k = A^{k+1}$ for some positive integer k. If A^k is an intersection of valuation ideals of J, then A is idempotent and is semi-prime.*

PROOF. By Corollary 2.9, A^k is semi-prime. And since $A \subseteq \sqrt{A^k}$, $A \subseteq A^k$. Hence $A = A^k = A^{k+1}$. In particular, $A = A^2$ and A is semi-prime. Q.E.D.

COROLLARY 2.11. *If A is an ideal of the Prüfer domain D such that $A^k = A^{k+1}$ for some positive integer k, then A is idempotent and semi-prime.*

PROOF. Since each ideal of a Prüfer domain is complete, Corollary 2.11 follows immediately from Corollary 2.10. Q.E.D.

COROLLARY 2.12. *If A is an idempotent ideal of an integrally closed domain J, then the completion of A coincides with the radical of A.*

PROOF. The completion of an ideal of an integrally closed domain is always contained in the radical of that ideal [22, p. 350]. But Theorem 2.8 shows that $\tilde{A} = \sqrt{\tilde{A}} \supseteq \sqrt{A}$. Hence $\tilde{A} = \sqrt{A}$ as we wished to show. Q.E.D.

From Corollary 2.11, questions naturally arise concerning the existence of idempotent ideals of an integral domain which are not semi-prime, as well as the existence of non-idempotent ideals A such that $A^k=A^{k+1}$ for some positive integer k. Theorem 2.13 relates to these questions.

THEOREM 2.13. *In a domain J, these conditions are equivalent:*
(1) *There is an idempotent ideal of J which is not semi-prime.*
(2) *There is an ideal A of J such that $A\supset A^2=A^3=\cdots$.*

PROOF. If (1) holds in J, there is an ideal B of J such that $B=B^2$ and $B\subset\sqrt{B}$. Hence there is an element x of $\sqrt{B}-B$ such that $x^2\in B$. If $A=B+(x)$, then $B\subset A$. But $A^2=B^2+Bx+(x^2)=B$. Therefore, $A\supset A^2=A^3=\cdots$, and (2) is valid. And if (2) holds, the ideal A^2 is idempotent but is not semi-prime.

We proceed to given an example of a domain in which condition (2) of Theorem 2.13 holds. We prove, in fact, the following stronger statement:

If k is a positive integer, there is a domain D_k and a maximal ideal M_k of D_k such that $M_k\supset M_k^2\supset\cdots\supset M_k^k=M_k^{k+1}=\cdots$.

To obtain such a domain D_k, we consider a field F and indeterminates X and Y over F. There is a unique rank one nondiscrete valuation v on $F(X, Y)$ such that v is trivial on F, $v(X)=1$, and $v(Y)=\sqrt{2}$. Let V be the valuation ring of v and let M be the maximal ideal of V; M is idempotent in this case. We let $\theta=\sqrt[k]{X}$ and $D_k=V[\theta]$. D_k is a domain with identity and $\{1, \theta, \cdots, \theta^{k-1}\}$ is a free module basis for D_k over V. The ideal

$$M_k=M+(\theta)=\{m_0+d_1\theta+\cdots+d_{k-1}\theta^{k-1}\mid m_0\in M,\ d_i\in V\}$$

is maximal in D_k and $D_k/M_k\simeq V/M$. Further, if $1\leq i\leq k-1$, then

$$\begin{aligned}(M_k)^i&=(M+(\theta))^i=M+M\theta+\cdots+M\theta^{i-1}+(\theta^i)\\&=\{m_0+\cdots+m_{i-1}\theta^{i-1}+d_i\theta^i+\cdots+d_{k-1}\theta^{k-1}\mid m_j\in M,\ d_j\in V\}.\end{aligned}$$

Moreover,

$$(M_k)^k=M+\cdots+M\theta^{k-1}+(\theta^k)=M+M\theta+\cdots+M\theta^{k-1}=MV[\theta]=M^2V[\theta]=\cdots.$$

It then follows that $M_k\supset M_k^2\supset\cdots\supset M_k^k=M_k^{k+1}=\cdots$.

3. Idempotent ideals in the union of a net of Prüfer domains

In this section, we use the following notation: D_0 is a Prüfer domain with quotient field K_0. K is an algebraic extension field of K_0 which may be expressed as the union of a net $\{K_\alpha\}_{\alpha\in A}$ of finite algebraic extension fields over K_0. By a *net*, we mean here that for $\alpha, \beta\in A$, there is an element γ of A such that K_α and K_β are subfields of K_γ. We also assume that $K_0\in\{K_\alpha\}$. (The assumption that K be expressible as the union of such a net is not restrictive; the family of all subfields of K which are finite extensions of K_0 is a net whose union is K. We shall not assume, however, that $\{K_\alpha\}$ is the family of *all* subfields of K which are finite extensions of K_0.) For each $\alpha \in A$, we denote by D_α the integral closure of D_0 in K_α. By Theorem 1.3, each D_α is a Prüfer domain. And we set $D=\bigcup_{\alpha\in A}D_\alpha$; D is the integral closure of D_0 in K, $D\cap K_\alpha=D_\alpha$ for each α in A, and D is also a Prüfer domain.

