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Gel'fand integrals and generalized derivatives of vector measures

Dedicated to Professor Takizo Minagawa on his 70th birthday

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Introduction

Let X be a real Banach space and (S, Σ, μ) a finite nonnegative complete measure space. For a general Banach space X, a μ-continuous vector measure $\nu: \Sigma \rightarrow X$ of finite variation need not be the indefinite Bochner integral of its derivative unless the values are suitably chosen. The purpose of this paper is to introduce a notion of generalized derivative which can be defined for any vector measure that is μ-continuous and of finite variation and investigate basic properties of the generalized derivatives.

The fundamental theorem of calculus for vector measures (called commonly the Radon-Nikodým theorem) need not be valid for all types of integrals. It is a notable pathology that the Bochner integral does not mimic the Lebesgue integral with regard to the fundamental theorem. In this connection various types of integrals which include the Bochner integral have been introduced through the duality theory by Birkhoff, Dunford, Gel'fand, Pettis and Phillips; and the Radon-Nikodým theorems have been established in the respective senses. Although each integral definition has its own features, the most general one among them is that of Gel'fand and the so-called Dunford-Pettis theorem is regarded as the associated fundamental theorem for vector measures with values in dual Banach spaces. Our notion of generalized derivative is also based on the Gel'fand integration theory.

Various examples of vector measures with values in nonreflexive Banach spaces such as $L^1(\mu)$ and $L^\infty(\mu)$ suggest that to an arbitrary vector measure only the differentiation in the sense of the weak*-topology of X (viewed as a subspace of its second dual unless X is a dual Banach space) may be applied. Indeed, if ν is a measure on Σ with values in a dual Banach space not possessing the Radon-Nikodým property, only the local boundedness of the set $\{\nu(E)/\mu(E): E \in \Sigma\}$ may be assumed, namely: There exists a sequence $\{S_n: n=0, 1, 2,...\}$ such that $\mu(S_0)=0$, $\mu(S_n)>0$ $(n \geqq 1)$, $S=\cup_{n=0}^{\infty} S_n$ and $\{\nu(E)/\mu(E): E \in \Sigma, E \in S_n\}$ is bounded for each $n \geqq 1$. Therefore, only the relative weak*-compactness is applied to find the derivative and the derivative (supposing it is defined) is to possess only the weak*-measurability. The Dunford-Pettis theorem may apply to X-valued measures since they can be embedded into its second dual. But in most cases the dual of a nonreflexive Banach space is not separable; hence, depending upon the choice of separable subspaces of X^*, their theorem would furnish with "uncountably" many (weak*-) derivatives for a single measure. This suggests that, in order to avoid this difficult situation, the derivative ought to be defined as a multi-valued function, rather than a single-valued function.

In virtue of the above-mentioned, we shall employ "multi-valued" derivatives defined in the following way: First let ν be an X^*-valued, μ-continuous measure ν of finite variation and Π a family of suitably chosen finite partitions of S on which a partial order is defined as follows: for $\pi_1, \pi_2 \in \Pi$ we write $\pi_1 \leqq \pi_2$ iff every set in π_1 is represented as a union of some sets in π_2. Now for $\pi \in \Pi$ we write $f_\pi(s) = \sum_{E \in \pi} (\nu(E)/\mu(E))\chi_E(s)$ and define the multi-valued function $\phi_\nu: S \to 2^{X^*}$ by

$$\phi_\nu(s) = \bigcap_{\pi \in \Pi} \overline{\mathrm{co}}^\sigma \{f_{\pi'}(s): \pi' \geqq \pi\}, \quad s \in S,$$

where $\overline{\mathrm{co}}^\sigma [W]$ denotes the weak*-closed convex hull of a set W. We then call ϕ_ν the generalized derivative of the X^*-valued measure ν. If X has no preduals and ν is an X-valued μ-continuous measure of finite variation, then ν is viewed as an X^{**}-valued measure and the generalized derivative ϕ_ν is defined as an X^{**}-valued function in the same way as above. Our notion of generalized derivative is therefore a generalization of the weak*-derivative for vector measures. Given an X-valued, μ-continuous measure ν of finite variation, it is proved that ϕ_ν is defined μ-a.e. on S. The function ϕ_ν may lie in the second dual of X and is perhaps genuinely multi-valued unless X is the dual of another separable Banach space. However, it will be shown that such multi-valued derivative ϕ_ν can be treated through a certain equivalence relation for weak*-measurable functions as if it were single-valued, and still possesses characteristic properties of the "derivative" of ν. Basic to this type of derivative is the fact that the fundamental theorem of calculus holds between ν and selection of ϕ_ν in the sense of the Gel'fand integral.

In 1968, Rieffel established a general Radon-Nikodým theorem for the Bochner integral. Since then important progress has been made in the study of Banach spaces with the Radon-Nikodým property. In fact, this class of spaces plays an important role in modern Banach space theory. Also it is noteworthy that Stegall gave various types of characterizations of dual Banach spaces with the Radon-Nikodým property. Moreover a wider class of Banach spaces (called spaces with the weak Radon-Nikodým property) was introduced in 1979 by Musiał and generalizations of the Radon-Nikodým theorem for the Bochner integral have been investigated by Musiał, Rybakov, Uhl, Jr., and Kupka. For the results as mentioned above we refer the reader to the distinguished survey of Diestel and Uhl, Jr. [20]. These important works not only provide us with information concerning exclusive classes of vector measures which are not Bochner differentiable, but also suggest deeper problems as to when our generalized derivatives become the Bochner derivative or else how our derivatives are related to the integral representation of vector measures in the sense of Pettis.

In Section 1 a notion of multi-valued weak*-measurable function is introduced and various integral definitions of scalarly measurable functions are discussed. Moreover, some basic results concerning the Gel'fand integration will be given.

Section 2 is devoted to the investigation of weak*-closed convex hulls of weak*-cluster points of bounded nets in dual Banach spaces. The main result of this section plays an essential role in our argument.

The aim of Section 3 is to introduce the notion of generalized derivative for general vector measures and examine basic properties of the generalized derivatives. The results given in this section are closely related to the work of Kupka [13]. He gave general integral representation theorems for vector measures with the help of the lifting theorem, while we base our argument on the relative weak*-compactness of the suitable average ranges of vector measures only; and it turns out that more precise aspects of the Radon-Nikodým theorem for the Gel'fand integral are found. The relations of our results to those of Kupka will be discussed in detail at the end of the section.

Section 4 deals with a Lebesgue type space (denoted $\mathscr{L}^1_G(\mu, X^*)$) of X^*-valued weak*-measurable functions. The space $\mathscr{L}^1_B(\mu, X^*)$ of Bochner integrable functions on S is isometrically embedded in $\mathscr{L}^1_G(\mu, X^*)$ and it is verified that the class of all X^*-valued μ-continuous vector measures of finite variation is isometrically isomorphic to the space $\mathscr{L}^1_G(\mu, X^*)$.

Section 5 continues with a series of definitions of Lebesgue type spaces $L^q_G(\mu, X^*)$, $1 \leqq q \leqq \infty$. The duals of the spaces $L^p_B(\mu, X)$, $1 \leqq p \leqq \infty$, of Bochner integrable functions on S are characterized by means of $L^q_G(\mu, X^*)$, $1 \leqq q \leqq \infty$.

Finally, Section 6 contains some applications of our results to the Bochner and Pettis integrals and it will be shown that there is a striking contrast between them. In this section we shall also make a few remarks on the relation of our results to the recent works of Geitz [7] and Hashimoto [9], in which Pettis integrable functions are discussed from the viewpoint of the sequential approximation of scalarly measurable functions by simple functions.

1. Vector integration of scalarly measurable functions

Throughout this paper the symbol X denotes an infinite dimensional real Banach space; and the symbols X^*, X^{**} and X^{***} represent the dual space of X, the second dual and the third dual space of X, respectively. We always identify X with the image of the natural embedding of X into X^{**}. For $x \in X$ and $x^* \in X^*$ we write $\langle x, x^* \rangle$ for the value $x^*(x)$ of the functional x^* at x. However when we focus our attention on the elements of X^* we sometimes write $\langle x^*, x \rangle$

for the pairing $\langle x, x^*\rangle$ by regarding x as an element of X^{**}. Given a subset K of a Banach space X, we mean the convex hull and the convex closure of K by co K and $\overline{\text{co}}\, K$, respectively. Moreover if $W \subset X^*$, $\overline{\text{co}}^{\sigma}\, W$ denotes the weak*-closure of co W.

Let $x \in X$ and $x^{**} \in X^{**}$. If f is an X^*-valued function on a set S, we write $\langle f, x^{**}\rangle$ and $\langle f, x\rangle$ for the real-valued functions defined respectively by

$$\langle f, x^{**}\rangle(s) = \langle f(s), x^{**}\rangle \qquad \text{and} \qquad \langle f, x\rangle(s) = \langle f(s), x\rangle$$

for $s \in S$. Likewise, given an X^*-valued measure defined on a σ-field Σ, we denote by $\langle v, x^{**}\rangle$ and $\langle v, x\rangle$ the real-valued measures $\langle v(\cdot), x^{**}\rangle$ and $\langle v(\cdot), x\rangle$, respectively. The triplet (S, Σ, μ) stands for a finite, complete and nonnegative measure space; and the symbol Σ^+ denotes the set of all elements $E \in \Sigma$ with positive measures. Moreover, for each $E \in \Sigma$, we write $\Sigma(E) = \{E \cap F: F \in \Sigma\}$ and call it the restriction of Σ to E.

1.1. Let X be a Banach space. We say that a function $f: S \to X$ is simple if there exist $x_1, \ldots, x_n \in X$ and $E_1, \ldots, E_n \in \Sigma$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$, where each χ_{E_i} denotes the characteristic function of E_i. A function $f: S \to X$ is said to be strongly μ-measurable if there exists a sequence (f_n) of simple functions such that $\lim_{n\to\infty} \|f_n(s) - f(s)\| = 0$ μ-almost everywhere. We say that f is weakly measurable if for each $x^* \in X^*$ the numerical function $\langle f, x^*\rangle$ is μ-measurable, and that f is weakly integrable if $\langle f, x^*\rangle$ is μ-integrable for each $x^* \in X^*$. One may introduce more general notions: Let $\Gamma \subset X^*$. A function $f: S \to X$ is said to be Γ-measurable (resp. Γ-integrable), if $\langle f, x^*\rangle$ is μ-measurable (resp. μ-integrable) for each $x^* \in \Gamma$. Let X be a Banach space, f an X^*-valued function on S, and let X be viewed as the image under the natural embedding of X into X^{**}. Then we say that f is weak*-measurable (resp. weak*-integrable) if it is X-measurable (resp. X-integrable) in the above sense. Two functions $f: S \to X$ and $g: S \to X$ are said to be Γ-equivalent, provided that $\langle f, x^*\rangle = \langle g, x^*\rangle$ μ-a.e. for each $x^* \in \Gamma$. If in particular $f: S \to X$ and $g: S \to X$ are X^*-equivalent, we say that f and g are weakly equivalent. Likewise, if $f: S \to X^*$ and $g: S \to X^*$ are X-equivalent, f and g are said to be weak*-equivalent. The space of all X^*-valued, weak*-integrable functions is denoted by $\mathfrak{G}(S, \Sigma, \mu, X^*)$, or simply $\mathfrak{G}(\mu, X^*)$. The space $\mathfrak{G}(\mu, X^*)$ is a vector space under the usual addition and scalar multiplication.

Remark. In contrast with the strong measurability, the notion of scalar measurability as mentioned above does not assume the existence of approximate sequence of simple functions. For instance, the weak*-measurability of an X^*-valued function f does not necessarily imply the existence of a sequence (f_n) of X^*-valued simple functions such that $\langle f_n, x\rangle \to \langle f, x\rangle$ μ-a.e. for each $x \in X$ (where the null set on which the convergence does not hold may vary with x) unless the Banach space X or else the base measure space (S, Σ, μ) is suitably chosen. In fact, even a weakly measurable function does not necessarily have an approximate sequence of simple functions when the base measure space is not perfect. It is an interesting problem in connection with the weak Radon-Nikodým property for Banach spaces to investigate such sequential approximation of weakly measurable functions by simple functions. For the results in this direction we refer to the recent works of Geitz [7] and Hashimoto [9].

