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# On the construction of spherical hyperfunctions on $\boldsymbol{R}^{p+q}$

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## Introduction

We consider $SO_0(p, q)$ (or $O(p, q)$-invariant solutions $u$ of the differential equation $(p + v)u = 0$, where $P = \sum_{1 \leq i \leq p} (\partial/\partial x_i)^2 - \sum_{1 \leq j \leq q} (\partial/\partial y_j)^2$ and $v$ is a complex number. There have appeared several papers dealing with the above solutions in the sense of distributions ([4], [9], [10], [14]). On the other hand, we find as a corollary of the result of A. Cerezo [2]: the dimension of the space of $O(p, q)$-invariant hyperfunctions $u$ on $\boldsymbol{R}^{p+q}$ which are solutions of the equation $(P + v)u = 0$ is 2 and only $SO_0(p, q)$-invariant is 2 if $p > 1$ and $q = 1$, or $p = 1$ and $q > 1$, 4 if $p = 1$, respectively.

In this paper, we call such hyperfunctions "spherical hyperfunctions" and will give integral representations of "spherical hyperfunctions". In the paper [3], Ehrenpreis' principle says that any solution $u$ of a differential equation $Pu = 0$ with constant coefficients has an integral representation by a suitable measure on the variety defined by the polynomial $\sigma_T(P)(i\xi)$, where $\sigma_T(P)$ is the total symbol of $P$. Thus spherical hyperfunctions may be represented through integrals with respect to $SO_0(p, q)$ (or $O(p, q)$)-invariant measures on the variety $\{(\xi, \eta) \in \boldsymbol{C}^{p+q}; \sum \xi_i^2 - \sum \eta_j^2 - v = 0\}$. But these integrals are not convergent at any point of $\boldsymbol{R}^{p+q}$. However, in his paper [11], Sato's idea enables us to justify these integrals. Thus we can construct spherical hyperfunctions explicitly. In this paper, when $v$ is not 0, we give integral representations of spherical hyperfunctions except for $p > 1$ and $q = 1$. But when $p > 1$ and $q = 1$ we can construct spherical hyperfunctions in the same way as in the case of $p = 1$ and $q > 1$.

I would like to express hearty thanks to Professor K. Okamoto who taught me Sato's idea.

## § 0. Notations

Let $G = O(p, q)$ and $G_0 = SO_0(p, q)$ for $p \geq 1$ and $q \geq 1$. Then both $G$ and $G_0$ are acting on $\boldsymbol{R}^{p+q}$ naturally. Let $v$ be a non-zero arbitrary complex number and put $\mu = (1/2)\mathrm{Arg}(v)$ (Arg is the principal value) and $\lambda = |v|^{1/2} e^{i\mu}$, where $i = (-1)^{1/2}$. Then $-\pi/2 < \mu \leq \pi/2$ and $v = \lambda^2$. Let $\mathfrak{g} = \mathfrak{so}_0(p, q)$ that is the Lie algebra of both $G$ and $G_0$. Let $\mathscr{B}^G(\boldsymbol{R}^{p+q})$ ($\mathscr{B}^{G_0}(\boldsymbol{R}^{p+q})$) be the space of all $G(G_0)$-invariant hyperfunctions on $\boldsymbol{R}^{p+q}$, respectively. From Lemma 1 in [2], $\mathscr{B}^{G_0}(\boldsymbol{R}^{p+q})=\mathscr{B}^{\mathfrak{g}}(\boldsymbol{R}^{p+q})$. Here $\mathscr{B}^{\mathfrak{g}}(\boldsymbol{R}^{p+q})$ is the space of all $\mathfrak{g}$-invariant hyperfunctios on $\boldsymbol{R}^{p+q}$. We denote by $\mathscr{B}_\nu^G(\boldsymbol{R}^{p+q})$ $(\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{p+q}))$ the space of all $G(G_0)$-invariant hyperfunctions $f$ such that $P_\nu f=0$, where $P_\nu=\sum_{1\le i\le p}(\partial/\partial x_i)^2-\sum_{1\le j\le q}(\partial/\partial y_j)^2+\nu$. In this paper, we denote by $\mathrm{ch}(t)$ (and $\mathrm{sh}(t)$) the real analytic function $(e^t+e^{-t})/2$ (and $(e^t-e^{-t})/2$) on $\boldsymbol{R}$, respectively.

## §1. $p=1$ and $q=1$

In this section, we give spherical hyperfunctions using an integral representation for the case in which $p=q=1$. That is $G=O(1,1)$, $G_0=SO_0(1,1)$. For each $\varepsilon=(\varepsilon_1,\varepsilon_2)$, where $\varepsilon_i\in\{1,-1\}$ $(i=1,2)$, we denote by $U_\varepsilon$ the set of all $(z_1,z_2)\in\boldsymbol{C}^2$ such that $\mathrm{Im}(\varepsilon_1 z_1+\varepsilon_2 z_2)>0$, where $\mathrm{Im}\,z$ is the imaginary part of $z$ $(\in\boldsymbol{C})$. Let

$$\mathscr{W}'=\{U_\varepsilon;\ \varepsilon=(\varepsilon_1,\varepsilon_2),\ \varepsilon_i\in\{\pm1\}\,(i=1,2)\}\ \text{and}\ \mathscr{W}=\{\boldsymbol{C}^2\}\cup\mathscr{W}'.$$

Then it is easily seen that $(\mathscr{W},\mathscr{W}')$ is a relative Stein covering of $(\boldsymbol{C}^2,\boldsymbol{C}^2\backslash\boldsymbol{R}^2)$ (see [7] for the relative Stein covering).

LEMMA 1.1. *For each* $\varepsilon=(\varepsilon_1,\varepsilon_2)$,

$$\psi_\varepsilon(z_1,z_2)=\int_0^\infty e^{i\lambda[\varepsilon_1 z_1\mathrm{ch}(t-i\mu)+\varepsilon_2 z_2\mathrm{sh}(t-i\mu)]}\,dt$$

*converges absolutely and uniformly on every compact subset of* $U_\varepsilon$ *and holomorphic on* $U_\varepsilon$. *Moreover,* $\psi_\varepsilon$ *satisfies the following differential equations on* $U_\varepsilon$;

$$1)\quad ((\partial/\partial z_1)^2-(\partial/\partial z_2)^2)\psi_\varepsilon=-\lambda^2\psi_\varepsilon,$$

$$2)\quad (z_2\partial/\partial z_1+z_1\partial/\partial z_2)\psi_\varepsilon=-\varepsilon_1\varepsilon_2 e^{i\lambda(\varepsilon_1 z_1\cos\mu-i\varepsilon_2 z_2\sin\mu)}.$$

PROOF. It is seen that the above integral converges absolutely and uniformly on every compact subset of $U_\varepsilon$ and holomorphic on $U_\varepsilon$, because

$$\begin{aligned}&\mathrm{Re}\,[i\lambda(\varepsilon_1 z_1\mathrm{ch}(t-i\mu)+\varepsilon_2 z_2\mathrm{sh}(t-i\mu))]\\&\quad=-|\lambda|[e^t\mathrm{Im}(\varepsilon_1 z_1+\varepsilon_2 z_2)+\mathrm{Im}\bar{e}^{t+2i\mu}(\varepsilon_1 z_1-\varepsilon_2 z_2)]/2.\end{aligned}$$

It is easily seen that $\psi_\varepsilon$ satisfies the differential equations 1) and 2), because

$$(z_2\partial/\partial z_1+z_1\partial/\partial z_2-\varepsilon_1\varepsilon_2\partial/\partial t)e^{i\lambda(\varepsilon_1 z_1\mathrm{ch}(t-i\mu)+\varepsilon_2 z_2\mathrm{sh}(t-i\mu))}=0.$$

Therefore the lemma is proved.

For each $\varepsilon=(\varepsilon_1,\varepsilon_2)$, we denote by $V_\varepsilon$ the set of all $(z_1,z_2)\in\boldsymbol{C}^2$ such that

$\mathrm{Re}(\varepsilon_1 z_1 + \varepsilon_2 z_2) > 0$. Here Re $z$ is the real part of $z$.

LEMMA 1.2. *$\psi_\varepsilon$ is analytically continued from $U_\varepsilon$ to $V_\varepsilon \cup V_{-\varepsilon}$ but is not holomorphic on any neighborhood of the point $(z_1, z_2) \in C^2$ such that $\varepsilon_1 z_1 + \varepsilon_2 z_2 = 0$.*

PROOF. Applying Cauchy's integral formula, for $R > 0$, we have

$$\begin{aligned}
&\int_0^R e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(t - i\mu) + \varepsilon_2 z_2 \mathrm{sh}(t - i\mu)]} dt \\
&= i \int_0^{\pi/2} e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(i\theta - i\mu) + \varepsilon_2 z_2 \mathrm{sh}(i\theta - i\mu)]} d\theta \\
&+ \int_0^R e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(t - i\mu + i\pi/2) + \varepsilon_2 z_2 \mathrm{sh}(t - i\mu + i\pi/2)]} dt \\
&- i \int_0^{\pi/2} e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(R - i\mu + i\theta) + \varepsilon_2 z_2 \mathrm{sh}(R - i\mu + i\theta)]} d\theta .
\end{aligned}$$

One can easily see that for each $(z_1, z_2) \in U_\varepsilon \cap V_\varepsilon$ the last integral converges to 0 when $R \to \infty$. Therefore for each $(z_1, z_2) \in U_\varepsilon \cap V_\varepsilon$ we have

$$\begin{aligned}
&\int_0^\infty e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(t - i\mu) + \varepsilon_2 z_2 \mathrm{sh}(t - i\mu)]} dt \\
&= i \int_0^{\pi/2} e^{i\lambda[\varepsilon_1 z_1 \cos(\theta - \mu) + i\varepsilon_2 z_2 \sin(\theta - \mu)]} d\theta \\
&+ \int_0^\infty e^{-\lambda[\varepsilon_1 z_1 \mathrm{sh}(t - i\mu) + \varepsilon_2 z_2 \mathrm{ch}(t - i\mu)]} dt .
\end{aligned}$$

Since the right-hand side of the above equality is holomorphic on $V_\varepsilon$, $\psi_\varepsilon$ is analytically continued from $U_\varepsilon$ to $V_\varepsilon$. On the other hand, from Cauchy's integral formula along another Jordan curve, we have for each $(z_1, z_2) \in U_\varepsilon \cap V_{-\varepsilon}$,

$$\begin{aligned}
&\int_0^\infty e^{i\lambda[\varepsilon_1 z_1 \mathrm{ch}(t - i\mu) + \varepsilon_2 z_2 \mathrm{sh}(t - i\mu)]} dt \\
&= i \int_0^{-\pi/2} e^{i\lambda[\varepsilon_1 z_1 \cos(\theta - \mu) + i\varepsilon_2 z_2 \sin(\theta - \mu)]} d\theta \\
&+ \int_0^\infty e^{\lambda[\varepsilon_1 z_1 \mathrm{sh}(t - i\mu) + \varepsilon_2 z_2 \mathrm{ch}(t - i\mu)]} dt .
\end{aligned}$$

Hence $\psi_\varepsilon$ is analytically continued from $U_\varepsilon$ to $V_{-\varepsilon}$ in the same way as

$V_\varepsilon$. Therefore the first assertion of the lemma is proved. But the above integral is not convergent at the point $(z_1, z_2) \in \boldsymbol{C}^2$ such that $\varepsilon_1 z_1 + \varepsilon_2 z_2 = 0$. Indeed, for fixed real numbers $a_1$, $a_2$ and $\delta$, we put $z_1(\delta) = \varepsilon_1(a_1 + ia_2 + i\delta)$ and $z_2(\delta) = \varepsilon_2(-a_1 - ia_2 + i\delta)$. If $\delta > 0$, then $(z_1(\delta), z_2(\delta)) \in U_\varepsilon$. It is easily seen that there are positive real numbers $M_1$, $M_2$ and $t_0$ such that if $t \geq t_0$ then $M_1 \leq \cos(ce^{-t}(a_1\cos 2\mu - a_2\sin 2\mu))$ and $M_2 \leq e^{-c\exp(-t)(a_1\sin 2\mu + a_2\cos 2\mu)}$, where $c = |\lambda|$ $(>0)$. Hence

$$\operatorname{Re}\psi_\varepsilon(z_1(\delta), z_2(\delta)) \geq M_1 M_2 \int_{t_0}^{\infty} e^{-c\delta\exp t}\, dt + \operatorname{Re}\int_0^{t_0} e^{i\lambda H(\delta,t)}\, dt,$$

where $H(\delta, t) = \varepsilon_1 z_1(\delta)\mathrm{ch}(t - i\mu) + \varepsilon_2 z_2(\delta)\mathrm{sh}(t - i\mu)$. The last term of the above inequality is convergent when $\delta \to +0$. But

$$\lim_{\delta \to +0} \int_{t_0}^{\infty} e^{-c\delta\exp t}\, dt = +\infty.$$

Therefore $\psi_\varepsilon$ is not holomorphic on any neighborhood of the point $(z_1, z_2) \in \boldsymbol{C}^2$ such that $\varepsilon_1 z_1 + \varepsilon_2 z_2 = 0$. This implies the second assertion of the lemma.