Suppose P_0 is a prime ideal of D_0 and P is a prime of D lying over P_0. We consider here the problems of determining when a given P, or when each such P, is not idempotent. The results we obtain generalize Nakano's results obtained in case $D_0=Z$ and $\{K_\alpha\}$ is a chain. The additional generality of our approach, however, seems to clarify the results obtained, for the question of idempotency of a prime P of D is unextricably connected to the structure of the valuation ring D_P, when considered as an extension of the valuation ring $(D_0)_{P_0}$.

Finally, we consider in this section the problems of determining when D is almost Dedekind or when D is a Dedekind domain. Our first two theorems are basic results which will be used throughout the remainder of this section.

THEOREM 3.1. *Suppose J is a Prüfer domain with quotient field F, that L is an algebraic extension field of F, and that $\bar{J}$ is the integral closure of J in L. If P is an idempotent prime ideal of J, then each prime ideal of $\bar{J}$ lying over P is also idempotent.*

PROOF. Because J is Prüfer, $\bar{J}$ is also Prüfer. and since $\bar{J}$ is integral over J and J is integrally closed, the prime ideals of $\bar{J}$ lying over P are the minimal primes of $P\bar{J}$ [11, Satz 9]. Because P is idempotent, $P\bar{J}$ is also idempotent. Theorem 2.1 then shows that each minimal prime of $P\bar{J}$ is idempotent. Q.E.D.

THEOREM 3.2. *Suppose J is an integrally closed domain with quotient field F, L is a finite algebraic extention field of F, and $\bar{J}$ is the integral closure of J in L. If P is a prime ideal of J, the number of primes of $\bar{J}$ lying over P is finite and is $\leq[L:F]_s$. If J is Prüfer and P is not idempotent, then no prime of $\bar{J}$ lying over P is idempotent.*

PROOF. We let V be a valuation overring of J associated with a valuation v such that v has center P on J. The number of extensions of v to L is finite and is not greater than $[L:F]_s$ [22, p. 29]. But if Q is any prime ideal of $\bar{J}$ lying over P, there is an extension v^* of v to L such that Q is the center of v^* on $\bar{J}$ [22, p. 31]. It then follows that the set of primes of $\bar{J}$ lying over P is finite and is not greater than $[L:F]_s$.

In case J is a Prüfer domain, we consider a normal closure E of L over F. E is a finite extension of F and the integral closure J^* of J in E is Prüfer. The prime ideals of J^* lying over P are conjugate under elements of the Galois group of E over F [15, p. 31]. It follows that either each prime of J^* lying over P is idempotent or no prime of J^* lying over P is idempotent. The prime ideals of J^* lying over P are the minimal primes of PJ^* in J^*. Further, their number is finite—say $\{P_1, \cdots, P_t\}$ is the set of minimal primes of PJ^*. Then $\sqrt{PJ^*}=P_1\cap\cdots\cap P_t=P_1P_2\cdots P_t$ and by Theorem 4 of [3], $(P_1P_2\cdots P_t)^n \subseteq PJ^*$ for some integer n. If each P_i were idempotent, we would then have $P_1P_2\cdots P_t\subseteq PJ^*\subseteq P_1P_2\cdots P_t$ so that $PJ^*=P_1P_2\cdots P_t$ and PJ^* is idempotent. But part (a) of Theorem 1.3 shows that $P^2J^*\cap J=P^2=PJ^*\cap J=P$, which contradicts the assumption that P is not idempotent. We conclude that no prime of J^* lying over P is idempotent.

We consider a prime ideal M of $\bar{J}$ lying over P. Each prime of J^* lying over M in $\bar{J}$ lies over P in J, and hence is not idempotent. By Theorem 3.1, this implies that M is not idempotent. Q.E.D.

We return now to the notation introduced in the beginning of this section in order to prove our next results.