We now state the following fact which is basic to the definition of Gel'fand integral.

LEMMA 1.1. *If a function $f: S \to X^*$ is weak*-integrable over S, then to each $E \in \Sigma$ there corresponds a constant $M(E) \geqq 0$ such that*

$$\int_E |\langle f(s), x\rangle| d\mu \leqq M(E)\|x\| \qquad \text{for} \quad x \in X. \tag{1.1}$$

The proof is obtained by applying the uniform boundedness theorem. See Gel'fand [8].

We then give the definition of Gel'fand integral of a weak*-integrable function. Let $f: S \to X^*$ be weak*-integrable over S and for each $E \in \Sigma$, set

$$M_f(E) = \sup \left\{ \int_E |\langle f(s), x\rangle| d\mu \colon \|x\| \leqq 1 \right\}, \quad E \in \Sigma. \tag{1.2}$$

The set function $M_f(\cdot): \Sigma \to [0, \infty)$ is monotone and countably subadditive. Moreover Lemma 1.1 states that for each $E \in \Sigma$ the mapping $x \mapsto \int_E \langle f(s), x\rangle d\mu$ defines a continuous linear functional $\nu(E)$ on X such that $\|\nu(E)\| \leqq M_f(E)$. Hence $\nu(E) \in X^*$ for $E \in \Sigma$ and we have

$$\langle \nu(E), x\rangle = \int_E \langle f(s), x\rangle d\mu \quad x \in X \quad \text{and} \quad E \in \Sigma.$$

As is easily seen, the set function $\nu(\cdot): \Sigma \to X^*$ is countably additive with respect to the weak*-topology of X^*. But it should be noted (Example 4 in [3], p. 53) that ν need not be norm-countably additive. Now given an $E \in \Sigma$ the value $\nu(E)$ is called the *Gel'fand integral* (or simply the *G-integral*) of f over E and is denoted as

$$\nu(E) = (G) - \int_E f d\mu.$$

For every pair f, $g \in \mathfrak{G}(\mu, X^*)$, f and g provide the same G-integral $\nu(E)$ for every $E \in \Sigma$ iff f and g are weak*-equivalent.

REMARK. Let X be an arbitrary Banach space and suppose that a function $f: S \to X$ is weakly integrable over S. Then Lemma 1.1 states that for each $E \in \Sigma$ there is a $\nu(E) \in X^{**}$ such that $\langle \nu(E), x^*\rangle = \int_E \langle f(s), x^*\rangle d\mu$ for every $x^* \in X^*$.

If in particular all of $\nu(E)$'s are determined in the original space X, the function f is said to be Pettis integrable over S. There is a notable difference between the Gel'fand integral and the Pettis integral; the absolute continuity in the sense of Saks. In fact, an indefinite G-integral need not be absolutely continuous in the sense of Saks, while any indefinite Pettis integral is absolutely continuous in that sense.

1.2. In this paper multi-valued functions play an important role in the discussion of the generalized derivatives of vector measures. Let X be a Banach space. By a multi-valued function ϕ from S into X we mean a mapping ϕ: $S\to 2^X$; and a selection of ϕ means a single-valued function $f: S\to X$ such that $f(s)\in\phi(s)$ for $s\in S$. For multi-valued functions as mentioned above, one may introduce natural notions of measurability as well as integrability. A multi-valued function $\phi: S\to 2^{X^*}$ is said to be weak*-measurable (resp. weak*-integrable over S) if

(i) any pair of selections of ϕ are weak*-equivalent to each other; and

(ii) there is a selection f of ϕ such that $\langle f, x\rangle$ is μ-measurable (resp. μ-integrable over S) for $x\in X$.

If ϕ is a multi-valued function from S into X, the weak measurability and weak integrability are defined in a similar manner.

Given a pair of X-valued (resp. X^*-valued) functions f and g on S, we usually identify f with g if $f=g$ μ-a.e. on S. In this sense every single-valued function may be regarded as a multi-valued function as mentioned above. However a multi-valued, weakly measurable (resp. weak*-measurable) function should mean a function $\phi: S\to 2^X$ (resp. $\phi: S\to 2^{X^*}$) with the property that ϕ has at least two selections f, g such that f and g are weakly equivalent (resp. weak*-equivalent), but for some $\varepsilon_0>0$ the outer measure of the set $\{s\in S: \|f(s)-g(s)\|\geqq\varepsilon_0\}$ is positive. In what follows, a function $\phi: S\to 2^X$ is said to be essentially single-valued if for any pair of selections f, g of ϕ we have $f(s)=g(s)$ μ-a.e.

1.3. Let Π be the set of all finite disjoint collections $\pi=\{E_1,\dots, E_n\}$ of elements of Σ^+. Let $1<p<\infty$ and $\nu: \Sigma\to X$ a vector measure such that $\nu(E)=0$ whenever $\mu(E)=0$. By the p-variation of ν we mean a mapping $|\nu|_p(\cdot): \Sigma\to [0, \infty]$ defined by

$$|\nu|_p(E) = \sup_{\pi\in\Pi}\{\textstyle\sum_{A\in\pi}\|\nu(A\cap E)\|^p/\mu(A\cap E)^{p-1}\}^{1/p}$$

for $E\in\Sigma$, where we use the convention that $\|\nu(E)\|/\mu(E)=0$ whenever $\mu(E)=0$.

If $p=1$, the 1-variation of ν is defined to be a mapping $|\nu|_1(\cdot): \Sigma\to[0, \infty]$ which assigns to each $E\in\Sigma$ the value

$$|\nu|_1(E) = \sup_{\pi\in\Pi}\{\textstyle\sum_{A\in\pi}\|\nu(A\cap E)\|\}.$$

Henceforth, $|\nu|_1$ is written as $|\nu|$ and is simply called the variation of ν. If $|\nu|$ is finite, ν is said to be of finite variation; and if $|\nu|$ is σ-finite, we say that ν is of σ-finite variation. For $1 \leqq p < \infty$, we denote by $V^p(\mu, X)$ the space of all vector measures $\nu: \Sigma \to X$ such that $\|\nu\|_p \equiv |\nu|_p(S) < \infty$. Moreover, $V^\infty(\mu, X)$ stands for the space of all vector measures $\nu: \Sigma \to X$ with $\|\nu\|_\infty \equiv \inf\{k > 0: \|\nu(E)\| \leqq k\mu(E)$ for $E \in \Sigma\} < \infty$.

Let $1 \leqq p < \infty$. We denote by $\mathscr{L}_B^p(\mu, X)$ the space of all X-valued strongly μ-measurable functions f such that $\|f(\cdot)\| \in L^p(\mu)$; hence in particular $\mathscr{L}_B^1(\mu, X)$ is the space of all X-valued Bochner integrable functions on S. In this paper the Bochner integral of $f \in \mathscr{L}_B^1(\mu, X)$ over $E \in \Sigma$ is written as

$$\nu(E) = (B) - \int_E f d\mu.$$

Given a pair f, g in $\mathscr{L}_B^p(\mu, X)$, we say that f is equivalent to g if $f(s) = g(s)$ μ-a.e. on S; and we write $L_B^p(\mu, X)$ for the space of all equivalence classes in $\mathscr{L}_B^p(\mu, X)$. If X and μ are fixed, we sometimes write $\mathscr{L}_B^p$ and L_B^p for $\mathscr{L}_B^p(\mu, X)$ and $L_B^p(\mu, X)$, respectively. The space $L_B^p(\mu, X)$ is a Banach space under the norm $\|\tilde{f}\| \equiv \left(\int_S \|f(s)\|^p d\mu\right)^{1/p}$ where $f \in \tilde{f}$ and $\tilde{f} \in L_B^p(\mu, X)$. It is well-known ([3], Theorem 1 on page 98) that $L_B^p(\mu, X)^* = L_B^q(\mu, X^*)$ where $p^{-1} + q^{-1} = 1$, iff X^* has the Radon-Nikodým property with respect to μ. Moreover the space of all X-valued strongly μ-measurable functions f on S with $\|f(\cdot)\| \in L^\infty(\mu)$ is denoted by $\mathscr{L}_B^\infty(\mu, X)$; and we write $L_B^\infty(\mu, X)$ for the space all equivalence classes in $\mathscr{L}_B^\infty(\mu, X)$. The space $L_B^\infty(\mu, X)$ is a Banach space under the norm $\|\tilde{f}\|_\infty = \operatorname{ess\,sup}_{s \in S} \|f(s)\|$, $f \in \tilde{f} \in L_B^\infty(\mu, X)$. Thus the dual space of $L_B^p(\mu, X)$ is no longer represented as Lebesgue type spaces of strongly μ-measurable functions on S. We then quote the following result which is originally due to Bochner and Taylor [1].

THEOREM 1.2. *Let $1 \leqq p < \infty$ and $p^{-1} + q^{-1} = 1$. Then $L_B^p(\mu, X)^*$ is isometrically isomorphic to $V^q(\mu, X^*)$ under the correspondence between $T \in L_B^p(\mu, X)^*$ and $\nu \in V^q(\mu, X)$ defined by*

$$T(\tilde{f}) = \int_S f d\nu \qquad \textit{for} \quad f \in \tilde{f} \quad \textit{and} \quad \tilde{f} \in L_B^p(\mu, X).$$

Let μ be a finite positive measure on (S, Σ) and $\nu: \Sigma \to X$ a vector measure. We say that ν is μ-continuous if for every $\varepsilon > 0$ there is a positive number δ such that $|\nu|(E) < \varepsilon$ for $E \in \Sigma$ with $\mu(E) < \delta$. The μ-continuity of ν is equivalent to the property that $\nu(E) = 0$ for $E \in \Sigma$ with $\mu(E) = 0$ if ν is countably additive and of finite variation.

For $1 < p < \infty$ the definition of p-variation implies that

$$\|\nu(E)\| \leqq |\nu|_p(S) \cdot \mu(E)^{1-1/p} \qquad \text{for} \quad E \in \Sigma^+.$$

Hence ν is μ-continuous. However the definition of 1-variation of ν does not yield the μ-continuity and $V^1(\mu, X)$ is simply the space of X-valued vector measures of finite variation. We then denote by $V_c^1(\mu, X)$ the subspace of $V^1(\mu, X)$ which consists of vector measures in $V^1(\mu, X)$ that are μ-continuous. The space $V_c^1(\mu, X^*)$ is a closed linear subspace of the Banach space $V^1(\mu, X^*)$ by Vitali-Hahn-Saks' theorem. Moreover, it is easy to see that $V^p(\mu, X^*) \subset V_c^1(\mu, X^*)$ for $p>1$ since μ is finite.

Finally, we state an important lemma which will often be applied in the sequel. Although this lemma is implicit in Musiał [14], we here give the complete statement as well as its proof.

LEMMA 1.3. *Let $\Gamma \subset X^*$. If $f: S \to X$ is μ-measurable, then there exists a nonnegative measurable function ψ_f with the following properties:*

(1.3) *For every $x^* \in X^*$, $|\langle x^*, f(s)\rangle| \leqq \psi_f(s)\|x^*\|$ μ-a.e.;*

(1.4) *$\psi_f(s) \leqq \|f(s)\|_\Gamma \equiv \sup\{|\langle x^*, f(s)\rangle|: \|x^*\| \leqq 1, x^* \in \Gamma\}$ μ-a.e. and*

(1.5) *if $\psi': S \to [0, \infty)$ is a measurable function satisfying* (1.3) *and* (1.4) *(with ψ_f replaced by ψ'), then $\psi_f(s) \leqq \psi'(s)$ μ-a.e..*

PROOF. Let $M(S, \Sigma, \mu)$ be the space of all μ-measurable, extended real valued functions endowed with the usual partial ordering "$\leqq$" (that is, $f \leqq g$ means that $f(s) \leqq g(s)$ μ-a.e. on S). The space $M(S, \Sigma, \mu)$ forms a complete lattice with respect to this partial ordering. Now consider the subset $N = \{|\langle x^*, f\rangle|: \|x^*\| \leqq 1, x^* \in \Gamma\}$ of μ-measurable functions. Since $M(S, \Sigma, \mu)$ is a complete lattice, there exists a least upper bound ψ_f in $M(S, \Sigma, \mu)$ of N. This ψ_f is the desired function. In fact, it is clear from the definition of ψ_f that (1.3) holds. Moreover, it follows from Theorem IV. 11.6 of [6] that there exists a sequence (x_n^*) with $x_n^* \in \Gamma$ and $\|x_n^*\| \leqq 1$ such that ψ_f is a least upper bound of $\{|\langle x_n^*, f\rangle|: n \geqq 1\}$, i.e., $\psi_f(s) = \sup\{|\langle x_n^*, f(s)\rangle|: n \geqq 1\}$ μ-a.e. on S. Hence (1.4) holds. This means that the least upper bound of N satisfies (1.4) automatically. It, thus, is clear that (1.5) also holds. q. e. d.