For the purpose of the construction of $\mathfrak{g}$-invariant hyperfunctions, we consider the following integral;

$$\chi(z_1, z_2; a, b) = i\int_a^b e^{i\lambda[z_1\cos\theta + iz_2\sin\theta]}\, d\theta.$$

Then $\chi(z_1, z_2; a, b)$ is an entire holomorphic function on $\boldsymbol{C}^2$ for any fixed $(a, b) \in \boldsymbol{R}^2$ and $((\partial/\partial z_1)^2 - \partial/\partial z_2)^2)\chi = -\lambda^2\chi$. Moreover, since

$$(z_2\partial/\partial z_1 + z_1\partial/z_2 + i\partial/\partial\theta)e^{i\lambda[z_1\cos\theta + iz_2\sin\theta]} = 0,$$

we have

$$(z_2\partial/\partial z_1 + z_1\partial/\partial z_2)\chi(z_1, z_2; a, b) = [e^{i\lambda(z_1\cos\theta + iz_2\sin\theta)}]_{\theta=a}^{\theta=b}.$$

Now we give spherical hyperfunctions by means of elements of the Čeck cohomology $H^1(\mathscr{W}'; \mathscr{O})$ as follows. Set $\Lambda = \{\varepsilon = (\varepsilon_1, \varepsilon_2); \varepsilon_i \in \{\pm 1\}\ (i = 1, 2)\}$ and $\Lambda_0 = \{(\varepsilon, \eta); \varepsilon \in \Lambda, \eta \in \Lambda, \varepsilon_1\varepsilon_2\eta_1\eta_2 = -1\}$. For each $(\varepsilon, \eta) \in \Lambda_0$, we define

$$\varphi_{\varepsilon,\eta}(z_1, z_2) = \psi_\varepsilon(z_1, z_2) + \psi_\eta(z_1, z_2) + \eta_1\eta_2\chi(z_1, z_2; c(\varepsilon), c(\eta)),$$

where $c(\varepsilon) = c(\varepsilon, \mu) = -\varepsilon_1\varepsilon_2\mu + (1 - \varepsilon_1)\pi/2$. Then $\varphi_{\varepsilon,\eta}(z_1, z_2)$ is a holomorphic function on $U_\varepsilon \cap U_\eta$ by Lemma 1.1. For given $U_i\,(i = 1, 2)$ in $\mathscr{W}'$ and a holomorphic function $\varphi$ on $U_1 \cap U_2$, we denote by $[(U_1 \cap U_2; \varphi)]$ the element in $H^1(\mathscr{W}'; \mathscr{O})$ which is given by the following 1-cocycle ; $\{(U_1 \cap U_2; \varphi), (U_2 \cap U_1; -\varphi), (\text{otherwise}; 0)\}$.

We define

$$f_0 = [(U_{(-1,1)} \cap U_{(1,1)};\ \chi(z_1, z_2;\ -\pi, \pi))]$$

and

$$f_{\varepsilon,\eta} = [(U_\varepsilon \cap U_\eta;\ \varphi_{\varepsilon,\eta})] \quad \text{for fixed } (\varepsilon, \eta) \in \Lambda_0.$$

PROPOSITION 1.3. *For any* $(\varepsilon, \eta) \in \Lambda_0$, $f_{\varepsilon,\eta}$ *is* $\mathfrak{g}$-*invariant and* $f_{\varepsilon,\eta} = -f_{\eta,\varepsilon}$. *Moreover,* $S.S\, f_{\varepsilon,\eta} = \{(x_1, x_2;\ i\varepsilon/2^{1/2}\infty) : \varepsilon_1 x_1 + \varepsilon_2 x_2 = 0\} \cup \{(x_1, x_2;\ i\eta/2^{1/2}\infty) : \eta_1 x_1 + \eta_2 x_2 = 0\}$, *where* $S.S\, f$ *is the singular spectrum of* $f$ (*see* [12], *for the singular spectrum*).

PROOF. From Lemma 1.1, we have

$$(z_2\partial/\partial z_1 + z_2\partial/\partial z_2)(\psi_\varepsilon + \psi_\eta)$$
$$= -\varepsilon_1\varepsilon_2 e^{i\lambda(\varepsilon_1 z_2 \cos\mu - i\varepsilon_2 z_2 \sin\mu)} - \eta_1\eta_2 e^{i\lambda(\eta_1 z_1 \cos\mu - i\eta_2 z_2 \sin\mu)}.$$

Since

$$\cos(c(\varepsilon, \mu)) = \varepsilon_1 \cos\mu \quad \text{and} \quad \sin(c(\varepsilon, \mu)) = -\varepsilon_2 \sin\mu,$$

we have

$$(z_2\partial/\partial z_1 + z_1\partial/\partial z_2)\chi(z_1, z_2;\ c(\varepsilon, \eta), c(\eta, \mu))$$
$$= -e^{i\lambda(\varepsilon_1 z_1 \cos\mu - i\varepsilon_2 z_2 \sin\mu)} + e^{i\lambda(\eta_1 z_1 \cos\mu - i\eta_2 z_2 \sin\mu)}.$$

Hence $(z_2\partial/\partial z_1 + z_1\partial/\partial z_2)\varphi_{\varepsilon,\eta} = 0$ for any $(\varepsilon, \eta) \in \Lambda_0$. Therefore the first assertion of the proposition is proved. Im view of the definition of $\chi$, we see that $\eta_1\eta_2\chi(z_1, z_2;\ c(\varepsilon), c(\eta)) = -\eta_1\eta_2\chi(z_1, z_2;\ c(\eta), c(\varepsilon)) = \varepsilon_1\varepsilon_2\chi(z_1, z_2;\ c(\eta), c(\varepsilon))$. Hence $\varphi_{\varepsilon,\eta}(z_1, z_2) = \varphi_{\eta,\varepsilon}(z_1, z_2)$ on $U_\varepsilon \cap U_\eta$. Therefore the second assertion of the proposition is proved. The third assertion of the proposition is clear from Lemma 1.2 and the definition of the singular spectrum.

Let $k_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $k_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $k_i \in G\ (i = 1, 2)$ and $G = G_0 \cup k_1 G_0 \cup k_2 G_0 \cup k_1 k_2 G_0$. For any hyperfunction $f$ on $\boldsymbol{R}^2$, we denote by $f^{k_i}$ the pull-back of $f$ by the transformation $;x \to k_i x\ (i = 1, 2)$.

PROPOSITION 1.4. *For each* $(\varepsilon, \eta) \in \Lambda_0$, *we have*

1) $f^{k_1}_{\varepsilon,\eta} = f_{k_1\eta, k_1\varepsilon}$,

2) $f^{k_2}_{\varepsilon,\eta} = f_{k_2\eta, k_2\varepsilon} + ((\varepsilon_1 - \eta_1)/2) f_0$.

PROOF. By virtue of the definition of $f_{\varepsilon,\eta}$ and the fact that $k_1^{-1} = k_1$, we have

$$f^{k_1}_{\varepsilon,\eta} = -[(U_{k_1\varepsilon} \cap U_{k_1\eta};\ \varphi_{\varepsilon,\eta}(-z_1, z_2))].$$

Since $c(k_1\varepsilon) = \pi - c(\varepsilon)$, it is easily seen that

$$\chi(-z_1, z_2; c(\varepsilon), c(\eta)) = \chi(z_1, z_2; c(k_1\eta), c(k_1\varepsilon)).$$

On the other hand, $\psi_\varepsilon(-z_1, z_2) = \psi_{k_1\varepsilon}(z_1, z_2)$. Hence,

$$\varphi_{\varepsilon,\eta}(-z_1, z_2) = \psi_{k_1\varepsilon}(z_1, z_2) + \psi_{k_1\eta}(z_1, z_2) + \eta_1\eta_2\chi(z_1, z_2; c(k_1\eta), c(k_1\varepsilon)).$$

Therefore $\varphi_{\varepsilon,\eta}(-z_1, z_2) = \varphi_{k_1\eta,k_1\varepsilon}(z_1, z_2)$, since $\eta_1\eta_2 = -\varepsilon_1\varepsilon_2$. Hence 1) of the proposition is proved. Next we show 2) of the proposition. Since, for any $\varepsilon$ and $\mu$,

$$\chi(z_1, z_2; -c(\varepsilon, \mu), c(k_2\varepsilon, \mu)) = (1-\varepsilon_1)\chi(z_1, z_2; -\pi, \pi)/2,$$

we have

$$\begin{aligned}
&\chi(z_1, -z_2; c(\varepsilon), c(\eta)) - \chi(z_1, z_2; c(k_2\eta), c(k_2\varepsilon)) \\
&\quad = \chi(z_1, z_2; -c(\eta), -c(\varepsilon)) + \chi(z_1, z_2; c(k_2\varepsilon), c(k_2\eta)) \\
&\quad = \chi(z_1, z_2; -c(\eta), c(k_2\eta)) - \chi(z_1, z_2; -c(\varepsilon), c(k_2\varepsilon)) \\
&\quad = (\varepsilon_1 - \eta_1)\chi(z_1, z_2; -\pi, \pi)/2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi_{\varepsilon,\eta}(z_1, z_2) = {}& \psi_{k_2\varepsilon}(z_1, z_2) + \psi_{k_2\eta}(z_1, z_2) + \eta_1\eta_2\chi(z_1, z_2; c(k_2\eta), c(k_2\varepsilon)) \\
&+ \eta_1\eta_2(\varepsilon_1 - \eta_1)\chi(z_1, z_2; -\pi, \pi)/2.
\end{aligned}$$

Therefore $\varphi_{\varepsilon,\eta}(z_1, -z_2) = \varphi_{k_2\eta,k_2\varepsilon}(z_1, z_2) + (\varepsilon_1 - \eta_1)\eta_1\eta_2\chi(z_1, z_2; -\pi, \pi)/2$. On the other hand, it is easily seen that

$$(\#) \quad [(U_{k_2\eta} \cap U_{k_2\varepsilon}; (\varepsilon_1 - \eta_1)\eta_1\eta_2\chi(z_1, z_2; -\pi, \pi)/2)] = (\varepsilon_1 - \eta_1)f_0/2$$

for any $(\varepsilon, \eta) \in \Lambda_0$. Indeed, we define a 0-cochain $\psi$ $(\in C^0(\mathscr{W}'; \mathscr{O}))$ such that $\psi = \{(U_{(1,1)}; \chi(z_1, z_2; -\pi, \pi)), (U_{(-1,1)}; 0), (U_{(1,-1)}; \chi(z_1, z_2; -\pi, \pi)), (U_{(-1,-1)}; 0)\}$. Then we have $\delta\psi =$

$$\begin{aligned}
&\{(U_{(-1,-1)} \cap U_{(1,-1)}; \chi(z_1, z_2; -\pi, \pi)), (U_{(1,1)} \cap U_{(1,-1)}; 0), \\
&\quad (U_{(-1,1)} \cap U_{(1,1)}; \chi(z_1, z_2; -\pi, \pi)), (U_{(-1,-1)} \cap U_{(-1,1)}; 0)\},
\end{aligned}$$

where $\delta$ is the coboundary operator. Hence

$$\begin{aligned}
&[(U_{(-1,-1)} \cap U_{(1,-1)}; -\chi(z_1, z_2; -\pi, \pi))] \\
&\quad = [(U_{(-1,1)} \cap U_{(1,1)}; \chi(z_1, z_2; -\pi, \pi))] = f_0.
\end{aligned}$$

This implies that the above equality (#) is true for the case $\varepsilon_1 = \varepsilon_2 = \eta_2 = 1$ and $\eta_1 = -1$. For the other cases, one can easily prove the equality (#) similarly. Therefore 2) of the proposition is proved.

Now, we can give a basis of $\mathscr{B}_\nu^{G_0}(R^2)$ and $\mathscr{B}_\nu^{G}(R^2)$, applying Cerezo's result ([2]): dim $\mathscr{B}_\nu^{G_0}(R^2)=4$ and dim $\mathscr{B}_\nu^{G}(R^2)=2$. We define hyperfunctions $g_j$ $(1\le j\le 4)$ as follows;

$$g_1=f_{(1,1),(1,-1)},\qquad g_2=f_{(1,1),(-1,1)},$$

$$g_3=f_{(-1,-1),(-1,1)},\quad g_4=f_{(-1,-1),(1,-1)}.$$

Then it is obvious that $g_j\in\mathscr{B}_\nu^{G_0}(R^2)$ for $1\le j\le 4$.

LEMMA 1.5. $g_1+g_2+g_3+g_4=0$.

PROOF. We can define a 0-cochain $\psi$ $(\in C^0(\mathscr{W}';\mathscr{O}))$ such that $\psi=\{(U_{(1,1)};-\psi_{(1,1)}(z_1,z_2)),(U_{(-1,1)};\psi_{(-1,1)}(z_1,z_2)-\chi(z_1,z_2;-\mu,\mu+\pi))$

$$(U_{(-1,-1)};-\psi_{(-1,-1)}(z_1,z_2)-\chi(z_1,z_2:-\mu,\pi-\mu)),$$

$$(U_{(1,-1)};\psi_{(1,-1)}(z_1,z_2)-\chi(z_1,z_2;-\mu,\mu))\}.$$

Then it is easily seen that $g_1+g_2+g_3+g_4=[(\delta\psi)]=0$. Therefore the lemma is proved.

PROPOSITION 1.6. *Any triple of $g_j$ $(1\le j\le 4)$ is linearly independent.*

PROOF. We prove the proposition for the case $g_1$, $g_2$, $g_3$. Let $c_1g_1+c_2g_2+c_3g_3=0$ $(c_j\in C)$. Then $c_1=c_3=0$, because $S.S\ g_1=\{(x_1,x_2;i(2^{-1/2},2^{-1/2})\infty);x_1+x_2=0\}\cup\{(x_1,x_2;i(2^{-1/2},-2^{-1/2})\infty);x_1-x_2=0\}$ and $S.S\ g_3=\{(x_1,x_2;i(2^{-1/2},2^{-1/2})\infty);x_1+x_2=0\}\cup\{(x_1,x_2;i(-2^{-1/2},2^{-1/2})\infty);-x_1+x_2=0\}$, by Proposition 1.3. Hence $c_2g_2=0$. Since $g_2$ is not 0, $c_2=0$. Thus $c_1=c_2=c_3=0$. In the same way, the linear independence is showed for the other cases. Hence the proposition is proved.

PROPOSITION 1.7.

$$g_1^{k_1}=g_3,\qquad g_1^{k_2}=g_1,$$

$$g_2^{k_1}=g_2,\qquad g_2^{k_2}=g_4+f_0,$$

$$g_3^{k_1}=g_1,\qquad g_3^{k_2}=g_3,$$

$$g_4^{k_1}=g_4,\qquad g_4^{k_2}=g_2-f_0.$$

PROOF. From Proposition 1.4, the proposition is clear.

Finally we define spherical hyperfunction $f_j$ $(1\le j\le 3)$ by

$$f_1=g_1+g_3,\ f_2=g_1-g_3\ \text{and}\ f_3=f_0-g_1-2g_2-g_3.$$

THEOREM 1.8.

1) $\{f_j; 0 \leq j \leq 3\}$ *is a basis of* $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^2)$.