LEMMA 3.3. *Suppose C is an ideal of D and α is a fixed element of A. We let $B=\{\beta\in A\,|\,K_\alpha\subseteq K_\beta\}$. For $\beta\in B$, we let $C_\beta=C\cap D_\beta$.*

(1) *If k is a positive integer, $C^k=\cup_{\beta\in B}C_\beta^k$.*

(2) *If for any $\beta\in B$, there is a γ in B such that $C_\beta\subseteq C_\gamma^2$, then C is idempotent.*

PROOF. The containment $\cup_{\beta\in B}C_\beta^k\subseteq C^k$ is clear. The reverse containment follows from the fact that if $x\in C^k$, then $x\in E^k$ for some finitely generated ideal E contained in C. (2) follows immediately from (1).

In order that a prime ideal P of D fail to be idempotent, Theorem 3.1 shows that it is necessary that P_0 not be idempotent, where $P_0=P\cap D_0$. Theorem 3.4 concerns the converse of this statement.

THEOREM 3.4. *Suppose P is a prime ideal of D lying over the prime ideal P_0 of D_0 and suppose that $P_0\supset P_0^2$. Then P is idempotent if and only if the following condition, which we label as (*), holds:*

(*) *For any α in A, there is an element β of A such that $K_\alpha\subseteq K_\beta$ and such that $P_\alpha\subseteq P_\beta^2$, where $P_\alpha=P\cap D_\alpha$ for any $\alpha\in A$.*

PROOF. Part (2) of Lemma 3.3 shows that if condition (*) holds, P is idempotent. To prove the converse, we suppose that condition (*) fails and we show that P is not idempotent. Hence there is an element α of A such that if $B=\{\beta\in A\,|\,K_\alpha\subseteq K_\beta\}$, then for any $\beta\in B$, $P_\alpha\not\subseteq P_\beta^2$. By part (1) of Lemma 3.3, $P^2=\cup_{\beta\in B}P_\beta^2$. To show $P\supset P^2$, it therefore suffices to show there is a fixed element of P_α which belongs to no P_β^2 for any $\beta\in B$. By Theorem 3.2, P_α is not idempotent. Therefore $P_\alpha D_{P_\alpha}$ is principal and is generated by any element x of $P_\alpha-P_\alpha^2$. Since P_β lies over P_α, D_{P_β} extends D_{P_α} to K_β. Further, P_β is not idempotent and $P_\alpha\not\subseteq P_\beta^2$. Consequently, $P_\alpha(D_\alpha)_{P_\alpha}(D_\beta)_{P_\beta}=P_\alpha(D_\beta)_{P_\beta}\not\subseteq P_\beta^2(D_\beta)_{P_\beta}$. It follows that $P_\alpha(D_\beta)P_\beta=x(D_\beta)_{P_\beta}=P_\beta(D_\beta)_{P_\beta}$. Hence $x\notin P_\beta^2(D_\beta)_{P_\beta}$ so that $x\notin P_\beta^2$. We conclude that $x\in P-P^2$ and that P is not idempotent.

Q.E.D.

Before proving Theorem 3.5, we introduce some new notation. We fix a prime ideal P_0 of D_0 and we consider collections $\{P_\alpha\}_{\alpha\in A}$ satisfying these two properties:

(a) P_α is prime in D_α and P_α lies over P_0.

(b) For $\alpha, \beta \in A$ with $K_\alpha \subseteq K_\beta$, P_β lies over P_α.

With each P_α we associate a positive integer e_α defined as follows: Since P_α lies over P_0, P_α is a minimal prime of P_0D_α. Thus if $V_\alpha=(D_\alpha)_{P_\alpha}$, $P_0D_\alpha V_\alpha = P_0V_\alpha$ is primary for the maximal ideal $P_\alpha V_\alpha$ of V_α. Because P_α lies over P_0 and P_0 is not idempotent, P_α is not idempotent. Consequently, $P_\alpha V_\alpha$ is not idempotent. Theorem 1.1 then shows that P_0V_α is a power of $P_\alpha V_\alpha : P_0V_\alpha = (P_\alpha V_\alpha)^{e_\alpha}$. We note that if $K_\alpha \subseteq K_\beta$, P_β lies over P_α so that V_β extends V_α. Therefore, $e_\alpha \leq e_\beta$ if $K_\alpha \subseteq K_\beta$. Hence with each collection $\{P_\alpha\}_{\alpha\in A}$ satisfying (a) and (b), we obtain the set $\{e_\alpha\}_{\alpha\in A}$. In terms of the sets $\{e_\alpha\}$ we state Theorem 3.5.