2. Convex hulls of weak*-cluster points of bounded nets

Let $(A, \leqq)$ be a directed set and let $(x_\alpha^*, \alpha \in A)$ be an arbitrary bounded net in X^*; and for each $\alpha \in A$, we denote $K_\alpha = \overline{\mathrm{co}}^\sigma\{x_{\alpha'}^*: \alpha' \geqq \alpha\}$ by the weak*-closed convex hull of $\{x_{\alpha'}^*: \alpha' \geqq \alpha\}$. Since each K_α is weak*-compact by Alaoglu's theorem and $\{K_\alpha: \alpha \in A\}$ is monotone nonincreasing with respect to the order $\geqq$, we have $K = \bigcap_{\alpha \in A} K_\alpha \neq \emptyset$ by the finite intersection property. On the other hand, we consider the set W of all weak*-cluster points of the net $(x_\alpha^*, \alpha \in A)$ and let $\tilde{K} = \overline{\mathrm{co}}^\sigma W$. The aim of this section is to establish the following result which is of use for the discussion of our generalized derivatives.

THEOREM 2.1. (1) *Every extremal point of K is a weak*-cluster point of the net $(x_\alpha^*, \alpha \in A)$. Namely, if the set of all extremal points of K is denoted by* $\operatorname{ext} K$ *then* $\operatorname{ext} K \subset W$. (2) *The set K coincides with $\tilde{K}$.*

To prove the above theorem we need the following lemma.

LEMMA 2.2. *The net $(x_\alpha^*, \alpha \in A)$ converges in X^* to an element x_0^* in the sense of the weak*-topology iff $K = \{x_0^*\}$.*

PROOF. Suppose that K contains two distinct points x_0^* and x_1^*. Then by the separation theorem one finds an element $x_0 \in X$ and real numbers c_1, c_2 such that $\langle x_0, x_0^*\rangle < c_1 < c_2 < \langle x_0, x_1^*\rangle$. Put $U(x_0^*) = \{x^* \in X^*: \langle x_0, x^*\rangle \leqq c_1\}$ and $U(x_1^*) = \{x^* \in X^*: \langle x_0, x^*\rangle \geqq c_2\}$. Then $U(x_0^*) \cap U(x_1^*) = \emptyset$, and $U(x_0^*)$ and $U(x_1^*)$ are weak*-closed convex neighborhoods of x_0^* and x_1^*, respectively. Since x_0^* is a weak*-limit of $(x_\alpha^*, \alpha \in A)$, there is an $\alpha_0 \in A$ such that $x_\alpha^* \in U(x_0^*)$ for every $\alpha \geqq \alpha_0$. It, thus, follows that $K_\alpha \subset U(x_0^*)$ for $\alpha \geqq \alpha_0$, and $K \subset U(x_0^*)$. This is a contradiction, and so we must have $x_0^* = x_1^*$ and $K = \{x_0^*\}$. To show the converse, suppose that x_α^* does not converge to x_0^* in the weak*-topology. Then one can choose an element $x_0 \in X$ and a positive number ε_0 so that for every $\alpha \in A$ there may exist an $\alpha' \geqq \alpha$ $(\alpha' \in A)$ satisfying

$$|\langle x_0, x_{\alpha'}^*\rangle - \langle x_0, x_0^*\rangle| \geqq \varepsilon_0 > 0 \tag{2.1}$$

Let B be the set of all such $\alpha' \in A$. Then B is a cofinal subset of A and $(x_\beta^*, \beta \in B)$ forms a subnet of $(x_\alpha^*, \alpha \in A)$. Therefore, by Alaoglu's theorem, there exists a subnet $(y_\gamma^*, \gamma \in \Gamma)$ of $(x_\beta^*, \beta \in B)$ such that y_γ^* converges to some element y_0^* in X^* in the sense of the weak*-topology. Let $\varphi: \Gamma \to B$ be any mapping with the following properties;

(i) for every $\gamma \in \Gamma$, $y_\gamma^* = x_{\varphi(\gamma)}^*$; and
(ii) for every $\beta_0 \in B$ there exists $\gamma_0 \in \Gamma$ such that $\gamma \geqq \gamma_0$ implies $\varphi(\gamma) \geqq \beta_0$.

Let $K' = \bigcap_{\gamma \in \Gamma} \overline{\mathrm{co}}^\sigma \{y_{\gamma'}^*: \gamma' \geqq \gamma\}$. Then we infer from the necessity part of this lemma that $K' = \{y_0^*\}$. We then demonstrate that $y_0^* = x_0^*$. To this end, let α be any element in A. Then there exists $\beta_0 \in B$ with $\beta_0 \geqq \alpha$. Hence, by use of (ii), one finds $\gamma_0 \in \Gamma$ such that $\gamma \geqq \gamma_0$ implies $\varphi(\gamma) \geqq \beta_0$, so that $\{y_\gamma^*: \gamma \geqq \gamma_0\} \subset \{x_\beta^*: \beta \geqq \beta_0\} \subset \{x_{\alpha'}^*: \alpha' \geqq \alpha\} \subset K_\alpha$. Therefore we have $K' \subset \overline{\mathrm{co}}^\sigma \{y_\gamma^*: \gamma \geqq \gamma_0\} \subset K_\alpha$. Since α is arbitrary, we obtain $K' \subset K = \{x_0^*\}$. So, $x_0^* = y_0^*$ and $y_\gamma^* \to x_0^*$ in the weak*-topology. But this contradicts (2.1), and it follows that $x_\alpha^* \to x_0^*$ in the weak*-topology of X^*. q. e. d.

PROOF OF THEOREM 2.1. (1): To show that $\operatorname{ext} K \subset W$, suppose the contrary and let $x_0^* \in \operatorname{ext} K - W$. Then there exist $x_0 \in X$, $\varepsilon_0 > 0$ and $\alpha_0 \in A$ such that $|\langle x_0, x_0^*\rangle - \langle x_0, x_\alpha^*\rangle| \geqq \varepsilon_0 > 0$ for every $\alpha \geqq \alpha_0$. We then take two sets $H_1 =$

$\{x^* \in X^*: \langle x_0, x^*\rangle \geqq \langle x_0, x_0^*\rangle + \varepsilon_0\}$ and $H_2 = \{x^* \in X^*: \langle x_0, x^*\rangle \leqq \langle x, x_0^*\rangle - \varepsilon - \varepsilon_0\}$. Then $H_1 \cap H_2 = \emptyset$ and $x_\alpha^* \in H_1 \cup H_2$ for $\alpha \geqq \alpha_0$. Also, both H_1 and H_2 intersect $\{x_\alpha^*\}$ in such a way that $\{\alpha: x_\alpha^* \in H_i\}$, $i=1, 2$ are cofinal subsets of A. In fact, suppose that $\{x_\alpha^*: \alpha \geqq \alpha_1\} \subset H_1$ for some $\alpha_1 \geqq \alpha_0$. Then $K_{\alpha_1} \subset H_1$ and so $K \subset H_1$, which contradicts $x_0^* \notin H_1$. Hence H_1 intersects $\{x_\alpha^*\}$ cofinally. Likewise, H_2 contains cofinally a subset of $\{x_\alpha^*\}$. Let $\alpha \geqq \alpha_0$, $A_\alpha = H_1 \cap \{x_{\alpha'}^*: \alpha' \geqq \alpha\}$, and let $B_\alpha = H_2 \cap \{x_\alpha^*: \alpha' \geqq \alpha\}$. It is obvious that $A_\alpha \neq \emptyset$, $B_\alpha \neq \emptyset$ and $\overline{\text{co}}^\sigma A_\alpha \cap \overline{\text{co}}^\sigma B_\alpha = \emptyset$ for every $\alpha \in A$ with $\alpha \geqq \alpha_0$. Since both A and B are relatively weak*-compact, we obtain the relation

$$K_\alpha = \overline{\text{co}}^\sigma (A_\alpha \cup B_\alpha) = \text{co}\,(\overline{\text{co}}^\sigma A_\alpha \cup \overline{\text{co}}^\sigma B_\alpha). \tag{2.2}$$

Moreover, the following holds:

$$K = \textstyle\bigcap_{\alpha \in A} K_\alpha = \text{co}\,((\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha) \cup (\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha)). \tag{2.3}$$

In fact, let $x^* \in \text{co}\,((\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha) \cup (\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha))$. Then there exist $y^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha$, $z^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha$ and $c \in [0, 1]$ such that $x^* = cy^* + (1-c)z^*$. Hence we infer from (2.2) that $x^* \in A_\alpha$ with $\alpha \in \alpha_0$, so that $x^* \in \bigcap_{\alpha \in A} K_\alpha = K$. To get the converse inclusion, let $x^* \in \bigcap_{\alpha \in A} K_\alpha$. Then it follows from (2.2) that for every $\alpha \in A$ with $\alpha \geqq \alpha_0$ there exist $y_\alpha^* \in \overline{\text{co}}^\sigma A_\alpha$, $z_\alpha^* \in \overline{\text{co}}^\sigma B_\alpha$ and $c_\alpha \in [0, 1]$ such that $x^* = c_\alpha y_\alpha^* + (1-c_\alpha) z_\alpha^*$. Since all of (c_α), (y_α^*) and (z_α^*) form bounded nets, one can choose subnets $(\tilde{c}_\gamma, \gamma \in \Gamma)$, $(\tilde{y}_\gamma^*, \gamma \in \Gamma)$ and $(\tilde{z}_\gamma^*, \gamma \in \Gamma)$, a number $c \in [0, 1]$, and elements $y^*, z^* \in X^*$ such that $\tilde{c}_\gamma \to c$ in $[0, 1]$, $\tilde{y}_\gamma^* \to y^*$ and $\tilde{z}_\gamma^* \to z^*$ in the weak*-topology of X^*. Hence we observe that $y^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha$ and $z^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha$. It, thus, follows that $x^* = cy^* + (1-c)z^* \in \text{co}\,((\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha) \cup (\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha))$. Now, using the relation (2.3), we conclude that $x_0^* \in \text{co}\,((\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha) \cup (\bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha))$. In virtue of this and the hypothesis $x_0^* \notin H_1 \cup H_2$ (hence $x_0^* \notin \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha \cup \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha$), we can take $c \in (0, 1)$, $y^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma A_\alpha$ and $z^* \in \bigcap_{\alpha \in A} \overline{\text{co}}^\sigma B_\alpha$ so that $x_0^* = cy^* + (1-c)z^*$. But $x_0^* \in \text{ext}\, K$; hence (2.3) yields that $x_0^* = y^* = z^*$. This contradicts the definitions of y^* and z^* since $y^* \neq z^*$. Thus $\text{ext}\, K \subset W$.

(2): First we prove that $\tilde{K} \subset K$. To this end, it is sufficient to show that $W \subset K$. Let x^* be any element of W. Then one can choose a subnet $(y_\beta^*, \beta \in B)$ of $(x_\alpha^*, \alpha \in A)$ so that $y_\beta^* \to x^*$ in the weak*-topology of X^*. Hence Lemma 2.2 shows that $\{x^*\} = \bigcap_{\beta \in B} \overline{\text{co}}^\sigma \{y_{\beta'}^*: \beta' \geqq \beta\}$. On the other hand, we see from the definition of $\{y_\beta^*: \beta \in B\}$ that to each $\alpha \in A$ there corresponds a $\beta_0 \in B$ such that $\{y_\beta^*: \beta \geqq \beta_0\} \subset \{x_{\alpha'}^*: \alpha' \geqq \alpha)$. So, $\{x^*\} \subset \overline{\text{co}}^\sigma \{y_\beta^*: \beta \geqq \beta_0\} \subset \overline{\text{co}}^\sigma \{x_{\alpha'}^*: \alpha' \geqq \alpha\} = K_\alpha$. Since α is arbitrary, we have $x^* \in \bigcap_{\alpha \in A} K_\alpha = K$. Thus $W \subset K$. To show the converse inclusion $K \subset \tilde{K}$, it suffices to show with the aid of the Krein-Milman's Theorem that $\text{ext}\, K \subset W$. But this has already been proved in the first step (1).

q. e. d.