2) $\{f_j; 0 \leq j \leq 1\}$ *is a basis of* $\mathscr{B}_\nu^{G}(\boldsymbol{R}^2)$.

PROOF. It is easily seen that $f_0$ and $g_j$ $(1 \leq j \leq 3)$ is linearly independent by the same proof as in Proposition 1.6, since $S.S\, f_0 = \phi$. Hence it is clear that $f_j$ $(0 \leq j \leq 3)$ is linearly independent. Therefore, since $\dim \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^2) = 4$, 1) of the theorem is proved (see [2]). From Proposition 1.7, $f_1$ is $G$-invariant. Moreover, it is obvious that $f_0$ is also $G$-invariant. Conversely, from Proposition 1.7, one can easily see that for any $f \in \mathscr{B}_\nu^{G}(\boldsymbol{R}^2)$, there exist complex numbers $c_0$ and $c_1$ such that $f = c_0 f_0 + c_1 f_1$. Therefore 2) of the theorem is proved.

REMARK. Since one can easily show that $f_2^{k_1} = -f_2, f_2^{k_2} = f_2, f_3^{k_1} = f_3$ and $f_3^{k_2} = -f_3$ from Proposition 1.7, we have that

$$\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^2) = \mathscr{B}_\nu^{G}(\boldsymbol{R}^2) \oplus \langle f_2 \rangle \oplus \langle f_3 \rangle$$

is the irreducible decomposition of the representation over $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^2)$ with respect to the finite group $\{e, k_1, k_2, k_1 k_2\}$.

## §2. $p = 1$ and $q > 1$

In this section, we give spherical hyperfunctions using integral representation for the case in which $p = 1$, $q > 1$. That is $G = O(1, q)$ and $G_0 = SO_0(1, q)$. For each $\varepsilon$ in $\{1, -1\}$, we denote by $U^{(\varepsilon)}$ the set of all $(z, w) \in \boldsymbol{C}^{1+q}$ (here $z \in \boldsymbol{C}$ and $w \in \boldsymbol{C}^q$) such that $\varepsilon \mathrm{Im}\, z > \|\mathrm{Im}\, w\|$, where $\|y\| = (\sum_{1 \leq j \leq q} y_j^2)^{1/2}$ for $y = (y_1, \dots, y_p) \in \boldsymbol{R}^q$ and $\mathrm{Im}\, w = (\mathrm{Im}\, w_1, \dots, \mathrm{Im}\, w_q)$ for $w = (w_1, \dots, w_q) \in \boldsymbol{C}^q$. Put

$$V_j^{(\pm)} = \{(z, w) \in \boldsymbol{C}^{1+q};\ \pm \mathrm{Im}\, w_j > 0\}.$$

Let

$$\mathscr{W}' = \{U^{(\varepsilon)}; \varepsilon \in \{\pm 1\}\} \cup \{V_j^{(\varepsilon)}; \varepsilon \in \{\pm 1\},\ 1 \leq j \leq q\} \text{ and } \mathscr{W} = \{\boldsymbol{C}^{1+q}\} \cup \mathscr{W}'.$$

Then it is easily seen that $(\mathscr{W}, \mathscr{W}')$ is a relative Stein covering of $(\boldsymbol{C}^{1+q}, \boldsymbol{C}^{1+q} \backslash \boldsymbol{R}^{1+q})$ (see [7] for the relative Stein covering).

LEMMA 2.1. *For each* $\varepsilon \in \{1, -1\}$,

$$\psi_\varepsilon(z, w) = \int_0^\infty \int_{S^{q-1}} e^{i\lambda[\varepsilon z \mathrm{ch}(t - i\mu) + \langle w, \eta \rangle \mathrm{sh}(t - i\mu)]} (\mathrm{sh}(t - i\mu))^{q-1} d\eta\, dt$$

*converges absolutely and uniformly on every compact subset of* $U^{(\varepsilon)}$ *and is holomorphic on* $U^{(\varepsilon)}$. *Here* $\langle u, v \rangle = \sum u_j v_j$ (*for* $u = (u_1, \dots, u_q) \in \boldsymbol{C}^q$ *and* $v$

$=(v_1,\dots,v_q)\in C^q$) *and* $d\eta$ *is the normalized* $SO(q)$*-invariant measure such that* $\int_{S^{q-1}} d\eta=1$. (*See* §0 *for the notations* $\lambda$, $\mu$, ch, sh.)

PROOF. Since

$$\begin{aligned}&\mathrm{Re}[i\lambda(\varepsilon z\mathrm{ch}(t-i\mu)+\langle w,\eta\rangle\mathrm{sh}(t-i\mu))]\\&\quad=-|\lambda|[e^t\mathrm{Im}(\varepsilon z+\langle w,\eta\rangle)+\mathrm{Im}e^{-t+2i\mu}(\varepsilon z-\langle w,\eta\rangle)]/2,\end{aligned}$$

it is clear that the above integral converges absolutely on every compact subset of $U^{(\varepsilon)}$ and is holomorphic on $U^{(\varepsilon)}$.

REMARK. It is easily seen that $\psi_\varepsilon$ satisfies the following differential equations in a way similar to Lemma 1.1;

$$((\partial/\partial z)^2-\textstyle\sum(\partial/\partial w_j)^2)\psi_\varepsilon=-\lambda^2)\psi_\varepsilon,$$

$$(w_j\partial/\partial w_k-w_k\partial/\partial w_j)\psi_\varepsilon=0\quad(1\le j\le q,\ 1\le k\le q),$$

$$(w_1\partial/\partial z+z\partial/\partial w_1)\psi_\varepsilon=-\varepsilon(-i\sin\mu)^{q-1}\int_{S^{q-1}} e^{i\lambda[\varepsilon z\cos\mu-i\langle w,\eta\rangle\sin\mu]}\eta_1\,d\eta.$$

Here $\eta_1$ is the first coordinate of $\eta$ ($\in S^{q-1}$). Indeed,

$$\{w_1\partial/\partial z+z\partial/\partial w_1-\varepsilon(\cos\tau_1\partial/\partial t-\sin\tau_1\coth(t-i\mu)\partial/\partial\tau_1)\}e^{i\lambda H(t,z,w)}=0,$$

where

$$H(t,z,w)=\varepsilon z\mathrm{ch}(t-i\mu)+\langle w,\eta(\tau)\rangle\mathrm{sh}(t-i\mu),$$

$$\eta(\tau)_j=\cos\tau_j\prod_{1\le k\le j-1}\sin\tau_k\ (1\le k\le q-1)\ \text{and}\ \eta(\tau)_q=\prod_{1\le k\le q-1}\sin\tau_k.$$

Hence, we have

$$(w_1\partial/\partial z+z\partial/\partial w_1)\psi_\varepsilon=\varepsilon\int_0^\infty\int_{S^{q-1}}(\mathrm{sh}(t-i\mu)^{q-1}(De^{i\lambda H(t,z,w)})dt\,d\eta,$$

where $D=\cos\tau_1\partial/\partial t-\sin\tau_1\coth(t-i\mu)\partial/\partial\tau_1$. By integration by parts in the above integral, we have the third equation of the Remark.

For the purpose of the construction of $\mathfrak{g}$-invariant hyperfunctions, we consider the following integral;

$$\chi(z,w;a,b)=-i\int_a^b\int_{S^{q-1}} e^{i\lambda[z\cos\theta-i\langle w,\eta\rangle\sin\theta]}(-i\sin\theta)^{q-1}\,d\theta\,d\eta.$$

It is easily seen that $\chi(z,w;a,b)$ is an entrire holomorphic function on $C^{1+q}$ for any fixed $(a, b) \in \boldsymbol{R}^2$. Moreover one can see that $\chi$ satisfies the following differential equations;

$$((\partial/\partial z)^2 - \textstyle\sum_{1 \le j \le q}(\partial/\partial w_j)^2)\chi = -\lambda^2\chi,$$

$$(w_j\partial/\partial w_k - w_k\partial/\partial w_j)\chi = 0,$$

$$(w_1\partial/\partial z + z\partial/\partial w_1)\chi = \int_{S^{q-1}} [(-i\sin\theta)^{q-1} e^{i\lambda[z\cos\theta - i\langle w,\eta\rangle\sin\theta]}]_{\theta=a}^{\theta=b}\eta_1 d\eta.$$

Here we obtain the third equality by the same calculation as in Remark on Lemma 2.1.

Put $\chi_1(z, w) = \chi(z, w; 0, \mu)$, $\chi_{-1}(z, w) = \chi(z, w; \pi - \mu, \pi)$ and $\varphi_\varepsilon(z, w) = \psi_\varepsilon(z, w) + \chi_\varepsilon(z, w)$ for each $\varepsilon$. Then $\varphi_\varepsilon$ is a holomorphic function on $U^{(\varepsilon)}$ by Lemma 2.1. Moreover, from the definition of $\varphi_\varepsilon$, it is clear that $\varphi_\varepsilon$ satisfies the following differential equations;

$$((\partial/\partial z)^2 - \textstyle\sum_{1 \le j \le q}(\partial/\partial w_j)^2)\varphi_\varepsilon = -\lambda^2\varphi_\varepsilon,$$

$$(w_j\partial/\partial w_k - w_k\partial/\partial w_j)\varphi_\varepsilon = 0 \quad (1 \le j \le q,\ 1 \le k \le q),$$

$$(w_1\partial/\partial z + z\partial/\partial w_1)\varphi_\varepsilon = 0.$$

Now we discuss the representation of $\varphi_\varepsilon$ in terms of special functions. Let $K_\nu(z)$ be the modified Bessel function of order $\nu$.

LEMMA 2.2. *For any $(z, w) \in U^{(\varepsilon)}$, we have*

$$\int_0^\infty \int_{S^{q-1}} e^{i[\varepsilon z\operatorname{ch}t + \langle w,\eta\rangle\operatorname{sh}t]}(\operatorname{sh}t)^{q-1}d\eta dt$$

$$= c_q(-z^2 + \langle w, w\rangle)^{-(q-1)/4} K_{(q-1)/2}((-z^2 + \langle w,w\rangle)^{1/2}),$$

*where $c_q = \pi^{-1/2}2^{(q-1)/2}\Gamma(q/2)$ ($\Gamma(z)$ is the gamma function).*

PROOF. The right-hand side of the above equality is an infinitely multi-valued holomorphic function. But it is easily seen that one can choose a single valued branch of the function on $U^{(\varepsilon)}$, because $\{\operatorname{Im}(-z^2 + \langle w, w\rangle) = 0, \operatorname{Re}(-z^2 + \langle w, w\rangle) \le 0\} \cap U^{(\varepsilon)} = \phi$. Since both sides of the equlity are holomorphic on $U^{(\varepsilon)}$, it is sufficient to prove that the above equality is true over the following real locus; $z = z(r, u) = i\varepsilon r\cos u$, $w = w(r, u, \alpha) = r\alpha\sin u$, where $r > 0$, $|u| < \pi/2$ and $\alpha \in S^{q-1}$. By easy calculation,

$$\int_0^\infty \int_{S^{q-1}} e^{i\lambda[\varepsilon z(r,u)\operatorname{ch}t + \langle w(r,u,\alpha),\eta\rangle\operatorname{sh}t]}(\operatorname{sh}t)^{q-1}d\eta dt$$

$$= c_q' \int_0^\infty \int_0^\pi e^{-r\cos u\operatorname{ch}t + ir\cos\tau\sin u\operatorname{sh}t}(\sin\tau)^{q-2}(\operatorname{sh}t)^{q-1}d\tau dt,$$

where $c_q' = \pi^{-1/2}\Gamma(q/2)/\Gamma((q-1)/2)$. But one can easily see that the above integral is independent of the value $u$. Indeed, since

$$(\partial/\partial u + i\cos\tau\partial/\partial t - i\sin\tau\coth t\partial/\partial\tau)e^{-r\cos u\operatorname{ch}t + ir\cos\tau\sin u\operatorname{sh}t} = 0$$

and

$$\int_0^\infty\int_0^\pi(\cos\tau\partial/\partial t - \sin\tau\coth t\partial/\partial\tau)e^{H_0(t,\tau;r,u)}(\sin\tau)^{q-2}(\operatorname{sh}t)^{q-1}\,d\tau dt = 0,$$

where $H_0(t, \tau; r, u) = -r\cos u\operatorname{ch}t + ir\cos\tau\sin u\operatorname{sh}t$, we have

$$\partial/\partial u\left(\int_0^\infty\int_{S^{q-1}} e^{i[\varepsilon z(r,u)\operatorname{ch}t + \langle w(r,u,\tau),\eta\rangle\operatorname{sh}t]}(\operatorname{sh}t)^{q-1}\,d\eta dt\right) = 0.$$

On the other hand, it is well known that for any $r > 0$,

$$\int_0^\infty e^{-r\operatorname{ch}t}(\operatorname{sh}t)^{q-1}dt = \pi^{-1/2}\Gamma(q/2)(r/2)^{-(q-1)/2}K_{(q-1)/2}(r).$$

Thus the equality of Lemma 2.2 is true over the above real locus. This completes the proof of the lemma.