Theorem 3.5. *In order that no prime of D lying over P_0 be idempotent, it is necessary and sufficient that each collection $\{e_\alpha\}_{\alpha\in A}$ obtained as described in the preceding paragraph be bounded.*

Proof. If the prime ideal P of D lies over P_0 and if P is idempotent, then if $P_\alpha = P\cap D_\alpha$ for each α in A, $\{P_\alpha\}_{\alpha\in A}$ satisfies conditions (a) and (b). Further, Theorem 3.4 shows that there is a sequence $\{\alpha_n\}_{n=1}^\infty$ of elements of A such that $K_{\alpha_i} \subset K_{\alpha_{i+1}}$ for each i and such that $e_{\alpha_{i+1}} \geq 2e_{\alpha_i}$ for each i. It follows that $\{e_{\alpha_i}\}_1^\infty$, and hence $\{e_\alpha\}$, is not bounded.

On the other hand, if no prime of D lying over P_0 is idempotent, then given a collection $\{P_\alpha\}_{\alpha\in A}$ satisfying (a) and (b), $P=\cup_{\alpha\in A}P_\alpha$ is a prime ideal of D lying over P_α in D_α for any $\alpha \in A$. Since P is not idempotent, Theorem 3.4 shows that there is an element $\alpha\in A$ such that for any $\beta \in A$ with $K_\alpha \subseteq K_\beta$, $P_\alpha \not\subseteq P_\beta^2$. As we have previously observed, this implies that $P_\alpha V_\beta = P_\beta V_\beta$. Hence $P_\beta^{e_\alpha} V_\beta = P_\alpha^{e_\alpha} V_\beta = P_\alpha^{e_\alpha} V_\alpha V_\beta = P_0 V_\alpha V_\beta = P_0 V_\beta$. It follows that $e_\alpha = e_\beta$ for any $\beta \in A$ such that $K_\alpha \subseteq K_\beta$. Now if γ is any element of A, there is an element β of A such that $K_\gamma \cup K_\alpha \subseteq K_\beta$. Hence $e_\gamma \leq e_\beta = e_\alpha$. It follows that $\{e_\gamma\}_{\gamma\in A}$ is bounded by e_α. Q.E.D.

We turn now to the problem of determining when D is almost Dedekind or when D is Dedekind. By Theorem 1.3, if D is almost Dedekind, so is D_0, and if D is a Dedekind domain, D_0 is also a Dedekind domain. Hence our question may be posed in this way: Suppose D_0 is almost Dedekind (respectively, Dedekind). Under what conditions is D almost Dedekind (resp., Dedekind)? Under either hypothesis, D_0 is one-dimensional Prüfer so that D is also one-dimensional and is Prüfer. Therefore, D is almost Dedekind if and only if D contains no idempotent maximal ideals [1, p. 270], and D is a Dedekind domain if and only if D is Noetherian. Hence, under the assumption that D_0 is almost Dedekind, we consider the problem of determining when D

contains no idempotent maximal ideals, and under the assumption that D_0 is Dedekind, we seek to determine necessary and sufficient conditions in order that D be Noetherian. Theorem 3.5 immediately yields one set of necessary and sufficient conditions in answer to the first question:

COROLLARY 3.6. *Suppose D_0 is an almost Dedekind domain. In order that D be almost Dedekind it is necessary and sufficient that for any maximal ideal P_0 of D_0 and any collection $\{P_\alpha\}_{\alpha\in A}$ satisfying* (a) *and* (b), *the set $\{e_\alpha\}$ is bounded.*

In case D_0 is almost Dedekind, the integer e_α may be related to the factorization of P_0D_α in D_α. To see this we first prove.

LEMMA 3.7. *If J is an almost Dedekind domain and if B is a proper ideal of J which is contained in only finitely many maximal ideals $M_1, M_2, \cdots, M_n$, then B may be expressed as a finite product of members of the set $\{M_1, \cdots, M_n\}$.*

PROOF. We have $B=\cap_{i=1}^n(BJ_{M_i}\cap J)$, where for each i, $BJ_{M_i}\cap J$ is M_i-primary. But in an almost Dekekind domain, primary ideals are prime powers [5, p. 813]. Hence there is a set $\{k_1, \cdots, k_n\}$ of positive integers such that $BJ_{M_i}\cap J=M_i^{k_i}$ for each i between 1 and n. Finally, because the $M_i^{k_i}$'s are pairwise comaximal we have $B=\cap_{i=1}^n M_i^{k_i}=\prod_{i=1}^n M_i^{k_i}$. Q.E.D.