3. Generalized derivatives of vector measures

In this section we introduce a notion of "multi-valued" generalized derivative of a vector measure and investigate some basic properties of the generalized derivatives. There are two distinct cases for defining such derivatives: Let X be a Banach space. If ν is X^*-valued, the generalized derivative ϕ_ν is defined as a possibly multi-valued function with values in the same space X^*. While the generalized derivative ϕ_ν of an X-valued measure is a possibly multi-valued function on S whose range may lie in the second dual X^{**}. In any case our notion of generalized derivative is understood to be a generalization of the so-called weak*-derivative for vector measures.

3.1. Let Π be a family of finite disjoint collections $\pi=\{E_1,\dots, E_n\}$ of Σ^+ such that $\cup_{i=1}^n E_i=S$. In what follows, a finite disjoint collection π as mentioned above is generically called a Σ^+-partition of S. On the family Π a partial order "$\leqq$" is defined in the following way: for $\pi_1, \pi_2 \in \Pi$ we write $\pi_1 \leqq \pi_2$ iff every set in π_1 is represented as a union of some sets in π_2. Employing this partial order the system $(\Pi, \leqq)$ forms a directed set, and if $x_\pi \in X$ for $\pi \in \Pi$ then $(x_\pi, \pi \in \Pi)$ forms a net in X.

The set, Σ_Π, of elements of Σ^+ which constitute the family Π is in general a proper (and even "very small") subset of Σ^+. Accordingly, it is necessary to require that Σ_Π consists of sufficiently many elements so that the family Π may induce "well-behaved" derivatives. In this section we are concerned with families of Σ^+-partitions of S satisfying condition (H) as mentioned below. Let Π be a family of Σ^+-partitions of S. We say that Π satisfies condition (H) if:

(H) for every $g \in L^1(\mu)$ we have $\lim_{\pi\in\Pi} g_\pi(s)=g(s)$ μ-a.e.,

where g_π is the simple function defined by

$$g_\pi(s) = \textstyle\sum_{\pi\in\Pi}\left(\int_E g \, d\mu/\mu(E)\right)\chi_E(s) \qquad \text{for} \quad s \in S.$$

Condition (H) is closely related to the existence theorem for the Lebesgue points of Bochner integrable functions. See Lemma 6.2 below. Moreover, given a lifting ρ for μ, the family Π_ρ of all $\rho(\Sigma^+)$-partitions of S endowed with the partial order as mentioned above does always satisfy condition (H). See Section 3.3. In fact, the class of those families defined through liftings for μ will play an important role in later arguments.

Let X be a Banach space and Π a family of Σ^+-partitions of S satisfying condition (H). First we define the generalized derivatives of X^*-valued measures. Let $\nu \in V_c^1(\mu, X^*)$. We write ψ for the Radon-Nikodým derivative $d|\nu|/d\mu$ and, for each $\pi \in \Pi$, we define a simple function f_π: $S \to X^*$ by

$$f_\pi(s) = \textstyle\sum_{E\in\pi}(\nu(E)/\mu(E))\chi_E(s) \qquad \text{for} \quad s \in S. \tag{3.1}$$

Then $\|f_\pi(s)\| \leqq \sum_{E\in\pi} (|v|(E)/\mu(E))\chi_E(s) \equiv \psi_\pi(s)$ for $s\in S$ and condition (H) yields that there exists a null set N_0 such that

$$\text{(3.2)} \qquad \limsup_\pi \|f_\pi(s)\| \leqq \psi(s) < \infty \qquad \text{for} \quad s\in S-N_0$$

Therefore we infer with the aid of Alaoglu's theorem that

$$\text{(3.3)} \qquad \bigcap_{\pi\in\Pi} \overline{\text{co}}^\sigma \{f_{\pi'}(s)\colon \pi' \geqq \pi\} \neq \emptyset \qquad \text{for} \quad s\in S-N_0.$$

We then define a set-valued function $\phi_v\colon S\to 2^{X^*}$ by

$$\text{(3.4)} \qquad \phi_v(s) = \begin{cases} \bigcap_{\pi\in\Pi} \overline{\text{co}}^\sigma \{f_{\pi'}(s)\colon \pi' \geqq \pi\}, & s\in S-N_0, \\ \{0\}, & s\in N_0. \end{cases}$$

The function ϕ_v as defined above is understood to be a generalized weak*-derivative of v. In fact, let $s\in S-N_0$, N_0 being the μ-null set which appeared in (3.2). Then Lemma 2.2 yields that $\phi_v(s)$ is a singleton set iff $\lim_\pi f_\pi(s)$ exists in the sense of the weak*-topology of X^*; and if $\phi_v(s)$ is a singleton set for a.e. $s\in S$, ϕ_v is the weak*-derivative of v in the usual sense. Although ϕ_v is in general genuinely multi-valued μ-a.e. on S, we call ϕ_v the *Π-generalized derivative* of the X^*-valued measure v. (If Π is fixed, we sometimes eliminate Π and simply call the (*generalized*) *derivative* of v since it will cause no confusion.)

We next suppose that the original Banach space X has no preduals and define the derivatives of X-valued measures. Let $v\in V_c^1(\mu, X)$. Then v can be viewed as a vector measure belonging to $V_c^1(\mu, X^{**})$ since $V_c^1(\mu, X)$ is isometrically embedded in $V_c^1(\mu, X^{**})$. Hence, one can define a set-valued function $\phi_v\colon S\to 2^{X^{**}}$ by (3.4), where the convex closure is taken with respect to the weak*-topology of X^{**}. This multi-valued function ϕ_v is in general properly X^{**}-valued. In fact, it will be shown in Section 6 that if ϕ_v is the Π_ρ-derivative of v (ρ being a lifting for μ) and $\phi_v(s)$ intersects X for a.e. $s\in S$ then ϕ_v gives a Bochner derivative of v. However it is interesting to note that the Π_ρ-derivative ϕ_v does not necessarily contain the Pettis derivative of v even if v is represented as the indefinite Pettis integral of an X-valued weakly measurable function on S. For more detailed argument concerning this problem, see the forthcoming paper [10].

Therefore, in the following, we are mainly concerned with vector measures which take their values in dual Banach spaces.

3.2. In order to investigate the properties of the generalized derivatives as defined above, we need the following lemmas which are derived from Theorem 2.1.

LEMMA 3.1. *Assume that Π satisfies condition* (H). *Let* $v\in V_c^1(\mu, X^*)$, ϕ_v *the generalized derivative defined by* (3.4), *and* $\psi\equiv d|v|/d\mu$ *the Radon-Nikodým derivative of* $|v|$ *with respect to* μ. *Then we have the relation*

$$\|f(s)\| \leqq \psi(s), \quad s \in S - N_0,$$

for every selection f of ϕ_v, where N_0 is the null set which appeared in (3.2).

PROOF. Let $\{f_\pi: \pi \in \Pi\}$ be a family of simple functions defined for v by (3.1). Then (3.2) holds. Now given $s \in S - N_0$ choose an arbitrary weak*-cluster point $f(s)$ of the net $(f_\pi(s), \pi \in \Pi)$. Then (3.2) yields that $\|f(s)\| \leqq \psi(s)$ for $s \in S - N_0$. Therefore, Alaoglu's theorem and Theorem 2.1 together imply that for each $s \in S - N_0$ we have $\|f(s)\| \leqq \psi(s)$ for $f(s) \in \phi_v(s)$. q.e.d.

LEMMA 3.2. *Assume that Π satisfies condition (H). Let $v \in V_c^1(\mu, X^*)$ and f any selection of ϕ_v. Then f is weak*-integrable over S and $v(E) = (G) - \int_E f(s)d\mu$ for $E \in \Sigma$.*

PROOF. Fix any $x \in X$. Put $g_x = d\langle v, x\rangle/d\mu$. Let $s \in S - N_0$ and $f(s)$ any weak*-cluster point of the net $(f_\pi(s), \pi \in \Pi)$. By virtue of condition (H) there exists a null set N_x such that $g_x(s) = \lim_\pi \langle f_\pi(s), x\rangle$ for $s \in S - N_x$. Hence we have

$$(3.5) \qquad \langle f(s), x\rangle = \lim_\pi \langle f_\pi(s), x\rangle = g_x(s) \qquad \text{for} \quad s \in S - (N_0 \cup N_x).$$

The relation (3.5) holds for every element of the weak*-closed convex hull of such weak*-cluster points $f(s)$. Therefore Theorem 2.1 implies that (3.5) holds for every $f(s) \in \phi_v(s)$, and so f is weak*-integrable and v is represented as the indefinite G-integral of f. q.e.d.

The above lemma suggests that we may treat ϕ_v as if it were a single-valued function. Since the G-integral $(G) - \int_E f d\mu$ does not depend upon the choice of selection f of ϕ_v, we sometimes write $(G) - \int_E \phi_v(s)d\mu$ for the value $(G) - \int_E f d\mu$. Our generalized derivative depends upon the choice of family Π of Σ^+-partitions, though Lemma 3.2 states that a selection of the Π-derivative of v is weak*-equivalent to any selection of another derivative, say Π'-derivative of v, so far as both Π and Π' satisfy condition (H). In this sense it is not so essential from our point of view to discuss the difference between generalized derivatives associated with distinct families of Σ^+-partitions.

REMARK. In connection with the Remark before Lemma 1.1, it is interesting to consider the approximation problem for the generalized derivative ϕ_v by means of simple functions. Applying the recent results of Hashimoto [9], we obtain the following

PROPOSITION. (i) *If X^* has the weak Radon-Nikodým property, then there is a sequence (π_n) of partitions of S such that $\langle f_{\pi_n}, x\rangle \to \langle f, x\rangle$ μ-a.e. for any $x \in X$ and any selection f of ϕ_v, where the null set on which the convergenece* *does not hold may vary with x and f.* (ii) *If the base measure space (S, Σ, μ) is separable (i.e., Σ is generated by a denumerable number of subsets of S), then the same conclusion as in* (i) *is valid.*

We are now in a position to state the main theorem of this section.

THEOREM 3.3. *Let $v \in V_c^1(\mu, X^*)$. Let Π be a family of Σ^+-partitions of S satisfying (H) and ϕ_v the Π-generalized derivative of v. Then for every selection f of ϕ_v there exists a μ-null set N_f such that $\|f(s)\| = \psi(s)$ for $s \in S - N_f$. Moreover, $\|f(\cdot)\|$ is μ-integrable and*

$$|v|(E) = \int_E \|f(s)\| d\mu \qquad \textit{for} \quad E \in \Sigma.$$

PROOF. Let f be any selection of ϕ_v. Then f is weak*-measurable by Lemma 3.2, and so Lemma 1.3 implies that there exists the greatest lower bound ψ_f of the family of μ-measurable functions ψ' with the following properties:

$$\text{for every} \quad x \in X,\ |\langle f(s), x\rangle| \leqq \psi'(s)\|x\| \quad \mu\text{-a.e., and} \tag{3.6}$$

$$\psi'(s) \leqq \|f(s)\| \quad \mu\text{-a.e..} \tag{3.7}$$

From this and Lemma 3.1 it follows that $\psi_f(s) \leqq \|f(s)\| \leqq \psi(s)$ μ-a.e.. But we infer from Proposition 1 in Musiał [14] that $\psi_f(s) = \psi(s)$ μ-a.e., and so there exists a null set N_f such that

$$\psi(s) = \|f(s)\| \qquad \text{for} \quad s \in S - N_f. \tag{3.8}$$

q. e. d.

REMARKS. 1) Given a vector measure $v \in V_c^1(\mu, X^*)$ and a family Π of Σ^+-partitions satisfying (H), we have defined a generalized derivative ϕ_v by taking weak*-cluster points of the bounded nets $(f_\pi(s))$, $s \in S$. Hence the values $\phi_v(s)$ are subsets of X^* which possibly span infinite-dimensional subspaces of X^*.