PROPOSITION 2.3. *For each* $(z, w)\in U^{(\varepsilon)}$, *we have*

$$\varphi_\varepsilon(z, w) = c_q(\lambda^2(-z^2 + \langle w, w\rangle))^{-(q-1)/4}K_{(q-1)/2}((\lambda^2(-z^2 + \langle w, w\rangle))^{1/2}).$$

PROOF. Let $U_\lambda^{(\varepsilon)} = \{(z, w)\in \boldsymbol{C}^{1+q}; (\lambda z, \lambda w)\in U^{(\varepsilon)}\}$. Then it is clear that if $\lambda$ is not zero, $U_\lambda^{(\varepsilon)}$ is holomorphically isomorphic to $U^{(\varepsilon)}$ and $U_\lambda^{(\varepsilon)}\cap U^{(\varepsilon)}$ is not $\phi$. By Cauchy's integral formula, for each $(z, w)\in U_\lambda^{(\varepsilon)}\cap U^{(\varepsilon)}$, we have

$$\begin{aligned}&\int_0^\infty\int_{S^{q-1}} e^{i\lambda[\varepsilon z\operatorname{ch}(t-i\mu) + \langle w,\eta\rangle\operatorname{sh}(t-i\mu)]}(\operatorname{sh}(t-i\mu))^{q-1}\,d\eta dt\\ &= i\int_0^\mu\int_{S^{q-1}} e^{i\lambda[\varepsilon z\operatorname{ch}(-i\theta) + \langle w,\eta\rangle\operatorname{sh}(-i\theta)]}(\operatorname{sh}(-i\theta))^{q-1}\,d\eta d\theta\\ &\quad + \int_0^\infty\int_{S^{q-1}} e^{i\lambda[\varepsilon z\operatorname{ch}t + \langle w,\eta\rangle\operatorname{sh}t]}(\operatorname{sh}t)^{q-1}\,d\eta dt.\end{aligned}$$

Thus from the definition of $\varphi_\varepsilon$,

$$\varphi_\varepsilon(z, w) = \int_0^\infty\int_{S^{q-1}} e^{i\lambda[\varepsilon z\operatorname{ch}t + \langle w,\eta\rangle\operatorname{sh}t]}(\operatorname{sh}t)^{q-1}\,d\eta dt$$

for each $(z, w)\in U_\lambda^{(\varepsilon)}\cap U^{(\varepsilon)}$. This implies that $\varphi_\varepsilon$ is analytically continued from $U^{(\varepsilon)}$ to $U_\lambda^{(\varepsilon)}$. Hence from Lemma 2.2,

$$\varphi_\varepsilon(z, w) = c_q(\lambda^2(-z^2 + \langle w, w\rangle))^{-(q-1)/4}K_{(q-1)/2}((\lambda^2(-z^2 + \langle w, w\rangle))^{1/2}).$$

Therefore the proposition is proved,

COROLLARY 2.4. *$\varphi_\varepsilon$ can be analytically continued over $\{(z, w);\ -z^2 + \langle w, w\rangle = 0\}$ but is not holomorphic on any neighborhood of the point $(z, w) \in \boldsymbol{C}^{1+q}$ scuh that $-z^2 + \langle w, w\rangle = 0$.*

PROOF. From the definition of the modified Bessel function, the corollary is clear.

Now, we give spherical hyperfunctions by means of the elements of the Čeck cohomology $H^q(\mathscr{W}'; \mathscr{O})$. For given $W_j$ $(1 \leq j \leq q+1)$ in $\mathscr{W}'$ and a holomorphic function $\varphi$ on $W_1 \cap \cdots \cap W_{q+1}$, we denote by $[(W_1 \cap \cdots \cap W_{q+1}; \varphi)]$ the element in $H^q(\mathscr{W}'; \mathscr{O})$ which is defined by the following $q$-cocycle;

$$\left\{\left(W_{j_1} \cap \cdots \cap W_{j_{q+1}};\ \operatorname{sgn}\begin{pmatrix}1, \dots, q+1\\ j_1, \dots, j_{q+1}\end{pmatrix}\varphi\right),\ (\text{otherwise};\ 0)\right\},$$

where sgn $\sigma$ is the signum of a permutation $\sigma$.

Let $f_0 = [(U^{(1)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \chi(z, w;\ -\pi, \pi))]$. Then it is clear that $f_0$ is a real analytic function on $\boldsymbol{R}^{1+q}$ and $f_0 \in \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$. For each $\varepsilon \in \{1, -1\}$, we define $g_\varepsilon = [(U^{(\varepsilon)} \cap V_1^{(1)} \cap \cdots \cap V_1^{(1)};\ \varepsilon\varphi_\varepsilon)]$.

REMARK. The hyperfunction $g_\varepsilon$ may be defined by the element; $[(U^{(\varepsilon)} \cap V_1^{(\eta_1)} \cap \cdots \cap V_q^{(\eta_q)};\ \varepsilon(\prod_{1 \leq j \leq q} \eta_j)\varphi_\varepsilon)]$ for fixed $\eta = (\eta_j)$ $(\eta_j \in \{1, -1\})$, because

$$[(U^{(\varepsilon)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \varphi)] = [(U^{(\varepsilon)} \cap V_1^{(\eta_1)} \cap \cdots \cap V_q^{(\eta_q)};\ \prod_{1 \leq j \leq q} \eta_j \varphi)]$$

for any holomorphic function $\varphi$ on $U^{(\varepsilon)}$. Indeed, let $\psi_{\eta,j}$ be a $q-1$ cochain defined as follows;

$$\psi_{\eta,j} = \{(U^{(\varepsilon)} \cap V_1^{(\eta_1)} \cap \cdots \cap V_{j-1}^{(\eta_{j-1})} \cap V_{j+1}^{(\eta_{j+1})} \cap \cdots \cap V_q^{(\eta_q)};\ (-1)^j\varphi),\ (\text{otherwise}\ ;\ 0)\}$$

for $\eta = (\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_q)$ $(\eta_j \in \{1, -1\})$ and $1 \leq j \leq q$. Then

$$\begin{aligned}&[(U^{(\varepsilon)} \cap V^{(\eta_1)} \cap \cdots \cap V_j^{(\eta_j)} \cap \cdots \cap V_q^{(\eta_q)};\ \varphi)]\\ &\quad + [(U^{(\varepsilon)} \cap V_1^{(\eta_1)} \cap \cdots \cap V_j^{(-\eta_j)} \cap \cdots \cap V_q^{(\eta_q)};\ \varphi)]\\ &\quad = [(\delta\psi_{\eta,j})] = 0.\end{aligned}$$

Here $\delta$ is the coboundary operator.

PROPOSITION 2.5. *For each $\varepsilon \in \{1, -1\}$, $g_\varepsilon \in \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$. Moreover, S.S $g_\varepsilon = \{(x, y;\ i(\varepsilon/2^{1/2}, \eta)\infty);\ x^2 = \|y\|,\ \|\eta\| = 1/2,\ x\eta/2^{1/2} + \varepsilon y = 0\ (1 \leq j \leq q)\}$.*

PROOF. It is clear that $g_\varepsilon \in \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$ from the definition of $g_\varepsilon$. From Sato's fundamental theorem (see [12]), we have that

$$S.S\ g_\varepsilon \subset \{(x, y; i(a, b)\infty);\ a^2 - \|b\|^2 = 0,\ ay_j + b_j x = 0,$$
$$y_j\eta_k = y_k\eta_j\ (1 \le j, k \le q)\}.$$

But, as seen from the definition of $g_\varepsilon$, if $(x, y; i(a, b)\infty) \in S.S\ g_\varepsilon$ then $a^2 = 1/2$, $\|b\| = 1/2$ and $a = \varepsilon/2^{1/2}$. Thus

$$S.S\ g_\varepsilon \subset \{(x, y; i(\varepsilon/2^{1/2}, \eta)\infty);\ x^2 = \|y\|^2,\ \|\eta\|^2 = 1/2,$$
$$x\eta_j/2^{1/2} + \varepsilon y_j = 0\ (1 \le j \le q)\}.$$

Conversely, it is easily seen that $g_\varepsilon$ is not microlocally analytic at the point $(x, y; i(\varepsilon/2^{1/2}, \eta)\infty)$ in $\sqrt{-1}S^*\boldsymbol{R}^{1+q}$ such that $x^2 = \|y\|^2$, $x\eta_j/2^{1/2} + \varepsilon y_j = 0$ $(1 \le j \le q)$ and $\|\eta\|^2 = 1/2$ from Corollary 2.4. Therefore the proposition is proved.

Let $k_1 = \begin{bmatrix} -1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{bmatrix}$, $k_2 = \begin{bmatrix} 1 & \ddots & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}$. Then $k_j \in G$ and $G = G_0 \cup k_1 G_0 \cup k_2 G_0 \cup k_1 k_2 G_0$. For any hyperfunction $f$ on $\boldsymbol{R}^{1+q}$, we denote by $f^{k_j}$ the pull back of $f$ by the transformation $x \to k_j x$.

PROPOSITION 2.6.

1) $f_0^{k_1} = f_0$ *and* $g_\varepsilon^{k_1} = g_{-\varepsilon}$ (*for any* $\varepsilon$),

2) $f_0^{k_2} = f_0$ *and* $g_\varepsilon^{k_2} = g_\varepsilon$ (*for any* $\varepsilon$).

PROOF. Since $\chi(-z, w; -\pi, \pi) = \chi(z, w; -\pi, \pi)$.

$$f_0^{k_1} = -[(U^{(-1)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \chi(z, w; -\pi, \pi))].$$

Let $\psi$ be a $q-1$ cochain defined as follows;

$$\psi = \{(V_1^{(1)} \cap V_2^{(1)} \cap \cdots \cap V_q^{(1)};\ \chi(z, w; -\pi, \pi)),\ \ (\text{otherwise};\ 0)\}.$$

Then it is easily seen that $f_0 - f_0^{k_1} = [(\delta\psi)] = 0$. Hence $f_0^{k_1} = f_0$. Since $\psi_\varepsilon(-z, w) = \psi_{-\varepsilon}(z, w)$ and $\chi_\varepsilon(-z, w) = \chi_{-\varepsilon}(z, w)$, we have

$$g_\varepsilon = -[(U^{(-\varepsilon)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \varepsilon\varphi_{-\varepsilon})] = g_{-\varepsilon}.$$

Therefore 1) of the proposition is proved. Since

$$\chi(z, w_1, \cdots, -w_q; -\pi, \pi) = \chi(z, w_1, \cdots, w_q; -\pi, \pi),$$

$$f_0^{k_2} = -[(U^{(1)} \cap (\bigcap_{1 \le j \le q-1} V_j^{(1)}) \cap V_q^{(-1)};\ \chi(z, w_1, \cdots, -w_q; -\pi, \pi))]$$

$$= -[(U^{(1)} \cap (\bigcap_{1 \le j \le q-1} V_j^{(1)}) \cap V_q^{(-1)};\ \chi(z, w_1, \cdots, w_q; -\pi, \pi))].$$

Let $\psi'$ be a $q-1$ cochain defined as follows;

$$\psi' = \{(U^{(1)} \cap (\bigcap_{1 \leq j \leq q-1} V_j^{(1)}); \chi(z, w; -\pi, \pi)), (\text{otherwise }; 0)\}.$$

Then it is easily seen that $f_0 - f_0^{k_2} = [(\delta\psi')] = 0$. Hence $f_0 = f_0^{k_2}$. Since $\varphi_\varepsilon(z, -w) = \varphi_\varepsilon(z, w)$, we obtain $g_\varepsilon^{k_2} = g_\varepsilon$ by the same proof as $f_0^{k_2} = f_0$. Therefore 2) of the proposition is proved.

PROPOSITION 2.7. *$f_0$, $g_1$ and $g_{-1}$ are linearly independent.*

PROOF. From Proposition 2.5 and $S.S\, f = \phi$, the assertion is clear.

Now, we give a basis of $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$ and $\mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q})$, since Cerezo proved in [2] that dim $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q}) = 3$ and dim $\mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q}) = 2$. We define hyperfunctions $f_j$ $(1 \leq j \leq 2)$ as follows;

$$f_1 = (g_1 + g_{-1})/2 \quad \text{and} \quad f_2 = (g_1 - g_{-1})/2.$$

THEOREM 2.8. 1) *$\{f_j; 0 \leq j \leq 2\}$ is a basis of $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$.*

2) *$\{f_j; 0 \leq j \leq 1\}$ is a basis of $\mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q})$.*

PROOF. From Proposition 2.7 and the fact that dim $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q}) = 3$, 1) is clear. By Proposition 2.6, $f_0$ and $f_1$ are both $G$-invariant. Conversely, from Proposition 2.6 and 2.7, one can easily see that for any $f \in \mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q})$ there exist complex numbers $\alpha_0$, $\alpha_1$ such that $f = \alpha_0 f_0 + \alpha_1 f_1$. Therefore 2) of the theorem is proved.

REMARK 1. Let $G_1 = G_0 \cup k_2 G_0$ and $G_2 = G_0 \cup k_1 k_2 G_0$. Then $G_j$ is Lie subgroups of $O(1, q)$ and $G_2 = SO(1, q)$. Let $\mathscr{B}_\nu^{G_j}(\boldsymbol{R}^{1+q})$ be the vector subspace $(\subset \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q}))$ of all $G_j$-invariants in $\mathscr{B}_\nu(\boldsymbol{R}^{1+q})$, for $j = 1, 2$. Then it is clear that $\mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q}) \subset \mathscr{B}_\nu^{G_2}(\boldsymbol{R}^{1+q})$ and $\mathscr{B}_\nu^{G_1}(\boldsymbol{R}^{1+q}) \subset \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$. But from Proposition 2.6 and Theorem 2.8, we have

$$\mathscr{B}_\nu^{G}(\boldsymbol{R}^{1+q}) = \mathscr{B}_\nu^{G_2}(\boldsymbol{R}^{1+q}) \subset \mathscr{B}_\nu^{G_1}(\boldsymbol{R}^{1+q}) = \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q}).$$

REMARK 2. Since $f_2^{k_1} = -f_2$ from Proposition 2.6,

$$\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q}) = \langle f_0 \rangle \oplus \langle f_1 \rangle \oplus \langle f_2 \rangle$$

is the irreducible decomposition of the respresentation over $\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{1+q})$ with respect to the finite group $\{e, k_1\}$.