In case D_0 is almost Dedekind and P_0 is a maximal ideal of D_0, then for any $\alpha\in A$, D_α is almost Dedekind, and by Theorem 3.2, there are only finitely many maximal ideals $M_1, \cdots, M_n$ of D_α lying over P_0. Hence $\{M_1, \cdots, M_n\}$ is the set of maximal ideals of D_α containing P_0D_α. By Lemma 3.7, $P_0D_\alpha=\prod_{i=1}^n M_i^{k_i}$ for some set $\{k_i\}_{i=1}^n$ of positive integers. But since, for any j between 1 and n, $\prod_{i=1}^n M_i^{k_i}$ extends to $[M_j(D_\alpha)_{M_j}]^{k_j}$ in $(D_\alpha)_{M_j}$, it follows that the positive integer e_j associated with any M_j is k_j, the power to which M_j occurs in the prime factorization of P_0D_α. In case K_α is a normal extension of K_0, the ideals $M_1, \cdots, M_n$ are conjugate under elements of the Galois group of K_α over K_0. Hence if K_α is normal over K_0, $k_1=k_2=\cdots=k_n$. This observation allows us to state Lemma 3.7 in a much more convenient form in terms of a normal closure L of K over K_0. Thus for $\alpha\in A$, we let L_α be a normal closure of K_α over K_0 in L. $\{L_\alpha\}_{\alpha\in A}$ is a net of subfields of L, $L=\cup_{\alpha\in A}L_\alpha$, and each L_α is a finite normal extension of K_0. If E_α is the integral closure of D_0 in L_α for each α and if E is the integral closure of D_0 in L, then $E=\cup_{\alpha\in A}E_\alpha$ and each E_α is almost Dedekind. Using this notation we state Theorem 3.8.

THEOREM 3.8. *In case D_0 is almost Dedekind and L is a normal extension of K, these statements are equivalent:*

(i) *D is almost Dedekind.*

(ii) *E is almost Dedekind.*

(iii) *For each maximal ideal P_0 of D_0, there is an element α_0 of A, de-*

pending on P_0, such that each maximal ideal of E_{α_0} lying over P_0 is unramified with respect to E—that is, no maximal ideal of E_{α_0} lying over P_0 is contained in the square of a maximal ideal of E.

(iv) *For any maximal ideal P_0 of D_0, there is an element α_0 of A such that each maximal ideal of E_{α_0} lying over P_0 is unramified with respect to E_β for any β in A such that $L_{\alpha_0} \subseteq L_\beta$.*

Proof. (i)→(ii): By Theorem 1.3.

(ii)→(iii): If P_0 is a maximal ideal of D_0 we consider a maximal ideal P of E lying over P_0. If $P_\alpha = P \cap E_\alpha$ for each α in A, and if e_α is the exponent to which P_α occurs as a factor of P_0E_α, Corollary 3.6 shows that the set $\{e_\alpha\}$ is bounded. We choose $\beta \in A$ such that $e_\beta \geq e_\alpha$ for each $\alpha \in A$. We show that no maximal ideal of E_β lying over P_0 is contained in the square of a maximal ideal of E. We first show that P_β is contained in the square of no maximal ideal of E. If $C = \{\gamma \in A \mid E_\beta \subseteq E_\gamma\}$, then $P^2 = \cup_{\gamma \in C} P_\gamma^2$. Hence by choice of e_β and from the fact that $P_\beta (D_\beta)_{P_\beta}$ is principal, it is clear that $P_\beta \not\subseteq P^2$. If M is any maximal ideal of E lying over P_β, then since L is normal over L_β, there is an element of the Galois group of L over L_β sending M onto P. Since $P_\beta \not\subseteq P^2$, it therefore follows that $P_\beta \not\subseteq M^2$. We have proved that P_β is contained in the square of no maximal ideal of E. If H_β is any maximal ideal of E_β lying over P_0, there is a K_0-automorphism σ of L_β such that $\sigma(H_\beta) = P_\beta$. Further, σ can be extended to a K_0-automorphism σ^* of L since L is normal over K_0 (compare [9, Vol III p. 42]). It follows that if H_β were contained in the square of a maximal ideal of E, P_β would also be contained in the square of a maximal ideal of E. Consequently, H_β is not contained in the square of a maximal ideal of E, and (iii) holds.

(iii)→(ii) : This is immediate from Corollary 3.6.

(iii)↔(iv): Trivial Q.E.D.

We conclude this section by considering the case when D_0 is a Dedekind domain. As we have previously remarked, D will be Dedekind in this case if and only if D is Noetherian. Further, D is Noetherian if and only if each prime ideal of D is finitely generated [2, p. 29]. And because D is one-dimensional, we are therefore led to the problem of determining when each maximal ideal of D is finitely generated. In Lemma 3.9 and 3.10 we need only assume that D_0 is a Prüfer domain. That is, we do not require that D_0 is Noetherian.