2) As mentioned in Section 3.1, the generalized derivative ϕ_v of an X-valued measure $v \in V_c^1(\mu, X)$ may be properly X^{**}-valued and it turns out that the indefinite G-integral v of an X^{**}-valued function on S is X-valued. This phenomenon is in contrast to the definition of Dunford integral since the indefinite Dunford integral of an X-valued function is in general X^{**}-valued.

3.3. In this subsection we discuss generalized derivatives defined by means of liftings for μ. Since (S, Σ, μ) is a finite nonnegative complete measure space, there is a lifting ρ for μ that is a mapping $\rho: \Sigma \to \Sigma$ with the following properties (hereafter called the lifting properties): (i) $\chi_E = \chi_{\rho(E)}$ μ-a.e.; (ii) $\chi_E = \chi_F$ μ-a.e. implies that $\rho(E) = \rho(F)$; (iii) $\rho(\phi) = \phi$, $\rho(S) = S$; (iv) $\rho(E \cap F) = \rho(E) \cap \rho(F)$; and (v) $\rho(E \cup F) = \rho(E) \cup \rho(F)$. See Dinculeanu's book [4] for the detailed argument. Let Π_ρ be the family of all Σ^+-partitions $\pi = \{E_1, \ldots, E_n\}$ such that $E_i \in \rho(\Sigma^+)$ (i.e., $\rho(E_i) = E_i \in \Sigma^+$ for $1 \leqq i \leqq n$ and $S = \bigcup_{i=1}^n E_i$). Then one gets a directed set $(\Pi_\rho, \leqq)$

by defining on the family Π_ρ the partial order as mentioned at the beginning of this section.

Now it is seen from the lifting properties (i)–(v) that for every $g \in L^1(\mu)$ we have $\lim_{\pi \in \Pi_\rho} g_\pi(s) = g(s)$ μ-a.e. on S. Hence Π_ρ satisfies condition (H) and all of the results obtained so far can be restated in terms of Π_ρ. In particular, the representation theorem for general vector measures due to Tulcea-Tulcea [11] is obtained from Lemma 3.2 in the following form (cf. [4], Theorem 5 in §13):

COROLLARY 3.4. *Let $\nu \in V_c^1(\mu, X^*)$, ρ any lifting for μ, and let ϕ_ν be the Π_ρ-generalized derivative of ν. Then $\nu(E) = (G) - \int_E \phi_\nu(s) d\mu$ for $E \in \Sigma$.*

In Theorem 3.3 we have assumed the existence of a family Π satisfying (H). However, the lifting theorem yields that there is at least one lifting for μ and the family Π_ρ which is defined through a lifting ρ and satisfies (H). In fact, if we employ such a particular family Π_ρ then we obtain stronger conclusions than those of Theorem 3.3.

THEOREM 3.5. *Let $\nu \in V_c^1(\mu, X^*)$, ρ a lifting for μ, and let ϕ_ν be the Π_ρ-generalized derivative of ν. Then there is a μ-null set N_ρ such that $\phi_\nu(s)$ is a singleton set $\{f(s)\}$ and*

$$(3.9) \qquad \lim_{\pi \in \Pi_\rho} \|f_\pi(s)\| = \|f(s)\| = \psi(s) \qquad \text{for} \quad s \in S - N_\rho.$$

Moreover $|\nu|(E) = \int_E \|f(s)\| d\mu$ for $E \in \Sigma$.

In order to prove this theorem we cite the following well-known result (see for instance [13]):

LEMMA 3.6. *Let $\varphi: S \to \boldsymbol{R}$ be μ-measurable and essentially bounded over S and $\rho(\varphi)$ the lifting for φ. Then $\lim_\pi \varphi_\pi(s) = \rho(\varphi)(s)$ for $s \in S$ and the convergence is uniform on S. In particular $\lim_\pi \varphi_\pi(s) = \varphi(s)$ μ-a.e. on S.*

PROOF OF THEOREM 3.5. In view of Theorem 3.3 and the fact that Π_ρ satisfies (H), it is sufficient to show that (3.9) holds. First it is seen that there exist sequences $(S_n)_{n=0}^\infty$ in Σ and $(M_n)_{n=1}^\infty$ in $(0, \infty)$ such that $\{S_n : n \geqq 0\}$ is a disjoint family, $\rho(S_n) = S_n$ for $n \geqq 1$, $\mu(S_0) = 0$, $\mu(S_n) > 0$ for $n \geqq 1$, $S = \cup_{n=0}^\infty S_n$, and $\|\nu(E)\| \leqq M_n \mu(E)$ for $E \in \Sigma(S_n)$ and $n \geqq 1$. Fix any $x \in X$. Let g_x denote the Radon-Nikodým derivative $d\langle \nu, x\rangle / d\mu$ of the scalar-valued measure $\langle \nu, x \rangle$ and let $g_x^0 = \sum_{n=1}^\infty \rho(g_x \cdot \chi_{S_n})$. Then we infer with the aid of Lemma 3.6 that $\lim_\pi \langle f_\pi(s), x \rangle = g_x^0(s)$ holds uniformly for $s \in S_n$ and $n \geqq 1$, and so the above convergence is valid pointwise on $S - S_0$. In view of this we show that $\phi_\nu(s)$ is a singleton set $\{f(s)\}$ for $s \in S - S_0$. Let $s \in S - S_0$ and $f(s)$ any weak*-cluster point of the net $(f_\pi(s))_{\pi \in \Pi}$. Then $\langle f(s), x \rangle = \lim_\pi \langle f_\pi(s), x \rangle = g_x^0(s)$ for $x \in X$. Hence $\langle g(s), x \rangle = g_x^0(s)$ for every $x \in X$ and every element $g(s)$ of the weak*-closed convex hull of weak*-cluster points of $(f_\pi(s))$. Therefore Theorem 2.1 implies that $\langle f(s), x\rangle = g_x^0(s)$ for $x \in X$ and $f(s) \in \phi_v(s)$. But this means that $\phi_v(s)$ can not contain two distinct points. Next Theorem 3.3 states that for every selection f of ϕ_v there is a μ-null set $\hat{N}_\rho$ such that

$$\|f(s)\| = \psi(s) = (d|v|/d\mu)(s) \qquad \text{for} \quad s \in S - \hat{N}_\rho \tag{3.10}$$

On the other hand, if $s \in S - S_0$ and $f(s)$ an arbitrary weak*-cluster point of the net $(f_\pi(s), \pi \in \Pi_\rho)$, then we have

$$|\langle f(s), x\rangle| = |g_x^0(s)| = \lim_{\pi \in \Pi_\rho} |\langle f_\pi(s), x\rangle| \leqq \|x\| \liminf_{\pi \in \Pi_\rho} \|f(s)\| .$$

Therefore, we get $\|f(s)\| \leqq \liminf_\pi \|f_\pi(s)\|$ for $s \in S - S_0$. Combining this with (3.2) as well as (3.10) we obtain

$$\lim_{\pi \in \Pi_\rho} \|f_\pi(s)\| = \|f(s)\| = \psi(s) \qquad \text{for} \quad s \in S - (S_0 \cup \hat{N}_\rho).$$

Now the set $N_\rho \equiv S_0 \cup \hat{N}_\rho$ is the desired μ-null set for (3.9). q.e.d.

REMARK. Given a lifting ρ for μ the Π_ρ-generalized derivative ϕ_v may be called a function of type ρ in the sense of Kupka; and conversely, every function of type ρ in the sense of Kupka is regarded as a selection of ϕ_v. This means that our notion of generlized derivative involves that of Kupka. On the other hand, Theorem 3.3 states that every selection of a Π-generalized derivative ϕ_v is strongly X-integrable in the sense of Kupka. It should be noted that a Π-generalized derivative ϕ_v is obtained without the lifting theorem and various properties of ϕ_v can be investigated in a rather elementary way.

4. Characterization of the space $V_c^1(\mu, X^*)$

Let X^* be the dual of a Banach space X and (S, Σ, μ) a finite, complete and nonnegative measure space. In this section we introduce a Lebesgue type space of weak*-integrable functions that characterizes the space $V_c^1(\mu, X^*)$ and then establish a generalized Radon-Nikodým theorem for general vector measures. We begin with the following.

DEFINITION 4.1. Given a function $f \in \mathfrak{G}(\mu, X^*)$, let $M_f(\cdot)\colon \Sigma \to [0, \infty)$ be the set function defined by (1.2). We denote by $|M_f|(S)$ the total variation of $M_f(\cdot)$; and we write $|M_f|(S) = +\infty$ if the total variation is infinite.

The functional $|M_f|(\cdot)$ defines an extended real-valued functional on the linear space $\mathfrak{G}(\mu, X^*)$. Let $f, g \in \mathfrak{G}(\mu, X^*)$. If $|M_f|(S) < \infty$ and g is weak*-equivalent to f, then $|M_f|(S) < \infty$ and $|M_f|(S) = |M_g|(S)$. Moreover, $|M_f|(S) = |M_g|(S)$ iff f is weak*-equivalent to g. This fact leads us to the following definition.

DEFINITION 4.2. The space $\mathscr{L}^1_G(\mu, X^*)$ is defined to be the subspace of $\mathfrak{G}(\mu, X^*)$ which consists of all elements f with $|M_f|(S)<\infty$. $L^1_G(\mu, X^*)$ is the space of weak*-equivalence classes in $\mathscr{L}^1_G(\mu, X^*)$. We write $\tilde{f}$ for the generic element of $L^1_G(\mu, X^*)$. If X^* and μ are fixed, we sometimes write $\mathscr{L}^1_G$ and L^1_G for $\mathscr{L}^1_G(\mu, X^*)$ and $L^1_G(\mu, X^*)$, respectively.

Let $f, g \in \mathfrak{G}(\mu, X^*)$. If f is weak*-equivalent to g and $f \in \mathscr{L}^1_G$, then $g \in \mathscr{L}^1_G$. Note that every pair f, g in $\mathscr{L}^1_G$ defines the same G-integral $\nu(E)$ for $E \in \Sigma$ iff both f and g belong to the same weak*-equivalence class $\tilde{f} \in L^1_G$. Therefore, to every $\tilde{f}$ in L^1_G and every $E \in \Sigma$, there corresponds a unique G-integral $\nu(E)$ in such a way that ν is the idefinite G-integral of every element f in $\tilde{f}$. Accordingly, ν is said to be the indefinite integral of $\tilde{f}$. Moreover, L^1_G forms a linear space in a natural way. We then set

$$(4.1) \qquad \|f\|_{G,1} = |M_f|(s) \qquad \text{for} \quad f \in \tilde{f} \quad \text{and} \quad \tilde{f} \in L^1_G.$$

In virtue of the remarks after Definition 4.1, it is seen that the functional $\|\cdot\|_{G,1}$ is well-defined as a real-valued functional on L^1_G and gives a norm on L^1_G. We then discuss the relation of the Gel'fand integral to the Bochner integral. First we need the following proposition.

PROPOSITION 4.3. *Given an $f \in \mathscr{L}^1_G$, let $\nu(E)=(G)-\int_E f d\mu$ for $E \in \Sigma$ and ψ_f the function defined for f through* (3.6) *and* (3.7). *Then we have*

$$|\nu|(E) = |M_f|(E) = \int_E \psi_f(s) d\mu \qquad \textit{for} \quad E \in \Sigma.$$

Consequently $\nu \in V^1_c(\mu, X^*)$ and $\psi_g = d|\nu|/d\mu$ a.e. for every $g \in \tilde{f}$.

PROOF. If $E \in \Sigma$, then we have $|\langle \nu(E), x\rangle| \leqq \int_E |\langle f(s), x\rangle| d\mu \leqq \|x\| \int_E \psi(s) d\mu = \|x\|\, |\nu|(E)$ for $x \in X$, where the last inequality follows from Musiał [14], Proposition 1. Hence $|\nu|(E) \leqq M_f(E) \leqq \int_E \psi_f(s) d\mu = |\nu|(E)$ for each $E \in \Sigma$, so that we have $|\nu|(E) = |M_f|(E) = \int_E \psi_f(s) d\mu$ for each $E \in \Sigma$. q.e.d.