## §3. $p > 1$ and $q > 1$

In this section, we give spherical hyperfunctions using integral repesentation for the case $p>1$, $q>1$. That is $G=O(p,q)$ and $G_0=SO(p,q)$. For each $\varepsilon\in\{1,-1\}$ and $j$ $(1\le j\le p)$, we denote by $U_j^{(\varepsilon)}$ the set of all $(z,w)\in \boldsymbol{C}^{p+q}$ (here $z\in\boldsymbol{C}^p$ and $w\in\boldsymbol{C}^q$) such that $\varepsilon\mathrm{Im}z_j>\|\mathrm{Im}z\|$, where $z=(z_1,\dots,z_p)$ and see §2 for the notation $\|\ \|$ and Im. Put $V_j^{(\pm 1)}=\{(z,w)\in\boldsymbol{C}^{p+q};\ \pm\mathrm{Im}w_j>0\}$, for $1\le j\le q$. Then $U_j^{(\varepsilon)}$ and $V_j^{(\varepsilon)}$ are both convex in $\boldsymbol{C}^{p+q}$. Let

$$\mathscr{W}'=\{U_j^{(\varepsilon)};\ \varepsilon\in\{1,-1\},\ 1\le j\le p\}\cup\{V_j^{(\varepsilon)};\ \varepsilon\in\{1,-1\},\ 1\le j\le q\}$$

and $\mathscr{W}=\mathscr{W}'\cup\{\boldsymbol{C}^{p+q}\}$. Then it is easily seen that $(\mathscr{W},\mathscr{W}')$ is relative Stein covering of $(\boldsymbol{C}^{p+q},\boldsymbol{C}^{p+q}\backslash\boldsymbol{R}^{p+q})$. (For the relative Stein covering, see [7]). Indeed, from the definition of $V_j^{(\varepsilon)}$,

$$(\cup\{V_j^{(\varepsilon)};\ \varepsilon\in\{\pm 1\},\ 1\le j\le q\})^c\subset\{(z,w)\in\boldsymbol{C}^{p+q};\ \mathrm{Im}w_j=0\,(1\le j\le q)\},$$

where $A^c$ is the complement of a set $A$. But since

$$\{(z,w)\in\boldsymbol{C}^{p+q};\ \mathrm{Im}z_j\ne 0,\ \mathrm{Im}w_k=0\ \ (1\le k\le q)\}\subset U_j^{(1)}\cup U_j^{(-1)}\quad\text{for each } j,$$

we have $\boldsymbol{C}^{p+q}\backslash\boldsymbol{R}^{p+q}\subset\cup\{W;\ W\in\mathscr{W}'\}$.

Let $e_j=(0,\cdots,\underset{j}{1},\cdots,0)\in\boldsymbol{R}^p$. For each $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$ such that $\varepsilon_j\in\{-1,1\}$ for $1\le j\le p$, we denote by $S_\varepsilon$ the set of all $\xi$ in $S^{p-1}$ such that $\langle\xi,\varepsilon_je_j\rangle\ge 0$ for any $j$ $(1\le j\le p)$ (for the notation $\langle\ \rangle$, see §2). For each $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$ let $D_\varepsilon$ be the set of all $(z,w)\in\boldsymbol{C}^{p+q}$ such that $\langle\mathrm{Im}z,\xi\rangle+\langle\mathrm{Im}w,\eta\rangle>0$ for any $\xi$ in $S_\varepsilon$ and $\eta$ in $S^{q-1}$, where $\mathrm{Im}z=(\mathrm{Im}z_1,\dots,\mathrm{Im}z_n)$ for each $z$ in $\boldsymbol{C}^n$.

LEMMA 3.1. *$D_\varepsilon=\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$ for any $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$.*

PROOF. Since $\varepsilon_je_j\in S_\varepsilon$ for any $j$ $(1\le j\le p)$ and the minimum value of $\langle\mathrm{Im}w,\eta\rangle$ $(\eta\in S^{q-1})$ is $-\|\mathrm{Im}w\|$, if $(z,w)\in D_\varepsilon$ then $\langle\mathrm{Im}z,\varepsilon_je_j\rangle>\|\mathrm{Im}w\|$. Hence $(z,w)\in U_j^{(\varepsilon_j)}$ for any $j$ $(1\le j\le p)$. Therefore $D_\varepsilon\subset\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$. Conversely, if $(z,w)\in\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$ then $\varepsilon_j\mathrm{Im}z_j>\|\mathrm{Im}w\|$ for any $j$ $(1\le j\le p)$. It is easily seen that $\langle\mathrm{Im}z,\xi\rangle>\|\mathrm{Im}w\|$ for any $\xi\in S_\varepsilon$ and $(z,w)\in\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$. Indeed, since $\varepsilon_1\xi_1+\cdots+\varepsilon_p\xi_p\ge 1$ for any $\xi\in S_\varepsilon$,

$$\langle\mathrm{Im}z,\xi\rangle>(\varepsilon_1\xi_1+\cdots+\varepsilon_p\xi_p)\|\mathrm{Im}w\|\ge\|\mathrm{Im}w\|$$

for any $\xi\in S_\varepsilon$ and $(z,w)\in\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$. Hence $(z,w)\in D_\varepsilon$, because the minumum of $\langle\mathrm{Im}w,\eta\rangle$ $(\eta\in S^{q-1})$ is $-\|\mathrm{Im}w\|$. Therefore $D_\varepsilon\supset\bigcap_{1\le j\le p}U_j^{(\varepsilon_j)}$. This completes the proof of the lemma.

Put $\Delta(z)=\Delta(z;p,q)=(\mathrm{ch}z)^{p-1}(\mathrm{sh}z)^{q-1}$ and $\pi_\varepsilon=\varepsilon_1\varepsilon_2\cdots\varepsilon_p$. (See §0 for the notation; ch, sh.)

LEMMA 3.2. *For each* $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_p)$ $(\varepsilon_j \in\{1, -1\})$, *the integral*

$$\psi_\varepsilon(z, w)=\pi_\varepsilon \int_0^\infty \int_{S_\varepsilon} \int_{S^{q-1}} e^{i\lambda[\langle z,\xi\rangle \mathrm{ch}(t-i\mu)+\langle w,\eta\rangle \mathrm{sh}(t-i\mu)]} \Delta(t-i\mu) d\xi d\eta dt$$

*converges absolutely and uniformly on every compact subset of* $D_\varepsilon$ *and is holomorphic on* $D_\varepsilon$. *Here* $d\xi$ $(d\eta)$ *is the normalized* $SO(p)$-*invariant* ($SO(q)$-*invariant*) *measure on* $S^{p-1}(S^{q-1})$ *such that* $\int_{S^{p-1}} d\xi=1$ $\left(\int_{S^{q-1}} d\eta=1\right)$, *respectively.*

PROOF. Since

$$\begin{aligned}
&\operatorname{Re}\{i\lambda[\langle z, \xi\rangle \mathrm{ch}(t-i\mu)+\langle w, \eta\rangle \mathrm{sh}(t-i\mu)]\} \\
&\quad=-|\lambda|[e^t \operatorname{Im}(\langle z, \xi\rangle+\langle w, \eta\rangle)+\operatorname{Im} e^{-t+2i\mu}(\langle z, \xi\rangle-\langle w, \eta\rangle)]/2,
\end{aligned}$$

the lemma is clear.

For each $(a, b) \in \boldsymbol{R}^2$ and $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_p)$ $(\varepsilon_j \in\{\pm 1\})$, we denote by $\chi_\varepsilon(z, w; a, b)$ an entire holomorphic functin on $\boldsymbol{C}^{p+q}$ defined by the following integral;

$$i\pi_\varepsilon \int_a^b \int_{S_\varepsilon} \int_{S^{q-1}} e^{i\lambda[\langle z,\xi\rangle \cos\zeta-i\langle w,\eta\rangle \sin\zeta]} \Delta(-i\zeta; p, q) d\xi d\eta d\zeta.$$

Put $\varphi_\varepsilon(z, w)=\psi_\varepsilon(z, w)-\chi_\varepsilon(z, w; 0, \mu)$. Then, by Lemma 3.2, $\varphi_\varepsilon$ is holomorphic on $D_\varepsilon$ for any $\varepsilon$. Moreover, from the definition of $\varphi_\varepsilon$, it is easily seen that $\varphi_\varepsilon$ satisfies the following differential equations

$$\begin{aligned}
&[(\partial/\partial z_1)^2+\cdots+(\partial/\partial z_p)^2-(\partial/\partial w_1)^2-\cdots-(\partial/\partial w_q)^2] \varphi_\varepsilon=-\lambda^2 \varphi_\varepsilon, \\
&(w_j \partial/\partial w_k-w_k \partial/\partial w_j) \varphi_\varepsilon=0 \qquad \text{for any } 1 \leq j,\ k \leq q .
\end{aligned}$$

Put $H(z, w; \xi, \eta, t)=\langle z, \xi\rangle \mathrm{ch} t+\langle w, \eta\rangle \mathrm{sh} t$ for $(z, w, \xi, \eta, t) \in \boldsymbol{C}^p \times \boldsymbol{C}^q \times S^{p-1} \times S^{q-1} \times \boldsymbol{C}$. Then $H$ is holomorphic with respect to the variables $(z, w, t)$ and real analytic with respect to the variables $(\xi, \eta)$. For fixed $\xi$ in $S_\varepsilon$, we denote by $h(z, w; \xi)$ a holomorphic function on $D_\varepsilon$ defined by the following integral:

$$\begin{aligned}
h(z, w; \xi)= & \int_0^\infty \int_{S^{q-1}} e^{i\lambda H(z,w;\xi,\eta,t-i\mu)} \Delta(t-i\mu; p, q) d\eta dt \\
& -i \int_0^\mu \int_{S^{q-1}} e^{i\lambda H(z,w;\xi,\eta,-i\zeta)} \Delta(-i\theta; p, q) d\eta d\zeta .
\end{aligned}$$

Then $h$ is real analytic with respect to $\xi$ in $S_\varepsilon$ and we have

$$\varphi_\varepsilon(z, w) = \pi_\varepsilon \int_{S_\varepsilon} h(z, w; \xi) d\xi$$

for any $(z, w) \in D_\varepsilon$.

For the purpose of the proof of the rotation invariance with respect to the variables $(x_1, \cdots, x_p)$, we use the following coordinate system on the sphere $S^{p-1}$;

$$\begin{cases} \xi_1(\theta) = \cos\theta_1, \\ \xi_2(\theta) = \sin\theta_1 \cos\theta_2 \\ \quad\vdots \\ \xi_{p-1}(\theta) = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\theta_{p-1}, \\ \xi_p(\theta) = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \sin\theta_{p-1}, \end{cases}$$

where $0 \leq \theta_j < \pi$ $(1 \leq j \leq p-2)$ and $0 \leq \theta_{p-1} < 2\pi$. It is well known that the normalized $SO(p)$-invariant measure $d\xi$ is represented with respect to this coordinate as follows;

$$d\xi = \frac{\Gamma(p/2)}{2\pi^{p/2}} (\sin\theta_1)^{p-2} (\sin\theta_2)^{p-3} \cdots \sin\theta_{p-2} d\theta_1 d\theta_2 \cdots d\theta_{p-1}.$$

Set $I^{(1)} = I^{(1,1)} = \{\theta; 0 \leq \theta \leq \pi/2\}$, $I^{(-1)} = I^{(-1,1)} = \{\theta; \pi/2 \leq \theta \leq \pi\}$, $I^{(1,-1)} = \{\theta; 3\pi/2 \leq \theta \leq 2\pi\}$ and $I^{(-1,-1)} = \{\theta; \pi \leq \theta \leq 3\pi/2\}$. Then it is easily seen that for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$ we have

$$S_\varepsilon = \{(\xi_1(\theta), \cdots \xi_p(\theta)); \theta_j \in I^{(\varepsilon_j)} (1 \leq j \leq p-2), \theta_{p-1} \in I^{(\varepsilon_{p-1}, \varepsilon_p)}\}.$$

Indeed, since if $(\xi_1(\theta), \cdots, \xi_p(\theta)) \in S_\varepsilon$ then $\varepsilon_j \xi_j(\theta) \geq 0$ $(1 \leq j \leq p)$, we have $\varepsilon_j \cos\theta_j \geq 0$ $(1 \leq j \leq p-1)$ and $\varepsilon_p \sin\theta_{p-1} \geq 0$. Hence $\theta_j \in I^{(\varepsilon_j)}$ $(1 \leq j \leq p-2)$ and $\theta_{p-1} \in I^{(\varepsilon_{p-1}, \varepsilon_p)}$ if and only if $(\xi_1(\theta), \cdots, \xi_p(\theta)) \in S_\varepsilon$. Put

$$S_\varepsilon^{(k)} = \{\xi(\theta) \in S_\varepsilon; \theta_k = \pi/2\} \quad \text{for each } k \in \{1, \cdots, p-2\}$$

$$S_\varepsilon^{(p-1)} = \{\xi(\theta) \in S_\varepsilon; \theta_{p-1} = \pi(2 - \varepsilon_p)/2\},$$

$$S_\varepsilon^{(p)} = \{\xi(\theta) \in S_\varepsilon; \theta_{p-1} = a_\varepsilon\},$$

where $\xi(\theta) = (\xi_1(\theta), \cdots, \xi_p(\theta))$, $a_\varepsilon = 0$ if $\varepsilon_{p-1} = \varepsilon_p = 1$, $a_\varepsilon = 2\pi$ if $\varepsilon_{p-1} = -\varepsilon_p = 1$ and $a_\varepsilon = \pi$ if $\varepsilon_{p-1} = -\varepsilon_p = -1$ or $\varepsilon_{p-1} = \varepsilon_p = -1$. Then one can easily see that $\partial S_\varepsilon = \bigcup_{1 \leq k \leq p} S_\varepsilon^{(k)}$ for each $\varepsilon$, where $\partial S_\varepsilon$ is the boundary of $S_\varepsilon$. Indeed, by virtue of the definition of $S_\varepsilon^{(k)}$, we have $S_\varepsilon^{(k)} = S_\varepsilon \cap \{\xi_k(\theta) = 0\}$ for any $\varepsilon$ and $k$ $(1 \leq k \leq p)$. We equip the sphere $S^{p-1}$ with the orientation which is induced by the canonical orientation of $\{\theta; 0 \leq \theta \leq \pi\}^{p-2} \times \{\theta; 0 \leq \theta < 2\pi\}$ and the map

$$(\theta_1, \cdots, \theta_{p-1}) \longmapsto (\xi_1(\theta), \cdots, \xi_p(\theta)).$$

Moreover, for any $\varepsilon$ and $k$ ($1 \leq k \leq p$), $S_\varepsilon^{(k)}$ can be equiped with the orientation which is compatible with the above orientation of $S^{p-1}$.