Lemma 3.9. *Let B be an ideal of D and for $\alpha \in A$, let $B_\alpha = B \cap D_\alpha$. If S is a finite subset of B, S generates B in D if and only if there is an element $\alpha \in A$ such that for any $\beta \in A$ for which $K_\alpha \subseteq K_\beta$, S generates B_β in D_β.*

Proof. It is clear that if an α can be found in A satisfying the condition described, then S generates B in D. And if $B = SD$, then because S is finite, there is an α in A such that $S \subseteq D_\alpha$. If $\beta \in A$ and if $D_\alpha \subseteq D_\beta$, then by Corollary 2 of [6] $SD\cap D_\beta = SD_\beta$ since D_β is a Prüfer domain. But $B=SD$ so that $SD\cap D_\beta = B_\beta$. It follows that S generates B_β in D_β for any β in A such that $D_\alpha \subseteq D_\beta$.

LEMMA 3.10. *Let B an ideal of D and for $\alpha \in A$, let $B_\alpha = B\cap D_\alpha$. B is finitely generated if and only if there exists α in A such that B_α is finitely generated and such that $B_\beta = B_\alpha D_\beta$ for any β in A such that $D_\alpha \subseteq D_\beta$.*

PROOF. Lemma 3.10 is a mere restatement of Lemma 3.9.

THEOREM 3.11. *Suppose D_0 is a Dedekind domain. These conditions are equivalent:*

(i) *D is a Dedekind domain.*

(ii) *For each maximal ideal P_0 of D_0, there exists an element α_0 of A, depending on P_0, such that each maximal ideal of D_{α_0} lying over P_0 is inertial with respect to D.*

(iii) *For each maximal ideal P_0 of D_0, there exists an element α_0 of A, depending on P_0, such that each maximal ideal of D_{α_0} lying over P_0 is inertial with respect to D_β for any β in A such that $D_{\alpha_0} \subseteq D_\beta$.*

PROOF. That (ii) and (iii) are equivalent is clear. To establish the equivalence of (i) and (iii) it suffices, in view of preceding remarks, to show that (iii) is equivalent to the condition that each maximal ideal of D is finitely generated. Hence if (i) holds and if P_0 is a maximal ideal of D_0, there are only finitely many maximal ideals $M_1, \cdots, M_r$ of D lying over P_0 (these are the maximal ideals which occur in the prime factorization of P_0D). Each M_i is generated by some finite set S_i, and there is an element α of A such that $\cup_{i=1}^r S_i \subseteq D_\alpha$. If for each i between 1 and r, $H_i = M_i \cap D_\alpha$, our proof of Lemma 3.9 shows that S_i generates H_i and H_i is inertial with respect to D_β for any $\beta \in A$ such that $D_\alpha \subseteq D_\beta$. To establish (iii), we note that $\{H_i\}_{i=1}^r$ is the set of maximal ideals of D_α lying over P_0. That this is true follows by choice of the set $\{M_1, \cdots, M_r\}$.

If (iii) holds and if P is a maximal ideal of D, we let $P_0 = P\cap D_0$. By hypothesis, there is an element α_0 in A such that each maximal ideal of D_{α_0} lying over P_0 is inertial with respect to D. Hence if $P_{\alpha_0} = P\cap D_{\alpha_0}$, $P_{\alpha_0}D$ is maximal in D and is contained in P. Thus $P = P_{\alpha_0}D$. But D_{α_0} is the integral closure of a Dedekind domain in K_{α_0}, where $[K_{\alpha_0} : K_0] < \infty$. Consequently, D_{α_0} is Dedekind [22, p. 281], and P_{α_0} is finitely generated. We conclude that P is finitely generated so that (i) is valid. Q.E.D.

REMARK. 3.12. Exercise 10, page 83, of [0] may also be used to obtain necessary and sufficient conditions in order that D be Dedekind. For it is known that a Krull domain is a Dedekind domain if and only if it has dimension ≤ 1. [22, p. 84].

4. Examples

Let D be a Dedekind domain with quotient field K. Under the assumptions that D/P is finite for each maximal ideal P of D and that the set of maximal ideal of D is countable (the integral closure of Z, the ring of integers, in any finite algebraic number field is a Dedekind domain with this property), we provide in this section a method for constructing an infinite algebraic extension field L of K such that the integral closure $\bar{D}$ of D in L is an almost Dedekind domain which is not Dedekind. For this construction we need Lemmas 4.1–4.2.