Now let $f \in \mathscr{L}^1_B(\mu, X^*)$ and $g \in \mathfrak{G}(\mu, X^*)$. If g is weak*-equivalent to f, then $(G)-\int_E g(s) d\mu = (B)-\int_E f(s) d\mu$ and $g \in \mathscr{L}^1_G$, and so $f \in \mathscr{L}^1_G$ and $|M_f|(S) = \int_E \|f(s)\| d\mu = \|f\|_1$ from the above proposition. This means that $\mathscr{L}^1_B \subset \mathscr{L}^1_G$ and that $L^1_B(\mu, X^*)$ is embedded isometrically and isomorphically in $L^1_G(\mu, X^*)$. It is well-known as the Bochner theorem that $f \in \mathscr{L}^1_B$ iff f is strongly μ-measurable and $\|f(\cdot)\|$ is μ-integrable over S. However, it should be noted that $\|f(\cdot)\|$ need not be μ-measurable even if $f(\cdot)$ is Pettis integrable. See Pettis [15], Example 9.1. As will be seen in later sections, the generalized derivative of an arbitrary vector measure in $V_c^1(\mu, X^*)$ can be treated within the framework of the space $\mathscr{L}_G^1$. This means that $\mathscr{L}_G^1$ is in general much larger than the space $\mathscr{L}_B^1$.

Theorem 3.5 and Proposition 4.3 together imply the following.

PROPOSITION 4.4. *Any weak*-equivalence class $\tilde{f} \in L_G^1$ contains at least one element f such that $\|f(\cdot)\|$ is μ-measurable.*

PROPOSITION 4.5. *The space L_G^1 is a Banach space under the norm $\|\cdot\|_{G,1}$.*

PROOF. It has already been observed that L_G^1 is a normed space. Hence it remains to prove that L_G^1 is complete. Let (f_n) be a sequence in $\mathscr{L}_G^1$ such that $|M_{f_m-f_n}|(S)\to 0$ as $m, n\to\infty$ and write $v_n(E)=(G)-\int_E f_n d\mu$ for $E\in\Sigma$. Then $|M_{f_m-f_n}|(S)=\|v_m-v_n\|_1=|v_m-v_n|(S)\to 0$ as $m, n\to\infty$ by Proposition 4.3. On the other hand, $V_c^1(\mu, X^*)$ is a Banach space with norm $\|v\|_1=|v|(S)$, so that there exists $v\in V_c^1(\mu, X^*)$ such that $\|v_n-v\|_1\to 0$ as $m, n\to\infty$. We now observe that there exists a weak*-integrable function f such that $v(E)=(G)-\int_E fd\mu$ for $E\in\Sigma$ by Corollary 3.4. Then we have $f\in\mathscr{L}_G^1$ since $\int_E |\langle x, f\rangle| d\mu \leqq \|x\|\,|v|(E)$ for $x\in X$ and $E\in\Sigma$; and thus $|M_{f_n-f}|(S)=\|v_n-v\|_1\to 0$ as $n\to\infty$. q.e.d.

THEOREM 4.6. *Let T: $L_G^1(\mu, X^*)\to V_c^1(\mu, X^*)$ be defined by the relation $T\tilde{f}= v$ for $\tilde{f}\in L_G^1(\mu, X^*)$, where v is the indefinite G-integral of $\tilde{f}$. Then T gives an isometric isomorphism between $L_G^1(\mu, X^*)$ and $V_c^1(\mu, X^*)$.*

PROOF. In virtue of Proposition 4.3 it suffices to show that T is onto. But the proof is involved in that of Proposition 4.5. q.e.d.

Theorem 4.6 is rewritten in the following form:

THEOREM 4.7 (*The generalized Radon-Nikodým theorem*). *Let v be an X^*-valued measure on Σ. Then $v\in V_c^1(\mu, X^*)$ if and only if there exists a weak*-integrable function f: $S\to X^*$ such that* (i) *there is an element $\tilde{f}\in L_G^1$ with $f\in\tilde{f}$;* (ii) *v is the indefinite Gel'fand integral of f; and* (iii) *we have $|v|(E)=\int_E \|f(s)\|d\mu$ for $E\in\Sigma$.*

5. Characterization of the dual spaces of $L_G^p(\mu, X)$, $1\leqq p<\infty$

In addition to the spaces $\mathscr{L}_G^1(\mu, X^*)$ and $L_G^1(\mu, X^*)$ other Lebesgue type spaces can be defined: Let $1<q<\infty$. The space $\mathscr{L}_G^q(\mu, X^*)$ is the subspace of $\mathfrak{G}(\mu, X^*)$ which consists of all elements f with $|M_f|_q(S)<\infty$; and $L_G^q(\mu, X^*)$ denotes the space of weak*-equivalence classes in $\mathscr{L}_G^q(\mu, X^*)$. Let $q=\infty$. The space $\mathscr{L}_G^\infty(\mu, X^*)$ is the subspace of $\mathfrak{G}(\mu, X^*)$ which consists of those elements f such that $\|f(\cdot)\|$ is essentially bounded on S; and $L_G^\infty(\mu, X^*)$ is the space of weak*-equivalence classes in $\mathscr{L}_G^\infty(\mu, X^*)$. In this section we discuss the relationship between the spaces $V^q(\mu, X^*)$ and $L_G^p(\mu, X^*)$ and characterize the dual spaces of the spaces $L_B^q(\mu, X)$. We first show the following result which extends Proposition 4.3.

PROPOSITION 5.1. *Let $1 \leqq p < \infty$. Given an $f \in \mathscr{L}_G^p$, let $\nu(E) = (G) - \int_E f d\mu$ for $E \in \Sigma$ and $\psi = d|\nu|/d\mu$. Then $|\nu|_p(E)^p = |M_f|_p(E)^p = \int_E \psi(s)^p d\mu$ for $E \in \Sigma$ Therefore, $\psi_g = \psi$ μ-a.e. for every $g \in \tilde{f}$.*

PROOF. Let $1 \leqq p < \infty, f \in \mathscr{L}_G^p$, $x \in X$, and $E \in \Sigma$. Then $|\langle \nu(E), x\rangle| \leqq \int_E |\langle f(s), x\rangle| d\mu \leqq \|x\| \int_E \psi_f(s) d\mu \leqq \|x\|\, |\nu|(E)$ by Proposition 4.3. Therefore $|\nu(E)| \leqq M_f(E) \leqq \int_E \psi_f(s) \leqq |\nu|(E)$ and so $|\nu|_p(E)^p \leqq |M_f|_p(E)^p \leqq \int_E \psi(s)^p d\mu \leqq |\nu|_p(E)^p$, where $\lambda = |\nu|$. Let $\psi = d\lambda/d\mu$; then it is easy to verify that $|\nu|_p(E) = \left(\int_E \psi(s)^p d\mu\right)^{1/p}$ for every $E \in \Sigma$. We then demonstrate that $|\nu|_p(E) = |\lambda|_p(E)$ for $E \in \Sigma$. Since it is obvious that $|\nu|_p \leqq |\lambda|_p$, we will show that $|\nu|_p \geqq |\lambda|_p$. Given an $\varepsilon > 0$ there exists a finite partition $\{E_k\colon k = 1,\dots, n\}$ with $E = \cup_{k=1}^n E_k$ and we have

$$|\lambda|_p(E)^p - \varepsilon/2 < \textstyle\sum_{k=1}^n |\nu|(E_k)^p/\mu(E_k)^{p-1}. \tag{5.1}$$

Moreover, for each $k \geqq 1$, there exists a partition $\{E_{k,i}\colon i = 1,\dots, i_k\}$ of E_k satisfying

$$|\nu|(E_k)^p \leqq (\textstyle\sum_{i=1}^{i_k} \|\nu(E_{k,i})\|)^p + \varepsilon\mu(E_k)^{p-1}/2n. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\begin{aligned} |\lambda|_p(E)^p - \varepsilon &< \textstyle\sum_{k=1}^n (\sum_{i=1}^{i_k} \|\nu(E_{k,i})\|)^p/(\sum_{i=1}^{i_k} \mu(E_{k,i}))^{p-1} \\ &\leqq \textstyle\sum_{k=1}^n (\sum_{i=1}^{i_k} \|\nu(E_{k,i})\|^p/\mu(E_{k,i})^{p-1}) \leqq |\nu|_p(E)^p. \end{aligned}$$

Hence we obtain $|\lambda|_p(E) \leqq |\nu|_p(E)$ for each $E \in \Sigma$. Consequently, $|\nu|_p(E)^p \leqq |M_f|_p(E)^p \leqq \int_E \psi_f(s)^p d\mu = \int_E h(s)^p d\mu = |\nu|_p(E)^p$ for every $E \in \Sigma$, and $|\nu|_p(E)^p = |M_f|_p(E)^p = \int_E \psi_f(s)^p d\mu$ for every $E \in S$. q.e.d.

PROPOSITION 5.2. *Let $1 \leqq p < \infty$. Then $L_G^p(\mu, X^*) = \{\tilde{f} \in L_G^1(\mu, X^*)\colon \|f(\cdot)\| \in L^p(\mu)$ and $|M_f|_p(E)^p = \int_E \|f(s)\|^p d\mu$ for $E \in \Sigma$ and some $f \in \tilde{f}\}$.*

PROOF. Let F be the space of those elements $\tilde{f}$ in $L_G^1(\mu, X^*)$ with the property that $\|f(\cdot)\| \in L^p(\mu)$ and $|M_f|(E) = \int_E \|f(s)\| d\mu$ for $E \in \Sigma$ and some $f \in \tilde{f}$. We shall show that $L_G^p(\mu, X^*) = F$. Let $\tilde{f} \in F$ and f an element of $\tilde{f}$ such that $\|f(\cdot)\| \in L^p(\mu)$ and $|M_f|(E) = \int_E \|f(s)\| d\mu$ for $E \in \Sigma$. Then for every $x \in X$ we have $\int_E |\langle x, y\rangle| d\mu \leqq \|x\| \int_E \|f(s)\| d\mu \leqq \|x\| \mu(E)^{1/q} \left(\int_E \|f(s)\|^p d\mu\right)^{1/p}$ for $E \in \Sigma$. Hence

$M_f(E) \leqq \mu(E)^{1/q} \int_E \|f(s)\|^p d\mu)^{1/p}$ for $E \in \Sigma$, so that $M_f(E)^p/\mu(E)^{p-1} \leqq \int_E \|f(s)\|^p d\mu$ for every $E \in \Sigma$. Therefore, we have

$$|M_f|_p(E)^p \leqq \int_E h^p d\mu \qquad \text{for} \quad E \in \Sigma.$$

Hence $|M_f|_p(S) < \infty$ and so $\tilde{f} \in L^p_G(\mu, X^*)$. This means that $F \subset L^p_G(\mu, X^*)$. To prove the reverse inclusion, let $\tilde{f} \in L^p_G(\mu, X^*)$. It is easy to see that $\mathscr{L}^p_G(\mu, X^*) \subset \mathscr{L}^1_G(\mu, X^*)$ for $p \geqq 1$. Hence, if v is the indefinite Gel'fand integral of $\tilde{f}$ and f_0 is any selection of ϕ^ρ_v (ρ being a lifting for μ), then Theorem 3.5 yields that $\|f_0(\cdot)\| \in L^1(\mu)$ and $|v|(E) = \int_E \|f_0(s)\| d\mu$ for every $E \in \Sigma$. Moreover, it is seen from the proof of Proposition 5.1 that $\|f_0(\cdot)\| \in L^p(\mu)$ and $|v|_p(E) = |M_{f_0}|_p(E) = \left(\int_E \|f_0(s)\|^p d\mu\|\right)^{1/p}$. q.e.d.

The following result gives a refinement of the representation theorem for the dual spaces of $L^p_B(\mu, X)$ due to Bochner and Taylor.