THEOREM 3.3 (*Stokes*). *Let $\omega$ be a differential form of the degree $p-2$ on $S^{p-1}$, then for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$,*

$$\int_{S_\varepsilon} d\omega = \sum_{1 \leq j \leq p-2} (-1)^{j+1} \varepsilon_j \int_{S_\varepsilon^{(j)}} \iota_{\varepsilon,j}^*(\omega) + (-1)^p \varepsilon_{p-1} \varepsilon_p \int_{S_\varepsilon^{(p-1)}} \iota_{\varepsilon,p-1}^*(\omega)$$
$$+ (-1)^{p+1} \varepsilon_{p-1} \varepsilon_p \int_{S_\varepsilon^{(p)}} \iota_{\varepsilon,p}^*(\omega),$$

*where $\iota_{\varepsilon,j}$ is the inclusion map from $S_\varepsilon^{(j)}$ to $S^{p-1}$ for each $\varepsilon$ and $j$ and $\iota_{\varepsilon,j}^*(\omega)$ is the pull-back of $\omega$ by the map $\iota_{\varepsilon,j}$.*

Now, we consider the natural action of $SO(p)$ on $\boldsymbol{R}^p$. Then the sphere $S^{p-1}$ is stable under this action. Let $\mathfrak{k} = \mathfrak{so}(p)$ be the Lie algebra of the Lie group $SO(p)$. For each $j$ ($1 \leq j \leq p-1$), set

$$E_j = (a_{ik}) \quad \text{and} \quad K_j(\theta_j) = \exp\, \theta_j E_j,$$

where

$$a_{ik} = \begin{cases} 0 & \text{if } (i, k) \neq (j, j+1),\ (j+1, j) \\ 1 & \text{if } (i, k) = (j+1, j) \\ -1 & \text{if } (i, k) = (j, j+1) \end{cases}$$

and exp is the exponential map of $\mathfrak{k}$ into $SO(p)$ and $\theta_j \in \boldsymbol{R}$. Then one can easily see that

$$\xi(\theta) = {}^t(K_{p-1}(\theta_{p-1}) \cdots K_1(\theta_1)^t e_1),$$

where ${}^tA$ is the transpose of a matrix $A$ and $e_1 = (1, 0, \cdots, 0)$.

For each $k$ ($1 \leq k \leq p-1$), we define the vector field $X_k$ ($X'_k$) on $\boldsymbol{R}^p$ ($S^{p-1}$) such that

$$(X_k f)(x) = -\frac{d}{dt}\bigg|_{t=0} f(\exp(tE_k)x) \qquad \text{for any } f \in C^\infty(\boldsymbol{R}^p)$$

$$((X'_k f)(\xi) = \frac{d}{dt}\bigg|_{t=0} f(\exp(tE_k)\xi) \qquad \text{for any } f \in C^\infty(S^{p-1})$$

for any $x \in \boldsymbol{R}^p$ ($\xi \in S^{p-1}$), respectively. Then

$$X_k = x_{k+1}\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_{k+1}} \quad \text{for any } k\ (1 \leq k \leq p-1)$$

and

$$X'_k = \cos\theta_{k+1}\frac{\partial}{\partial\theta_k} - \cot\theta_k \sin\theta_{k+1}\frac{\partial}{\partial x_{k+1}} \quad (1 \leq k \leq p-2),\ X'_{p-1} = \frac{\partial}{\partial x_{p-1}}.$$

Indeed, the first and second assertion for $k = p-1$ are simply seen. For the second assertion except for $k = p-1$, we need some calculations. Since $K_k(t)K_j(\theta_j) = K_j(\theta_j)K_k(t)$ for $j \geq k+2$, we have

$$\begin{aligned} &K_k(t)K_{p-1}(\theta_{p-1})\cdots K_1(\theta_1) \\ &\quad = K_{p-1}(\theta_{p-1})\cdots K_{k+2}(\theta_{k+2})K_k(t)K_{k+1}(\theta_{k+1})K_k(\theta_k)\cdots K_1(\theta_1). \end{aligned}$$

On the other hand, we can choose $\tilde{\theta}_k = \tilde{\theta}_k(t, \theta_k, \theta_{k+1})$, $\tilde{\theta}_{k+1} = \tilde{\theta}_{k+1}(t, \theta_k, \theta_{k+1})$ and $\varphi = \varphi(t, \theta_k, \theta_{k+1})$ such that

$$K_k(t)K_{k+1}(\theta_{k+1})K_k(\theta_k) = K_{k+1}(\tilde{\theta}_{k+1})K_k(\tilde{\theta}_k)K_{k+1}(\varphi).$$

In fact, such $\tilde{\theta}_k$, $\tilde{\theta}_{k+1}$ are given as follows;

$$\begin{cases} \cos\tilde{\theta}_k = \cos t\cos\theta_k - \sin t\sin\theta_k\cos\theta_{k+1}, \\ \sin\tilde{\theta}_k\cos\tilde{\theta}_{k+1} = \sin t\cos\theta_k + \cos t\sin\theta_k\cos\theta_{k+1}, \\ \sin\tilde{\theta}_k\sin\tilde{\theta}_{k+1} = \sin\theta_k\sin\theta_{k+1}. \end{cases}$$

Hence $\partial\tilde{\theta}_k/\partial t|_{t=0} = \cos\theta_{k+1}$ and $\partial\tilde{\theta}_{k+1}/\partial t|_{t=0} = -\cot\theta_k\sin\theta_{k+1}$. Since $K_{k+1}(\varphi)K_j(\theta_j) = K_j(\theta_j)K_{k+1}(\varphi)$ for $j \leq k-1$ and $K_{k+1}(\varphi)\,{}^te_1 = {}^te_1$ for $1 \leq k \leq p-2$, we have the second assertion.

Since $X_k$ is a real analytic vector field on $\boldsymbol{R}^p$, we can extend it on the holomorphic vector field on $\boldsymbol{C}^p$, uniquely. In this section, we use the same notation $X_k$ for such a vector field. Let $F$ be a $C^\infty$-function on $\boldsymbol{C}$. Set $G(z, \xi) = F(\langle z, \xi\rangle)$ for $z \in \boldsymbol{C}^p$ and $\xi \in S^{p-1}$. Then we have $X_k G(z, \xi) = X'_k G(z, \xi)$. Indeed, snce $\langle\ \rangle$ is $SO(p)$-invariant,

$$\left.\frac{d}{dt}\right|_{t=0} G(exp(tX)z,\ exp(tX)\xi) = 0 \quad \text{for any } X \in \mathfrak{k}.$$

Here we extend the action of $SO(p)$ on $\boldsymbol{R}^p$ to $\boldsymbol{C}^p$, naturally. Hence we have the assertion from the definition of $X_k$ and $X'_k$.

Put $\omega(\theta) = \Gamma(p/2)/(2\pi^{p/2})\ (\sin\theta_1)^{p-2}(\sin\theta_2)^{p-3}\cdots(\sin\theta_{p-2})$. Then $d\xi = \omega(\theta)d\theta_1 \wedge \cdots \wedge d\theta_{p-1}$. We denote by $\iota(X)(\omega)$ the interior product of $X$ and $\omega$.

Lemma 3.4. *We have*

1) $\iota(X'_k)(d\xi)=(-1)^{k-1}\omega(\theta)[d\theta_k(X'_k)d\theta_1\wedge\overset{k}{\cdots\vee\cdots}\wedge d\theta_{p-1}$

$$-d\theta_{k+1}(X'_k)d\theta_1\wedge\overset{k+1}{\cdots\vee\cdots}\wedge d\theta_{p-1}]\quad (\textit{for any } k(1\le k\le p-2))$$

*and* $\iota(X'_{p-1})(d\xi)=(-1)^p\omega(\theta)d\theta_1\wedge\cdots\wedge d\theta_{p-2}$,

2) *for any* $\varepsilon$ *and* $j$ $(1\le j\le p)$

$$\iota^*_{\varepsilon,j}(\iota(X'_k)(d\xi))=\delta_{k,j}(-1)^{k-1}[\omega(\theta)d\theta_k(X'_k)]_{\theta_k=\pi/2}\,d\theta_1\wedge\overset{k}{\cdots\vee\cdots}\wedge d\theta_{p-1}$$

$$+\delta_{k+1,j}(-1)^k[\omega(\theta)d\theta_{k+1}(X'_k)]_{\theta_{k+1}=\pi/2}\,d\theta_1\wedge\overset{k+1}{\cdots\vee\cdots}\wedge d\theta_{p-1}$$

(*for any* $k$ $(1\le k\le p-3)$),

$$\begin{aligned}&\iota^*_{\varepsilon,j}(\iota(X'_{p-2})(d\xi))\\ &=\delta_{p-2,j}(-1)^{p-3}[\omega(\theta)d\theta_{k-2}(X'_{p-2})]_{\theta_{p-2}=\pi/2}\,d\theta_1\wedge\cdots\wedge d\theta_{p-3}\wedge d\theta_{p-1}\\ &\quad+\delta_{p-1,j}(-1)^{p-2}[\omega(\theta)d\theta_{p-1}(X'_{p-2})]_{\theta_{p-1}=\pi(2-\varepsilon_p)/2}\,d\theta_1\wedge\cdots\wedge d\theta_{p-2},\end{aligned}$$

$$\begin{aligned}\iota^*_{\varepsilon,j}(\iota(X'_{p-1})(d\xi))&=\delta_{p-1,j}(-1)^p[\omega(\theta)]_{\theta_{p-1}=\pi(2-\varepsilon_p)/2}\,d\theta_1\wedge\cdots\wedge d\theta_{p-2}\\ &\quad+\delta_{p,j}(-1)^p[\omega(\theta)]_{\theta_{p-1}=a_\varepsilon}\,d\theta_1\wedge\cdots\wedge d\theta_{p-2},\end{aligned}$$

*where* $d\theta_1\wedge\cdots\vee\cdots\wedge d\theta_{p-1}=d\theta_1\wedge\cdots\wedge d\theta_{k-1}\wedge d\theta_{k+1}\wedge\cdots\wedge d\theta_{p-1}$ *and* $\delta_{k,j}$ *is the Kronecker's* $\delta$.

PROOF. 1) Put $\iota(X'_k)(d\xi)=\sum_{1\le j\le p-1}a_j(\theta)d\theta_1\wedge\overset{j}{\cdots\vee\cdots}\wedge d\theta_{p-1}$. Then we see from the definiton of the interior product that for any $j$ $(1\le j\le p-1)$,

$$a_j(\theta)=\omega(\theta)\det\begin{bmatrix} d\theta_1(X'_k) & c_{1,1} & \cdots & c_{1,j-1} & c_{1,j+1} & \cdots & c_{1,p-1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ d\theta_{p-1}(X'_k) & c_{p-1,1} & \cdots & c_{p-1,j-1} & c_{p-1,j+1} & \cdots & c_{p-1,p-1}\end{bmatrix}$$

where $c_{i,j}=d\theta_i(\partial/\partial\theta_j)$ and det $A$ is the determinant of a matrix $A$. Since $c_{i,j}=d\theta_i(\partial/\partial\theta_j)=\delta_{i,j}$ $(1\le i,j\le p-1)$, if $1\le k\le p-2$ then $a_k(\theta)=(-1)^{k-1}\omega(\theta)d\theta_k(X'_k)$, $a_{k+1}(\theta)=(-1)^k\omega(\theta)d\theta_{k+1}(X'_k)$ and $a_j(\theta)=0$ for $1\le j\le k-1$, $k+2\le j\le p-1$. If $k=p-1$ then $a_{p-1}(\theta)=(-1)^p\omega(\theta)$ and $a_j(\theta)=0$ for $1\le j\le p-2$. Thus 1) of the lemma is proved.

2) From the definition of $\iota^*_{\varepsilon,j}$ and 1), 2) is easily obtained.

Now we recall the functions $\varphi_\varepsilon$, $h$ and the vector field $X_k$ on $\boldsymbol{R}^p$ or $\boldsymbol{C}^p$. In view of the remark on the vector fields $X_k$ and $X'_k$, we have

$$X_k\varphi_\varepsilon(z, w) = \pi_\varepsilon \int_{S_\varepsilon} (X'_k h)(z, w; \xi)d\xi \qquad \text{for any } \varepsilon \text{ and } k.$$

Let $L_{X'_k}$ be the Lie derivative over $S^{p-1}$ with respect to $X'_k$. Then $L_{X'_k}(d\xi) = 0$, because $d\xi$ is an invariant measure. Hence we have for any $\varepsilon$ and $k$,

$$\int_{S_\varepsilon} (X'_k h)(z, w; \xi)d\xi = \int_{S_\varepsilon} L_{X'_k}(h(z, w; \xi)d\xi) \qquad \text{for } (z, w)\in D_\varepsilon.$$

Let $d$ be the exterior derivative over $S^{p-1}$. Since

$$L_{X'_k} = d\circ \iota(X'_k) + \iota(X'_k)\circ d \quad \text{and} \quad d(hd\xi) = 0,$$

we have for any $\varepsilon$ and $k$.

$$\int_{S_\varepsilon} L_{X'_k}(h(z, w; \xi)d\xi) = \int_{S_\varepsilon} d(\iota(X'_k)(h(z, w; \xi)d\xi)) \qquad \text{for } (z, w)\in D_\varepsilon.$$

Thanks to Stokes' Theorem 3.3 and from Lemma 3.4, we have

LEMMA 3.5. *For any* $\varepsilon$ *and* $(z, w)\in D_\varepsilon$,

$$(X_k\varphi_\varepsilon)(z, w) = \varepsilon_k\pi_\varepsilon \int_{S_\varepsilon^{(k)}} [h(z, w; \xi(\theta))\omega(\theta)\cos\theta_{k+1}]_{\theta_k=\pi/2}\, d\theta_1 \overset{k}{\cdots\check{}\cdots} d\theta_{p-1}$$

$$- \varepsilon_{k+1}\pi_\varepsilon \int_{S_\varepsilon^{(k+1)}} [h(z, w; \xi(\theta))\omega(\theta)\cot\theta_k]_{\theta_{k+1}=\pi/2}\, d\theta_1 \overset{k+1}{\cdots\check{}\cdots} d\theta_{p-1}$$