LEMMA 4.1. *Let R be a commutative ring with identity and let $\{A_i\}_{i=1}^n$ be a collection of pairwise comaximal ideals of R (that is, $R=A_i+A_j$ for $i\neq j$). If $\{f_i\}_{i=1}^n$ is a finite subset of $R[X]$, where each f_i is monic of degree k, then there exists $f\in R[X]$, f monic of degree k, such that $f\equiv f_i$ $(A_i[X])$, $i=1, 2, \cdots, n$.*

We omit the proof of Lemma 4.1 since it is essentially that of Theorem 31 (9) in [21, p. 177].

If F is a finite algebraic extension of K, then the integral closure $\bar{D}$ of D in F is a Dedekind domain. Therefore, if P is a maximal ideal of D, $P\bar{D}$ is a product of maximal ideals of $\bar{D}$; we write $P\bar{D}=M_1^{e_1}\cdots M_g^{e_g}$, where the maximal ideals M_i are all distinct. The integer e_i is called the *reduced ramification index* of M_i over P, and the degree $[\bar{D}/M_i : D/P]=f_i$ is called the *relative degree of* M_i over P; $\sum_{i=1}^g e_i f_i \leq [F:K]$, and in particular, $\bar{D}/M$ is finite for each maximal ideal M of $\bar{D}$ [21, pp. 284–285]. If $P\bar{D}$ is maximal in $\bar{D}$, we say that P is *inertial* with respect to $\bar{D}$; if $g=1$ but $e_1>1$, we say that P *ramifies* with respect to $\bar{D}$; and if $g>1$, we say that P *decomposes* with respect to $\bar{D}$. Using this notation and terminology, we state and prove Lemma 4.2.

LEMMA 4.2. *Let $\{P_i\}_{i=1}^r$, $\{Q_i\}_{i=1}^s$, and $\{U_i\}_{i=1}^t$ be finite collections of distinct maximal ideals of D. Then there exists a simple quadratic extension $K(t)$ of K such that each P_i is inertial with respect to $\bar{D}$, each Q_i ramifies with respect to $\bar{D}$, and each U_i decomposes with respect to $\bar{D}$; here $\bar{D}$ denotes the integral closure of D in $K(t)$.*

PROOF. For each i between 1 and r, D/P_i is a finite field and $D[X]/P_i[X]\simeq(D/P_i)[X]$. Hence we can find $f_1, \cdots, f_r\in D[X]$, f_i monic of degree 2, such that f_i is irreducible modulo $P_i[X]$ for each i. For each i between 1 and s, let $q_i\in Q_i-Q_i^2$. Since the ideals $\{P_i\}_{i=1}^r$, $\{Q_i^2\}_{i=1}^s$, $\{U_i\}_{i=1}^t$ are pairwise comaximal, there exists, by Lemma 4.1, an element f of $D[X]$, f monic of degree 2, such that

$$\begin{aligned} &f\equiv f_i \quad (P_i[X]), && 1\leq i\leq r\\ &f\equiv X^2+q_i \quad (Q_i^2[X]), && 1\leq i\leq s\\ &f\equiv X(X+1) \quad (U_i[X]), && 1\leq i\leq t. \end{aligned}$$

Let α be a root of f in an extension field of K. f is irreducible (since f is monic and is irreducible modulo $P_1[X]$) so that $K(\alpha)$ is a quadratic extension of K.

From Theorem 5 [21, p. 260], it follows that $\{g \in D[X] \mid g(\alpha)=0\}=(f)$, the principal ideal of $D[X]$ generated by f. Then from fundamental properties of ring isomorphisms we have, for any maximal ideal P of D,

$$D[\alpha]/P[\alpha] \simeq [D[X]/(f(X))]/[(P[X]+(f(X)))/(f(X))]$$
$$\simeq D[X]/(P[X]+(f(X))) \simeq [D[X]/P[X]]/[(P[X]+(f(X)))/P[X]]$$
$$\simeq (D/P)[X]/(\bar{f}(X)).$$

where $\bar{f}(X)$ is the canonical image of $f(X)$ in $(D/P)[X]$.