THEOREM 5.3. *Let $1 \leqq p < \infty$. Then $L^p_B(\mu, X)^* = L^q_G(\mu, X^*)$, where $1/p + 1/q = 1$ if $p > 1$ and $q = \infty$ if $p = 1$.*

PROOF. Let $1 \leqq p < \infty$ and $1/p + 1/q = 1$ (or $q = \infty$ if $p = 1$). In view of Theorem 1.2, it is sufficient to show that $V^q(\mu, X^*)$ is isometrically isomorphic to $L^q_G(\mu, X^*)$. First we consider the case $p = 1$ and let $\tilde{f} \in L^\infty_G(\mu, X^*)$. Then by virtue of Propositions 5.1 and 5.2 one finds an $f_0 \in \tilde{f}$ such that $\|f_0(\cdot)\| \in L^1(\mu)$ and $|v|(E) = |M_f|(E) = \int_E \|f_0(s)\| d\mu$, where v is the indefinite Gel'fand integral of f_0. Also, we see from the proof of Proposition 4.3 that $\|v(E)\| \leqq M_{f_0}(E) \leqq \int_E \psi_{f_0}(s) d\mu \leqq |v|(E)$ for every $E \in \Sigma$. Noting that $\sup_{E \in \Sigma} \|v(E)\|/\mu(E) = \sup_{E \in \Sigma} |v|(E)/\mu(E)$, we have $\|f\|_{G,\infty} = \sup_{E \in \Sigma} M_{f_0}(E)/\mu(E) = \|v\|_\infty$. Next, let $1 < p < \infty$, $1/p + 1/q = 1$, and $\tilde{f} \in L^q_G(\mu, X^*)$. Then by Proposition 5.2 there exists an element $f_0 \in \tilde{f}$ such that $\|f_0(s)\| \in L^q(\mu)$ and $|v|_q(E)^q = |M_f|(E)^q = \int_E \psi_f(s)^q d\mu$ for each $E \in \Sigma$, where v denotes the indefinite Gel'fand integral of f_0. Hence $\|\tilde{f}\|_{G,q} = |v|_q(S) = \|v\|_q$. Consequently $L^q(\mu, X^*)$ is isometrically embedded in $V^q(\mu, X^*)$ for $1 < q \leqq \infty$. To prove that $L^q(\mu, X^*)$ is isometrically isomorphic to $V^q(\mu, X^*)$, let $v \in V^q(\mu, X^*)$. Since $V^q(\mu, X^*) \subset V^1_c(\mu, X^*)$ as mentioned before Lemma 1.3, Theorem 3.5 yields that there exists an $f \in \mathfrak{G}(\mu, X^*)$ such that $\|f(\cdot)\| \in L^1(\mu)$ and $|v|(E) = \int_E \|f(s)\| d\mu$ for each $E \in \Sigma$; and by an argument similar to the proof of Proposition 5.2 we have $|v|_q(E)^q = \int_E \|f(s)\|^q d\mu$ for each $E \in \Sigma$. Now let $\tilde{f}$ be the weak*-equivalence class in $\mathfrak{G}(\mu, X^*)$ containing f. Then $\tilde{f} \in L^q_G(\mu, X^*)$ and $\|\tilde{f}\|_{G,q} = |v|_q(S) = \|v\|_q$. q.e.d.

The next result is a counterpart of Theorem 5.3.

PROPOSITION 5.4. *$L^1_G(\mu, X^*)$ is isometrically embedded in $L^\infty_B(\mu, X^*)$.*

PROOF. Let $\tilde{f} \in L^1(\mu, X^*)$. Then, by Theorem 4.7, we can choose an element $f \in \tilde{f}$ such that $\|f(\cdot)\| \in L^1(\mu)$ and $\|\tilde{f}\|_{G,1} = \int_S \|f(s)\| d\mu$. Also the real-valued function $\langle f(\cdot), g(\cdot)\rangle$ is μ-integrable for every $g \in \mathscr{L}^\infty_B(\mu, X)$. We then define a linear functional T over $\mathscr{L}^\infty_B(\mu, X)$ by $Tg = \int_S \langle f, g\rangle d\mu$ for $g \in \mathscr{L}^\infty_B(\mu, X)$. Then we have $|Tg| \leqq \int_S |\langle f(s), g(s)\rangle| d\mu \leqq \int_S \|f(s)\| \|g(s)\| d\mu \leqq \|\tilde{f}\|_{G,1} \|\tilde{g}\|_\infty$ for each $\tilde{g} \in L^\infty_B(\mu, X)$, and so $\|T\| \leqq \|\tilde{f}\|_{G,1}$. To show the reverse inequality, let ν be the indefinite Gel'fand integral of $\tilde{f}$. Then we get $|\nu|(S) = \|\tilde{f}\|_{G,1}$ by Theorem 4.7. Let $g = \sum_{i=1}^n x_k \chi_{E_k}$, where $\|x_k\| \leqq 1$ $(k = 1, 2, \ldots)$ and $E_i \cap E_j = \emptyset$ $(i \neq j)$. Then $\|g\|_\infty \leqq 1$ and $\sum_{i=1}^n \langle \nu(E_i), x_i\rangle = \int_S \langle \sum_{k=1}^n x_k \chi_{E_k}, f\rangle d\mu = Tg \leqq \|T\|$. Taking appropriate suprema yields that $|\nu|(S) \leqq \|T\|$, which show that $\|\tilde{f}\|_{G,1} \leqq \|T\|$. q.e.d.

6. Applications to Bochner and Pettis integrals

Let X be a Banach space that has no preduals and let ν be an X-valued measure belonging to $V^1_c(\mu, X)$. If ν is the indefinite Bochner integral of an X-valued, strongly μ-measurable function f on S, we call f a Bochner derivative of ν. Likewise, if ν is the indefinite Pettis integral of an X-valued, weakly μ-measurable function g on S, we call g a Pettis derivative of ν. The aim of this section is to discuss Bochner and Pettis derivatives of vector measures from the point of view of our generalized derivatives. In this section we are concerned with the applications of generalized derivatives defined through liftings for μ to those typical derivatives. Let ρ be any lifting for μ, Π_ρ the family of all $\rho(\Sigma^+)$-partitions of S, and ϕ_ν the Π_ρ-generalized derivative as mentioned in Section 3.3. We shall see that any Bochner derivative of ν is regarded as a selection of ϕ_ν, while ϕ_ν never contains a Pettis derivative as its selection; and this phenomenon will be discussed to some extent in the latter half of this section.

In what follows, we sometimes regard the Π_ρ-generalized derivative ϕ_ν as an (X^{**}-valued) *single-valued* function for convenience in notation, since $\phi_\nu(s)$ is a singleton set for a.e. $s \in S$ by Theorem 3.5.

LEMMA 6.1. *Let $\nu \in V^1_c(\mu, X)$, ρ a lifting for μ, and let ϕ_ν be the Π_ρ-generalized derivative of ν. Suppose that $\nu(\Sigma)$ is separable and $\phi_\nu(s) \in X$ μ-a.e. on S. Then ϕ_ν is Bochner integrable and gives a Bochner derivative of ν.*

PROOF. Let X_0 be the closed linear span of $\nu(\Sigma)$. Then X_0 is separable. Now given a $\pi \in \Pi_\rho$ let f_π be defined by (3.1). Then $f_\pi(s) \in X_0$ for $s \in S$ and we see in the same way as in the proof of Theorem 3.5 that there is a μ-null set S_0

such that weak*-$\lim_{\pi\in\Pi_\rho} f_\pi(s)=\phi_v(s)$ for $s\in S-S_0$. But this means that $\phi_v(s)$ is a weak limit in X of $(f_\pi(s))$ since $\phi_v(s)\in X$, and so $\phi_v(s)$ belongs to the separable closed subspace X_0 for a.e. $s\in S$. From this it follows that ϕ_v is essentially separably-valued. Moreover, it is clear that ϕ_v is weakly measurable. Hence ϕ_v is strongly μ-measurable by the Pettis measurability theorem. On the other hand, $\|\phi_v(\cdot)\|$ is μ-integrable by Theorem 3.5. Therefore the Pettis theorem implies that ϕ_v is Bochner integrable over S and the Gel'fand integral of ϕ_v turns out to be the Bochner integral. In other words, ϕ_v gives a Bochner derivative of v. q.e.d.

LEMMA 6.2. *Let f be an X-valued, Bochner integrable function on S and Π a family of Σ^+-partitions of S satisfying (H). Then we have*

$$\lim_\pi \|f_\pi(s)-f(s)\| = 0 \quad \mu\text{-a.e.}. \tag{6.1}$$

PROOF. Since f is essentially separably-valued, there exists a μ-null set N_0 such that $f(S-N_0)$ is separable. Let $\{x_n\colon n\geqq 1\}$ be a dense subset of $f(S-N_0)$ and put $g^n(s)=\|f(s)-x_n\|$ for $s\in S$ and $n\geqq 1$. Then $g^n\in L^1(\mu)$ for all $n\geqq 1$ since $f\in\mathscr{L}^1_B(\mu, X)$. Therefore condition (H) implies that for each $n\geqq 1$ there is a μ-null set N_n such that the convergence

$$\lim_\pi g^n_\pi(s) = g^n(s) \tag{6.2}$$

holds for $s\in S-N_n$. Set $N=\cup_{n=1}^\infty N_n$. Then $\mu(N)=0$ and (6.2) holds for $s\in S-N$ and $n\geqq 1$. Now let $\varepsilon>0$ and $s\in S-N$. First we observe that $g^n(s)=\|f(s)-x_n\|<\varepsilon/2$ for some $n\geqq 1$. Hence, if $\pi\in\Pi$ and $s\in E$ for some $E\in\pi$ then we have

$$\begin{aligned}\Big(\int_E \|f(t)-f(s)\|\mu(dt)\Big)/\mu(E) &\leqq \Big(\int_E \|f(t)-x_n\|\mu(dt)\Big)/\mu(E)\\ &+\Big(\int_E g^n(s)\mu(dt)\Big)/\mu(E) < g^n(s)+\varepsilon/2;\quad\text{and so}\\ \|f_\pi(s)-f(s)\| &\leqq \textstyle\sum_{E\in\pi}\Big(\Big(\int_E \|f(s)-f(t)\|\mu(dt)\Big)/\mu(E)\Big)\cdot\chi_E(s)\\ &< g^n_\pi(s)+\varepsilon/2\end{aligned}$$

for every $\pi\in\Pi$. From this it follows that

$$\begin{aligned}\limsup_\pi \|f_\pi(s)-f(s)\| &\leqq \lim_\pi g^n_\pi(s)+\varepsilon/2\\ &= g^n(s)+\varepsilon/2<\varepsilon.\end{aligned}$$

Since ε is arbitrary, we obtain the convergence (6.1) for $s\in S-N$. q.e.d.

We now state the main theorem of this section.

THEOREM 6.3. *Let $v\in V^1_c(\mu, X)$, ρ a lifting for μ, and let ϕ_v be the Π_ρ-*

generalized derivative of ν. Then the following are equivalent:

(i) *ν is an indefinite Bochner integral*;

(ii) *$\phi_\nu(s) \subset X$ for μ-a.e. $s \in S$*;

(iii) *there exists a strongly μ-measurable function $f: S \to X$ such that f is X^*-equivalent to ϕ_ν viewed as a single-valued function*;

(iv) *every element $E \in \Sigma^+$ has a subset F such that $F \in \Sigma^+$ and there exists a weakly compact convex subset K_F of X^{**} with the property that for each $x^* \in X^*$, $\langle \phi_\nu(s), x^* \rangle \leqq \sup \{\langle x^{**}, x^* \rangle : x^* \in K_F\}$ μ-a.e. on F.*