(*for any* $k$ $(1\le k\le p-3)$),

$$(X_{p-2}\varphi_\varepsilon)(z, w) =$$

$$\varepsilon_{p-2}\pi_\varepsilon \int_{S_\varepsilon^{(p-2)}} [h(z, w; \xi(\theta))\omega(\theta)\cos\theta_{p-1}]_{\theta_{p-2}=\pi/2}\, d\theta_1 \cdots d\theta_{p-3} d\theta_{p-1}$$

$$- \varepsilon_{p-1}\pi_\varepsilon \int_{S_\varepsilon^{(p-1)}} [h(z, w; \xi(\theta))\omega(\theta)\cot\theta_{p-2}]_{\theta_{p-1}=\pi(2-\varepsilon_p)/2}\, d\theta_1 \cdots d\theta_{p-2},$$

$$(X_{p-1}\varphi_\varepsilon)(z, w) =$$

$$\varepsilon_{p-1}\varepsilon_p\pi_\varepsilon \int_{S_\varepsilon^{(p-1)}} [h(z, w; \xi(\theta))\omega(\theta)]_{\theta_{p-1}=\pi(2-\varepsilon_p)/2}\, d\theta_1 \cdots d\theta_{p-2}$$

$$- \varepsilon_{p-1}\varepsilon_p\pi_\varepsilon \int_{S_\varepsilon^{(p)}} [h(z, w; \xi(\theta))\omega(\theta)]_{\theta_{p-1}=a_\varepsilon}\, d\theta_1 \cdots d\theta_{p-2}.$$

Set $Y = w_1\partial/\partial z_1 + z_1\partial/\partial w_1$. Then it is easily seen that

$$\{Y - D(t, \theta_1, \tau_1; \partial/\partial t, \partial/\partial\theta_1, \partial/\partial\tau_1)\}\ e^{i\lambda H(z,w;\xi(\theta),\eta(\tau),t-i\mu)} = 0,$$

where $D(t, \xi, \eta) = D(t, \theta_1, \tau_1; \partial/\partial t, \partial/\partial\theta_1, \partial/\partial\tau_1) = \cos\theta_1\cos\tau_1\partial/\partial t - \sin\tau_1\cos\theta_1 \coth(t - i\mu)\partial/\partial\tau_1 - \sin\theta_1\cos\tau_1\tanh(t - i\mu)\partial/\partial\theta_1$ and $\eta(\tau) = (\eta_1(\tau), \cdots, \eta_q(\tau))$ ($\in S^{q-1}$) is defined in a way similar to $\xi(\theta)$.

Let $\omega_p(\theta) = \omega(\theta)$ and $\omega_q(\tau)$ be defined in a way similar to $\omega_p(\theta)$. Then $d\eta = \omega_q(\tau)d\tau_1 \wedge \cdots \wedge d\tau_{q-1}$. Now, we calculate $Y\varphi_\varepsilon$. First we have

$$Y\psi_\varepsilon(z, w) = \pi_\varepsilon\int_0^\infty\int_{S_\varepsilon}\int_{S^{q-1}} (D(t, \xi, \eta)e^{i\lambda H(z,w;\xi,\eta,t-i\mu)})\Delta(t - i\mu)d\eta d\xi dt$$

$$= \pi_\varepsilon\int_{S_\varepsilon}\int_{S^{q-1}} [\Delta(t - i\mu)e^{i\lambda H(\cdot,t-i\mu)}]_{t=0}^{t=\infty}\ \omega_p(\theta)\cos\theta_1\cos\tau_1 d\theta_1\cdots d\theta_{p-1}d\eta$$

$$- \pi_\varepsilon\int_0^\infty\int_{S_\varepsilon}\int_{S^{q-1}} \frac{d\Delta(t - i\mu)}{dt}\ e^{i\lambda H}\,\omega_p(\theta)\cos\theta_1\cos\tau_1 d\theta_1\cdots d\theta_{p-1}d\eta dt$$

$$+ \pi_\varepsilon\int_0^\infty\int_{S_\varepsilon}\int_{S^{q-1}} e^{i\lambda H}\,\Delta(t - i\mu)\coth(t - i\mu)\cos\theta_1\frac{\partial}{\partial\tau_1}(\sin\tau_1\omega_q(\tau))d\xi d\eta dt$$

$$- \varepsilon_1\pi_\varepsilon\int_0^\infty\int_{S_\varepsilon^{(1)}}\int_{S^{q-1}} v(\theta', \tau, t)\Delta(t - i\mu)\mathrm{th}(t - i\mu)\cos\tau_1 d\theta_2\cdots d\theta_{p-1}d\eta dt$$

$$+ \pi_\varepsilon\int_0^\infty\int_{S_\varepsilon}\int_{S^{q-1}} e^{i\lambda H}\,\Delta(t - i\mu)\mathrm{th}(t - i\mu)\cos\tau_1\frac{\partial}{\partial\theta_1}(\sin\theta_1\omega_p(\theta))d\theta_1\cdots d\theta_{p-1}\,d\eta dt,$$

where $v(\theta', \tau, t; z, w) = v(\theta', \tau, t) = [\omega_p(\theta)^{i\lambda H(\cdot,t-i\mu)}]_{\theta_1=\pi/2}$ and $\mathrm{th}(t) = \tanh(t)$. But

$$- \cos\theta_1\cos\tau_1\omega_p(\theta)\omega_p(\tau)\frac{d\Delta(t - i\mu)}{dt}$$

$$+ \Delta(t - i\mu)\coth(t - i\mu)\cos\theta_1\omega_p(\theta)\frac{\partial}{\partial\tau_1}(\sin\tau_1\omega_q(\tau))$$

$$+ \Delta(t - i\mu)\tanh(t - i\mu)\cos\tau_1\omega_q(\tau)\frac{\partial}{\partial\theta_1}(\sin\theta_1\omega_p(\theta)) = 0.$$

Thus we have

$$Y\psi_\varepsilon(z, w) = -\pi_\varepsilon\int_{S_\varepsilon}\int_{S^{q-1}} e^{i\lambda H(z,w;\xi(\theta),\eta(\tau),-i\mu)}\,\Delta(-i\mu; p, q)\cos\theta_1\cos\tau_1 d\xi d\eta$$

$$- \varepsilon_1\pi_\varepsilon\int_0^\infty\int_{S_\varepsilon^{(1)}}\int_{S^{q-1}} v(\theta', \eta, t; z, w)\Delta(t - i\mu; p - 1, q + 1)\cos\tau_1 d\theta_2\cdots d\theta_{p-1}dndt.$$

By the same calculation for $Y\chi_\varepsilon(z, w; a, b)$, we have

$$Y\chi_\varepsilon(z, w; a, b) =$$

$$-\pi_\varepsilon\int_{S_\varepsilon}\int_{S^{q-1}}[e^{i\lambda H(\cdot,-i\zeta)}\Delta(-i\zeta; p, q)]_{\zeta=a}^{\zeta=b}\cos\theta_1\cos\tau_1 d\xi d\eta$$

$$-i\varepsilon_1\pi_\varepsilon\int_a^b\int_{S_\varepsilon^{(1)}}\int_{S^{q-1}}v(\theta', \eta, -i\zeta)\Delta(-i\zeta; p-1, q+1)\cos\tau_1 d\theta_2\cdots d\theta_{p-1}d\eta d\zeta.$$

Therefore we have

LEMMA 3.6. *For any $\varepsilon$ and $(z, w)\in D_\varepsilon$,*

$$Y\varphi_\varepsilon(z, w) =$$

$$-\varepsilon_1\pi_\varepsilon\int_0^\infty\int_{S_\varepsilon^{(1)}}\int_{S^{q-1}}v(\theta', \tau, t; z, w)\Delta(t-i\mu; p-1, q+1)\cos\tau_1 d\theta_2\cdots d\theta_{p-1}d\eta dt$$

$$+i\varepsilon_1\pi_\varepsilon\int_0^\mu\int_{S_\varepsilon^{(1)}}\int_{S^{q-1}}v(\theta', \tau, -i\zeta)\Delta(-i\zeta; p-1, q+1)\cos\tau_1 d\theta_2\cdots d\theta_{p-1}d\eta d\zeta.$$

Now, we give spherical hyperfunctions by the elements of the Čeck cohomology $H^{p+q-1}(\mathscr{W}'; \mathscr{O})$. Under the same notation as in §2, we put

$$f = [(U_1^{(1)}\cap\cdots\cap U_p^{(1)}\cap V_1^{(1)}\cap\cdots\cap V_q^{(1)}; \chi(z, w))],$$

where

$$\chi(z, w) = \int_{-\pi}^{\pi}\int_{S_\varepsilon}\int_{S^{q-1}}e^{i\lambda[\langle z,\xi\rangle\cos\zeta - i\langle w,\eta\rangle\sin\zeta]}\Delta(-i\zeta; p, q)d\xi d\eta d\zeta.$$

Then it is clear that $f$ is a real analytic function on $\boldsymbol{R}^{p+q}$ and $f\in\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{p+q})$. Let

$$g = [(U_1^{(\varepsilon_1)}\cap\cdots\cap U_p^{(\varepsilon_p)}\cap V_1^{(1)}\cap\cdots\cap V_q^{(1)}; \varphi_\varepsilon)].$$

Then we have

PROPOSITION 3.7. $g\in\mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{p+q})$.

PROOF. It is clear that $g$ satisfies the following differential equations;

$$[(\partial/\partial x_1)^2+\cdots+(\partial/\partial x_p)^2-(\partial/\partial y_1)^2-\cdots-(\partial/\partial y_q)^2]g = -\lambda^2 g,$$

$$(y_j\partial/\partial y_k - y_k\partial/\partial y_j)g = 0 \quad \text{for any } 1\le j,\ k\le q.$$

Since the Lie algebra $\mathfrak{g}$ is spanned by the differential operators $x_k\partial/x_{k+1} - x_{k+1}\partial/x_k$ $(1\le k\le p-1)$, $y_k\partial/y_{k+1} - y_{k+1}\partial/y_k$ $(1\le k\le q-1)$, $y_1\partial/\partial x_1 + x_1\partial/\partial y_1$, we must prove that

$(x_k\partial/\partial x_{k+1} - x_{k+1}\partial/\partial x_k)g = 0$ $(1 \le k \le p-1)$ and $(y_1\partial/\partial x_1 + x_1\partial/\partial y_1)g = 0$.

First we prove that $(x_{k+1}\partial/\partial x_k - x_k\partial/\partial x_{k+1})g = 0$. For each $k$ $(1 \le k \le p)$, set $\varepsilon(k) = (\varepsilon_1, \cdots, \varepsilon_{k-1}, 0, \varepsilon_{k+1}, \cdots, \varepsilon_p)$, where $\varepsilon_j \in \{\pm 1\}$ for $j \ne k$ and $U(\varepsilon(k)) = \bigcap_{\substack{1 \le j \le p \\ j \ne k}} U_j^{(\varepsilon_j)}$ for any $1 \le k \le p$ and $\varepsilon(k)$. Put

$$\varphi_{\varepsilon(k)}(z, w) = \varepsilon_k \pi_\varepsilon \int_{S_\varepsilon^{(k)}} [h(z, w; \xi(\theta))\omega_p(\theta)\cos\theta_{k+1}]_{\theta_k = \pi/2}\, d\theta_1 \cdots \overset{k}{\vee} \cdots d\theta_{p-1} \qquad (\text{if } 1 \le k \le p-2),$$

$$\psi_{\varepsilon(k)}(z, w) = \varepsilon_k \pi_\varepsilon \int_{S_\varepsilon^{(K)}} [h(z, w; \xi(\theta))\omega_p(\theta)\cot\theta_{k-1}]_{\theta_k = \pi/2}\, d\theta_1 \cdots \overset{k}{\vee} \cdots d\theta_{p-1} \qquad (\text{if } 2 \le k \le p-2),$$

where $d\theta_1 \cdots \overset{k}{\vee} \cdots d\theta_{p-1} = d\theta_1 \cdots d\theta_{k-1} d\theta_{k+1} \cdots d\theta_{p-1}$ and

$$\varphi_{\varepsilon(p-1)}(z, w) = \varepsilon_{p-1}\varepsilon_p \pi_\varepsilon \int_{S_\varepsilon^{(p-1)}} [h(z, w; \xi(\theta))\omega_p(\theta)]_{\theta_{p-1} = b_\varepsilon}\, d\theta_1 \cdots d\theta_{p-2},$$

$$\psi_{\varepsilon(p-1)}(z, w) = \varepsilon_{p-1} \pi_\varepsilon \int_{S_\varepsilon^{(p-1)}} [h(z, w; \xi)\omega_p(\theta)\cot\theta_{p-2}]_{\theta_{p-1} = b_\varepsilon}\, d\theta_1 \cdots d\theta_{p-2},$$

$$\psi_{\varepsilon(p)}(z, w) = \varepsilon_{p-1}\varepsilon_p \pi_\varepsilon \int_{S_\varepsilon^{(p)}} [h(z, w; \xi(\theta))\omega_p(\theta)]_{\theta_{p-1} = a_\varepsilon}\, d\theta_1 \cdots d\theta_{p-2},$$

where $b_\varepsilon = \pi(2 - \varepsilon_p)/2$.

Then it is easily seen that $\varphi_{\varepsilon(k)}$ and $\psi_{\varepsilon(k)}$ are holomorphic on $U(\varepsilon(k))$ for $1 \le k \le p-1$ and $2 \le k \le p$, respectively. In fact, we see from the same proof as in Lemma 3.1 that if $(z, w) \in U(\varepsilon(k))$ then $\langle \mathrm{Im} z, \xi \rangle + \langle \mathrm{Im} w, \eta \rangle > 0$ for any $\xi \in S_\varepsilon^{(k)}$ and $\eta \in S^{q-1}$, where we set $\varepsilon(k) = (\varepsilon_1, \cdots, \varepsilon_{k-1}, 0, \varepsilon_{k+1}, \cdots, \varepsilon_p)$ for $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$. Thus, by the same proof as in Lemma 3.2, $\varphi_{\varepsilon(k)}$ and $\psi_{\varepsilon(k)}$ are both holomorphic on $U(\varepsilon(k))$. For each $k$ $(1 \le k \le p-1)$, let $c_k$ be a $p+q-2$ cochain defined as follows:

$$\{(U(\varepsilon(k)) \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)}; (-1)^{k+1}\varphi_{\varepsilon(k)}) \quad \text{for each } \varepsilon(k),$$
$$(U(\varepsilon(k+1)) \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)}; (-1)^{k+1}\psi_{\varepsilon(k+1)}) \text{ for each } \varepsilon(k+1),$$
$$(\text{otherwise};\ 0)\}.$$

Then $\delta(c_k) = \{(U_1^{(\varepsilon_1)} \cap \cdots \cap U_p^{(\varepsilon_p)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)}; \varphi_{\varepsilon(k)} - \psi_{\varepsilon(k+1)})$, (otherwise ; 0), for $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)\}$. On the other hand, by Lemma 3.5 and the definition of

$\varphi_{\varepsilon(k)}$ and $\psi_{\varepsilon(k)}$, we have

$$X_k\varphi_\varepsilon = \varphi_{\varepsilon(k)} - \psi_{\varepsilon(k+1)} \quad \text{for any } \varepsilon \text{ and } 1 \le k \le p-1.$$

Thus $(x_{k+1}\partial/\partial x_k - x_k\partial/\partial x_{k+1})g = [\delta(c_k)] = 0$ for any $1 \le k \le p-1$.