If $P=P_i$, $1 \leq i \leq r$, then $\bar{f}(X)=\bar{f}_i(X)$ is irreducible; consequently $P_i[\alpha]$ is a maximal ideal of $D[\alpha]$. Further, since $\bar{f}_i(X)$ has degree 2, $[D[\alpha]/P_i[\alpha]: D/P_i]=2$. If $P=U_i$, $1 \leq i \leq t$, then $\bar{f}(X)=X(X+1)$ so there exist two distinct maximal ideals of $D[\alpha]$ containing $U_i[\alpha]$. Finally, if $P=Q_i$, $1 \leq i \leq s$, then $\bar{f}(X)=X^2$ so that $Q=(X)/(\bar{f}(X))$ is a maximal ideal of $(D/Q_i)[X]/(\bar{f}(X))$ such that $Q^2=(0)$. Therefore, $(Q_i[\alpha], \alpha)$ is a maximal ideal of $D[\alpha]$ such that $(Q_i[\alpha], \alpha)^2 \subseteq Q_i[\alpha] \subset (Q_i[\alpha], \alpha)$. We show that $(Q_i[\alpha], \alpha)^2 = Q_i[\alpha]$. Thus suppose $f(X)=X^2+pX+q$. Since $f(X) \equiv X^2(Q_i[X])$ and $f(X) \equiv X^2+q_i(Q_i^2[X])$, it follows that $p \in Q_i$ and $q \in Q_i - Q_i^2$, $1 \leq i \leq s$. Then $Q_i^2 \subset Q_i^2+(q) \subseteq Q_i$ and since D is a Dedekind domain, $Q_i^2+(q)=Q_i$. But $q=-\alpha^2-p\alpha \in (Q_i[\alpha], \alpha)^2$ so that $Q_i=Q_i^2+(q) \subseteq (Q_i[\alpha], \alpha)^2$. Hence $Q_i[\alpha]=Q_iD[\alpha] \subseteq (Q_i[\alpha], \alpha)^2$, and consequently, equality holds.

Since $\bar{D}$ is integral over $D[\alpha]$, it now follows that there exist maximal ideals $\{M_i\}_{i=1}^r$, $\{N_i\}_{i=1}^s$, $\{H_i^{(j)}\}_{i=1}^t$, $j=1, 2$, such that $P_i \subseteq M_i$, $1 \leq i \leq r$, $Q_i \subseteq N_i^2$, $1 \leq i \leq s$ and $U_i \subseteq H_i^{(j)}$, $1 \leq i \leq t$, $j=1, 2$. For $1 \leq i \leq r$, the relative degree of M_i over P_i is greater than or equal to 2 since $[\bar{D}/M_i: D/P_i]=[\bar{D}/M_i: D[\alpha]/P_i[\alpha]][D[\alpha]/P_i[\alpha]: D/P_i]=2[\bar{D}/M_i: D[\alpha]/P_i[\alpha]]$. It now follows from Theorem 21 of [21, p. 285] that $P_i\bar{D}=M_i$, $1 \leq i \leq r$, $Q_i\bar{D}=N_i^2$, $1 \leq i \leq s$, and $U_i\bar{D}=H_i^{(1)}H_i^{(2)}$, $1 \leq i \leq t$.

Now let $\{P_i\}_{i=1}^\infty$ be the collection of all the maximal ideals of D. Using the method described in Lemma 2.4, we may construct a sequence $K \subset K(\alpha_1) \subset \cdots \subset K(\alpha_n) \subset \cdots$ of simple algebraic extensions of K such that the following properties hold:

(1) $[K(\alpha_1): K]=2$ and $[K(\alpha_{i+1}): K(\alpha_i)]=2$, $i=1, 2, \cdots$.

(2) If D_n is the integral closure of D in $K(\alpha_n)$ and if $\{M_{n_1}^{(r)}, \cdots, M_{n_{\lambda_n}}^{(r)}\}$ is the set of maximal ideals of D_n lying over P_r, $1 \leq r \leq n+2$, then $M_{n_1}^{(1)}D_{n+1} = M_{n+1_1}^{(1)}M_{n+1_2}^{(1)}$ and for any $M_{n_j}^{(i)} \neq M_{n_1}^{(1)}$, $M_{n_j}^{(i)}$ is inertial with respect to D_{n+1}.

For any positive integer n, no prime factor of P_nD_n ramifies with respect to D_m for $m>n$. However, for any positive integer m, $M_{m_1}^{(i)}$ is a prime factor of P_1D_m which decomposes with respect to D_{m+1}. Therefore, by Corollary 3.6 and Theorem 3.11, if $\bar{D}$ is the integral closure of D in $L=\bigcup_{i=1}^\infty K(\alpha_i)$, $\bar{D}$ is an almost Dedekind domain which is not Dedekind. Further, by Lemma 3.10, $M=\cup_{i=1}^{\infty}M_{i_1}^{(1)}$ is the unique maximal ideal of $\bar{D}$ which is not finitely generated.

Similarly, using Lemma 4.2, L may be constructed in such a manner that $\bar{D}$ is a Dedekind domain. This has been done for the case in which $D=Z$ by Maclane and Schilling in [13] by a similar method of construction. Also, if D is not a local domain, L may be constructed so that $\bar{D}$ is not an almost Dedekind domain. In fact, L may be constructed so that $\bar{D}$ contains a unique maximal ideal which is idempotent.

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