PROOF. First assume that (i) holds and let f be a Bochner derivative of ν. Since Π_ρ satisfies condition (H) as mentioned in Section 3.3, we infer from Lemma 6.2 that $f(s) \in \phi_\nu(s)$ for a.e. $s \in S$. This means that (ii) is satisfied. To prove that (ii) implies (i), it suffices to show with the aid of Lemma 6.1 that $\nu(\Sigma)$ is relatively norm-compact. To this end, let $\{\nu(E_n): n=1, 2,\dots\} \subset \nu(\Sigma)$, where we may suppose that $E_n \in \rho(\Sigma^+)$ for $n \geqq 1$ without loss of generality. We shall show that $\{\nu(E_n): n=1, 2,\dots\}$ has a convergent subsequence. Let $S_0 = \bigcup_{n=1}^\infty E_n$, Σ_0 the σ-field on S_0 generated by $\{E_n: n=1, 2,\dots\}$; $\mu_0 = \mu \mid \Sigma_0$, the restriction of μ to Σ_0; $\nu_0 = \nu \mid \Sigma_0$, the restriction of ν to Σ_0; and let $\rho_0 = \rho \mid \Sigma_0$, the restriction of ρ on Σ_0. Then ρ_0 is a lifting for μ_0; $\rho_0(\Sigma_0^+) = \rho(\Sigma_0^+) \subset \rho(\Sigma^+)$; $X_0 = \overline{\mathrm{sp}}\,[\nu(\Sigma_0)]$ is a separable Banach space; and $\nu_0 \in V_c^1(\mu_0, X_0)$. Therefore the Π_{ρ_0}-generalized derivative ϕ_{ν_0} of ν_0 is defined and $\phi_\nu(s) \subset \phi_{\nu_0}(s)$ μ_0-a.e. on S_0. Therefore $\phi_{\nu_0}(s) \subset X$ μ_0-a.e. on S_0 by (ii). Since $\nu_0(\Sigma_0)$ is separable, we infer from Lemma 6.1 that there exists a Bochner integrable function $f_0: S_0 \to X_0$ such that $\nu_0(E) = (B) - \int_E f_0 d\mu_0$ for every $E \in \Sigma_0$. Since the range of an indefinite Bochner integral is relatively norm-compact, so is $\nu_0(\Sigma_0)$. This $\nu_0(\Sigma_0)$ contains the original sequence $\{\nu(E_n): n=1, 2,\dots\}$, which therefore has a convergent subsequence. Thus (i) is obtained. It is easy to show with the aid of Lemma 6.2 that (i) implies (iii). Suppose then that (iii) holds. Then the function f is a Pettis derivative of ν that is strongly μ-measurable. Therefore it is proved in the same way as in [3], Theorem III.2.6 on page 71 that f becomes a Bochner derivative of ν. This means that (i) follows from (iii). In order to prove the equivalence between (iii) and (iv), we recall a result due to Uhl, Jr. [20]. If (iii) holds then ϕ_ν viewed as single-valued function is X^*-equivalent to f, and so Theorem 1 in [20] implies that condition (iv) holds for ϕ_ν. Conversely, assume (iv) is satisfied. Then Theorem 1 of [20] again yields that ϕ_ν is X^*-equivalent to some strongly μ-measurable function $g: S \to X^*$. Hence ν becomes the indefinite Bochner integral of g, and it turns out that $g(s) \in X$ for μ-a.e. $s \in S$. q.e.d.

COROLLARY 6.4. *Let $\nu \in V_c^1(\mu, X)$, ρ a lifting for μ, and let ϕ_ν be the Π_ρ-generalized derivative of ν. If $\phi_\nu(s)$ intersects X for μ-a.e. $s \in S$, then ϕ_ν gives a Bochner derivative of ν and $\lim_{\pi \in \Pi_\rho} f_\pi(s) = \phi_\nu(s)$ in norm for μ-a.e. $s \in S$.*

PROOF. Let $\phi_v(s) \in X$ for μ-a.e. $s \in S$. Then ϕ_v becomes a Bochner integrable derivative of v by Theorem 6.3. Hence it follows from Lemma 6.2 that $\lim_\pi f_\pi(s) = \phi_v(s)$ holds in norm for μ-a.e. $s \in S$. q.e.d.

The above results suggest several points in the theory of Pettis integration: Firstly the equivalence between conditions (i) and (ii) mentioned in Theorem 6.3 yields the following.

COROLLARY 6.5. *Let $v \in V_c^1(\mu, X)$ and ϕ_v the Π_ρ-generalized derivative of v. If v has a Pettis derivative $f: S \to X$ that is not Bochner integrable on each $E \in \Sigma^+$, then $f(s) \notin \phi_v(s)$ for a.e. $s \in S$.*

The above proposition states that a Pettis derivative is not necessarily determined by the so-called average range of its indefinite Pettis integral. In other words, a proper Pettis derivative is inaccessible in the sense of Kupka [13]; and it seems to the authors that this fact makes it difficult to give any useful characterization of Pettis integrable functions. Extremely speaking, the definition of Pettis integral might be of use for the well-shaped integral representation of vector measures, rather than the differentiation theory for vector measures. Nevertheless, it is interesting to investigate as to when a selection of X^{**}-valued derivative ϕ_v is weak*-equivalent to some X-valued weakly measurable function.

Recently Geitz [7] has shown that if (S, Σ, μ) is a perfect measure space, then a norm-bounded weakly integrable function $f: S \to X$ is Pettis integrable iff there exists a sequence (f_n) of X-valued simple functions on S such that

$$\langle f_n, x^* \rangle \longrightarrow \langle f, x^* \rangle \ \mu\text{-a.e.} \qquad \text{for each } x^* \in X^*, \tag{6.3}$$

where the null set (hereafter called the exceptional set) on which the convergence (6.3) does not hold may vary with x^*. Such sequential approximation property for vector-valued functions is closely related to the weak Randon-Nikodým property for Banach spaces. See Hashimoto [9], Section 4.

On the other hand, if the exceptional set for (6.3) is independent of $x^* \in X^*$ then the function f is necessarily Bochner integrable over S. Combining this fact with Theorem 6.3 and Corollary 6.5, we obtain the following result which gives an aspect of the above results:

COROLLARY 6.6. *Let $f: S \to X$ be a Pettis integrable function, $v: \Sigma \to X$ the indefinite Pettis integral of f, and let ρ be a lifting for μ. Let f_π, $\pi \in \Pi_\rho$, be X-valued simple functions defined by* (3.1). *If there is a countable sequence (π_n) in Π_ρ such that $f_n \equiv f_{\pi_n}$, $n \geqq 1$, satisfy* (6.3) *and the exceptional set for* (6.3) *is independent of $x^* \in X^*$, then f is Bochner integrable over S and $\phi_v(s) = \{f(s)\}$ for μ-a.e. $s \in S$.*

REMARKS. A more precise aspect of the above-mentioned conspicuous phenomena (all of which show the contrast between the Bochner and Pettis integrals) may be obtained by investigating the generalized derivatives of dual-Banach-space-valued measures.

Let X be the dual of another Banach space Y, i.e. $X=Y^*$, and let $\nu\in V_c^1(\mu, X)$. Then given a lifting ρ for μ two sorts of generalized derivatives are defined for ν. The first one is the X-valued, Π_ρ-derivative ϕ_ν defined by (3.4), i.e., $\phi_\nu(s)=\bigcap_{\pi\in\Pi_\rho}\overline{\mathrm{co}}^{\sigma(X,Y)}\{f_{\pi'}(s)\colon \pi'\geqq\pi\}$; and the other one is the X^{**}-valued derivative, say $\tilde{\phi}_\nu$, of ν viewed as the X^{**}-valued measure, namely:

$$\tilde{\phi}_\nu(s)=\bigcap_{\pi\in\Pi_\rho}\overline{\mathrm{co}}^{\sigma(X^{**},X^*)}\{f_{\pi'}(s)\colon \pi'\geqq\pi\},\qquad \text{for}\quad s\in S.$$

Apparently, $\tilde{\phi}_\nu$ coincides on Y with ϕ_ν μ-a.e. on S. Moreover, if $\tilde{\phi}_\nu(s)$ intersects X for a.e. $s\in S$ then $\tilde{\phi}_\nu$ is regarded as a Bochner derivative of ν by Corollary 6.4 and therefore coincides with ϕ_ν. However $\tilde{\phi}_\nu$ is in general completely distinct from ϕ_ν even if ϕ_ν is viewed as an X^{**}-valued function and ν is the indefinite Pettis integral of a weakly measurable function $f\colon S\to X$. In fact, suppose that ν is the indefinite Pettis integral of a function $f\colon S\to X$ that is not Bochner integrable on each $E\in\Sigma^+$. Then $f(s)\notin\tilde{\phi}_\nu(s)$ for μ-a.e. $s\in S$ by Corollary 6.5. But, as mentioned in the proof of Lemma 6.1, $\lim_{\pi\in\Pi_\rho}f_\pi(s)$ exists for μ-a.e. $s\in S$ in the sense of the weak*-topology of X and $\phi_\nu(s)=\{f(s)\}$ for μ-a.e. $s\in S$. Thus $\tilde{\phi}_\nu(s)\cap\phi_\nu(s)=\emptyset$ for μ-a.e. $s\in S$.

Finally, as a simple consequence of the above observation, we get a specific result which contrasts with Corollary 6.5.

PROPOSITION 6.7. *Let $\nu\in V_c^1(\mu, X)$ and ρ a lifting for μ. Suppose that $X(=Y^*)$ has the weak Radon-Nikodým property. Then the Π_ρ-derivative ϕ_ν gives a Pettis derivative of ν.*

Added in Proof: After this paper was submitted for publication, the authors were called their attention to the monograph of M. Sion entitled "A theory of semigroup-valued measures" (Lect. Notes in Math., Springer-Verlag, 355 (1973)), in which a general theory of differentiation for semigroup-valued measures is developed by using the concept of Vitali system. He introduced a notion of "outer derivative" for general vector measures through a differentiation basis and gave fundamental properties of such derivatives. Although his approach is different from ours, his results are closely related to the present work in some respects. For instance it is seen from Corollary 3.4 that an outer derivative in the sense of Sion of $\nu\in V_c^1(\mu, X^*)$ is weak*-equivalent to a Π-generalized derivative of ν if the Vitali system and the family Π of partitions are defined through a lifting ρ. For details concerning the relations of our results to the work of Sion, we shall publish them elsewhere.

References

[1] S. Bochner and A. Taylor, Linear functionals on certain spaces of abstractly-valued functions, Ann. of Math., (2) **39** (1938), 913–944.

[2] J. Diestel and J. Uhl, Jr., The Radon-Nikodým theorem for Banach space valued measures, Rocky Mountain J. Math., **6** (1) (1976), 1–46.

[3] J. Diestel and J. Uhl, Jr., Vector measures, Math. Surveys 15, Amer. Math. Soc., (1977).

[4] N. Dinculeanu, Vector measures, Int. Series of Monographs in Pure and Appl. Math., 95, Pergamon Press, (1967).

[5] N. Dunford and B. Pettis, Linear operators on summable functions, Trans. Amer. Math. Soc., **47** (1940), 323–392.

[6] N. Dunford and J. Schwartz, Linear operators, Part I, Interscience, New York, (1958).

[7] R. Geitz, Pettis integration, Proc. Amer. Math. Soc., **82** (1981), 81–86.

[8] I. Gel'fand, Abstrakte Funktionen und lineare Operatoren, Mat. Sb. (N. S.), **4** (46) (1938), 235–286.

[9] K. Hashimoto, On the sequential approximation of scalarly measurable functions by simple functions, to appear in Tokyo J. Math..

[10] K. Hashimoto, S. Oharu and Y. Shimohira, Some theorems and counterexamples in the theory of differentiation for vector-valued functions, in preparation.

[11] A. Ionescu Tulcea and C. Ionescu Tulcea, On the lifting property II. Representation of linear operators on spaces L_E^r, $1 \leqq r < \infty$, J. Math. Mech., **11** (1962), 773–795.

[12] L. Janicka, Some measure theoretical characterization of Banach spaces not containing l^1, Bull. Acad. Polon. Sci., **27** (1979), 561–565.

[13] J. Kupka, Radon-Nikodým theorems for vector valued measures, Trans. Amer. Math. Soc., **169** (1972), 197–217.

[14] K. Musiał, The weak Radon-Nidodým property in Banach spacecs, Studia Math., **64** (1979), 151–173.

[15] B. Pettis, On integration in vector space, Trans. Amer. Math. Soc., **44** (1938), 227–304.

[16] M. Rieffel, The Radon-Nikodým theorem for the Bochner integral, Trans. Amer. Math. Soc., **131** (1968), 466–487.

[17] S. Saks, Theory of the integral, (Second ed.); (English Transl.), Dover Publication, Inc., New York (1937).

[18] C. Stegall, The Radon-Nikodým property in conjugate Banach spaces, Trans. Amer. Math. Soc., **206** (1975), 213–233.

[19] C. Stegall, The duality between Aslund spaces and spaces with the Radon-Nikodým property, Israel. J. Math., **29** (4) (1978), 408–412.

[20] J. Uhl, Jr., Vector valued functions equivalent to measurable functions, Proc. Amer. Math. Soc., **68** (1978), 32–36.

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