Next, we prove that $(y_1\partial/\partial x_1 + x_1\partial/\partial y_1)g = 0$. For any $\varepsilon(1) = (0, \varepsilon_2, \cdots, \varepsilon_p)$, let $\chi_{\varepsilon(1)}(z, w)$ be the holomorphic function on $D_\varepsilon$ defined by the right-hand side of the equality of Lemma 3.6. Then in a way similar to the proof of Lemma 3.2, we see that $\chi_{\varepsilon(1)}$ is holomorphic on $U(\varepsilon(1))$. Let $c$ be a $p+q-2$ cochain defined as follows;

$$c = \{(U(\varepsilon(1)) \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \chi_{\varepsilon(1)}),\ \ (\text{otherwise}\,;\,0);\ \text{for } \varepsilon(1) = (0, \varepsilon_2, \cdots, \varepsilon_p)\}.$$

Then

$$\delta(c) = \{(U_1^{(\varepsilon_1)} \cap \cdots \cap U_p^{(\varepsilon_p)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \chi_{\varepsilon(1)}),\ (\text{otherwise}; 0);$$
$$\text{for } \varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)\}.$$

Thus $(y_1\partial/\partial x_1 + x_1\partial/\partial y_1)g = 0$, because $Y\varphi_\varepsilon = \chi_{\varepsilon(1)}$ for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$. Therefore the proposition is proved.

Now, we consider the singular spectrum of the hyperfunction $g$. For any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$ $(\varepsilon_j \in \{\pm 1\})$, let

$$g_\varepsilon = [(U_1^{(\varepsilon_1)} \cap \cdots \cap U_p^{(\varepsilon_p)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)};\ \varphi_\varepsilon)].$$

Then $g = \sum g_\varepsilon$. For each $\varepsilon$ and $(x, y) \in \boldsymbol{R}^{p+q}$, let $\Gamma_\varepsilon(x, y)$ be the dual cone of $D_\varepsilon(x, y)$, where $D_\varepsilon(x, y) = \{(a, b) \in \boldsymbol{R}^{p+q};\ (x+ia,\ y+ib) \in D_\varepsilon\}$. Here $\Gamma_\varepsilon(x, y)$ is regarded as the subset of $\sqrt{-1}\ T^*_{(x,y)}\boldsymbol{R}^{p+q}$. We put

$$\tilde{\Gamma}_\varepsilon(x, y) = \{\tilde{p} \in \sqrt{-1}\ S^*_{(x,y)}\boldsymbol{R}^{p+q};\ p \in \Gamma_\varepsilon(x, y)\}$$

for each $\varepsilon$ and $(x, y) \in \boldsymbol{R}^{p+q}$, where $\tilde{p}$ is the projection of $p \in \sqrt{-1}\ T^*_{(x,y)}\boldsymbol{R}^{p+q}$ to $\sqrt{-1}\ S^*_{(x,y)}\boldsymbol{R}^{p+q}$. Then one can easily see that

$$\tilde{\Gamma}_\varepsilon(x, y) = \{i(a, b)\infty\,;\ \varepsilon_1 a_1 + \cdots + \varepsilon_p a_p \ge \|b\| \text{ and } \varepsilon_j a_j \ge 0 \text{ for } 1 \le j \le p\}.$$

In fact, if $\varepsilon_1\xi_1 + \cdots + \varepsilon_p\xi_p \ge \|\eta\|$ and $\varepsilon_j\xi_j \ge 0$ $(1 \le j \le p)$ then $\xi_1 a_1 + \cdots + \xi_p a_p + \eta_1 b_1 + \cdots + \eta_q b_q \ge \|b\|(\varepsilon_1\xi_1 + \cdots + \varepsilon_p\xi_p) + \eta_1 b_1 + \cdots + \eta_q b_q \ge \|b\|\,\|\eta\| + \eta_1 b_1 + \cdots + \eta_q b_q \ge 0$ for any $(a, b) \in D_\varepsilon(x, y)$. Conversely, if $\xi_1 a_1 + \cdots + \xi_p a_p + \eta_1 b_1 + \cdots + \eta_q b_q \ge 0$ for any $(a, b) \in D_\varepsilon(x, y)$ then $\varepsilon_1\xi_1 + \cdots + \varepsilon_p\xi_p \ge r\|\eta\|$ for any $0 \le r < 1$, because we can choose $(a, b) \in D_\varepsilon(x, y)$ such that $a_j = \varepsilon_j$ and $b_j = -\eta_j r/\|\eta\|$ for $0 \le r < 1$ and $1 \le j \le p$. Thus $\varepsilon_1\xi_1 + \cdots + \varepsilon_p\xi_p \ge \|\eta\|$ and $\varepsilon_j\xi_j \ge 0$. In view of the definition of the singular spectrum, we have

$$S.S\ g_\varepsilon \subset \bigcup_{(x,y)\in \boldsymbol{R}^{p+q}} \tilde{\Gamma}_\varepsilon(x, y) \quad \text{for each } \varepsilon.$$

We put $S = S_{(1,\cdots,1)}$, $\varphi_0(z, w) = \varphi_{(1,\cdots,1)}(z, w)$ and $g_0 = g_{(1,\cdots,1)}$. We shall prove that $(0, 0; i(a, b)\infty) \in S.S\ g_0$ for any $(a, b) \in \boldsymbol{R}^{p+q}$ such that $\|a\| = \|b\| = 2^{-1/2}$ and $a_j \geq 0$ for any $1 \leq j \leq p$. Let

$$D_0 = D_{(1,\cdots,1)} \text{ and } D_1 = \{(z, w) \in \boldsymbol{C}^{p+q};\ \mathrm{Re} z_j > \|\mathrm{Re} w\| \text{ for any } 1 \leq j \leq p\}.$$

Put

$$\varphi_1(z, w) =$$

$$i^{p+q-2} \int_0^\infty \int_S \int_{S^{q-1}} e^{-\lambda[\langle z,\xi\rangle \mathrm{sh}(t-i\mu) + \langle w,\eta\rangle \mathrm{ch}(t-i\mu)]} \Delta(t - i\mu; q, p)\, d\xi d\eta dt.$$

Then $\varphi_j$ is a holomorphic function on $D_j$ $(j = 1, 2)$. Moreover, it is easily seen that

$$\varphi_0(z, w) = \varphi_1(z, w) - \chi_0(z, w; 0, \mu - \pi/2) \quad \text{for any } (z, w) \in D_0 \cap D_1$$

by the same proof as in Proposition 2.3 (or Lemma 1.2), where $\chi_0 = \chi_{(1,\cdots,1)}$. But we have

PROPOSITION 3.8. *Let $(a, b) \in \boldsymbol{R}^{p+q}$ be such that $\|a\| = \|b\| \neq 0$ and $\|a\|^{-1} a \in S$ or $-\|a\|^{-1} a \in S$. Then $\varphi_1$ is not holomorphic on any neighborhood of the point $(z, w) = (ia_1, \cdots, ia_p, ib_1, \cdots, ib_q)$. Hence $\varphi_0$ can't be analytically continued to the previous point.*

COROLLARY 3.9. $(0, 0; i(a, b)\infty) \in S.S\ g_0$ *for any $(a, b) \in \boldsymbol{R}^{p+q}$ such that $\|a\| = \|b\| = 2^{-1/2}$ and $a_j \geq 0$ $(1 \leq j \leq p)$.*

PROOF. Since $S.S\ g_0 \subset \cup \tilde{\Gamma}_{(1,\cdots,1)}(x, y)$, the corollary follows from Proposition 3.8.

For the proof of Proposition 3.8, we need some lemmas. Let $\boldsymbol{N} = \{1, 2, \cdots\}$ and $J_\nu(z)$ $(\mathscr{H}_\nu^{(2)})$ be the Bessel function (Hankel) of order $\nu$.

LEMMA 3.10. *If* $\mathrm{Re}\beta > |\mathrm{Im}\alpha|$, $\nu \in \boldsymbol{N}$ *and* $2\mu \in \boldsymbol{N} \cup \{0\}$ *then we have*

$$\int_0^\infty e^{-\beta \mathrm{sh} t} (\mathrm{ch} t)^{\mu+1} (\mathrm{sh} t)^\nu J_\mu(\alpha \mathrm{ch} t) dt$$

$$= c_1(\nu, \mu)(\partial/\partial\beta)^\nu \{\alpha^\mu (\alpha^2 + \beta^2)^{-\mu/2-1/4} \mathscr{H}^{(2)}_{-\mu-1/2}((\alpha^2 + \beta^2)^{1/2})\}$$

$$- \int_0^1 e^{-\beta(x^2-1)^{1/2}} x^{\mu+1} (x^2 - 1)^{(\nu-1)/2} J_\mu(\alpha x) dx,$$

*where* $c_1(\nu, \mu) = (\pi/2)^{1/2} e^{i\pi(\nu+\mu)}$ *and* $\arg(x^2 - 1) = \pi/2$ *if* $x < 1$.

PROOF. We put $x = \mathrm{ch}t$. Then

$$\int_0^\infty e^{-\beta \mathrm{sh}t}(\mathrm{ch}t)^{\mu+1}(\mathrm{sh}t)^\nu J_\mu(\alpha \mathrm{ch}t)dt$$

$$= \int_1^\infty e^{-\beta(x^2-1)^{1/2}} x^{\mu+1}(x^2-1)^{(\nu-1)/2} J_\mu(\alpha x)\,dx.$$

On the other hand, it is well known that for $\mathrm{Re}\beta > |\mathrm{Im}\alpha|$

$$\int_0^\infty e^{-\beta(x^2-1)^{1/2}} x^{\mu+1}(x^2-1)^{-1/2} J_\mu(\alpha x)dx$$

$$= (\pi/2)^{1/2} e^{i\pi\mu}\alpha^\mu(\alpha^2+\beta^2)^{-\mu/2-1/4}\mathscr{H}^{(2)}_{-\mu-1/2}((\alpha^2+\beta^2)^{1/2}),$$

where $\arg(x^2-1)^{1/2} = \pi/2$ if $x < 1$ (see [1]). This implies the lemma.

Let $U$ be a relatievely compact open subset of $\boldsymbol{C}$. Then for each $\alpha \in \boldsymbol{C}$ we have

LEMMA 3.11. *If $\nu \in \boldsymbol{N}$ and $2\mu \in \boldsymbol{N}$ then there exists a positive number $M$ such that for any $\beta \in U \setminus \{\pm i\alpha\}$*

$$|(\partial/\partial\beta)^\nu\{(\alpha^2+\beta^2)^{-\mu/2}\mathscr{H}^{(2)}_{-\mu}((\alpha^2+\beta^2)^{1/2})\} - c_2(\nu,\mu)\beta^\nu(\alpha^2+\beta^2)^{-\nu-\mu}|$$

$$\leq M|\alpha^2+\beta^2|^{-\nu-\mu+1},$$

*where* $c_2 = c_2(\nu,\mu) = (-1)^\nu 2^{\nu+\mu}\Gamma(\nu+\mu)/\Gamma(\mu)\Gamma(1-\mu)$ *if* $\nu \in \boldsymbol{N}$ *and* $\mu - 1/2 \in \boldsymbol{N} \cup \{0\}$, $(-1)^{\nu+\mu+1/2}\pi^{-1}2^{\nu+\mu}\Gamma(\nu+\mu)$ *if* $\nu \in \boldsymbol{N}$ *and* $\mu \in \boldsymbol{N}$.

PROOF. 1) Let $\mu - 1/2 \in \boldsymbol{N} \cup \{0\}$. It is well known that

$$\mathscr{H}^{(2)}_{-\mu}(z) = J_{-\mu}(z) - (-1)^\mu J_\mu(z).$$

Hence from the definition of $J_{\pm\mu}(z)$, we have $z^{-\mu}\mathscr{H}^{(2)}_{-\mu}(z) =$

$$2^\mu z^{-2\mu}\sum_{k=0}^\infty \frac{(-1)^k 2^{-2k}}{\Gamma(k+1)\Gamma(-\mu+k+1)} z^{2k}$$

$$- (-1)^\mu 2^{-\mu}\sum_{k=0}^\infty \frac{(-1)^k 2^{-2k}}{\Gamma(k+1)\Gamma(\mu+k+1)} z^{2k}.$$

Hence, we have

$$(\partial/\partial\beta)^\nu\{(\alpha^2+\beta^2)^{-\mu/2}\mathscr{H}^{(2)}_{-\mu}((\alpha^2+\beta^2)^{1/2})\}$$

$$= \frac{(-1)^\nu 2^{\nu+\mu}\Gamma(\nu+\mu)}{\Gamma(\mu)\Gamma(1-\mu)}\beta^\nu(\alpha^2+\beta^2)^{-\nu-\mu} + (\alpha^2+\beta^2)^{-\nu-\mu+1}\sum_{k=0}^\infty u_k(\beta)(\alpha^2+\beta^2)^k,$$

where $u_k(\beta)$ is a polynomial of $\beta$ and the last term of the above equality is uniformly convergent on every compact subset of $\boldsymbol{C}$ with respect to the variable $\beta$. Thus there exists a positive number $M$ such that $|\sum_{k=0}^{\infty} u_k(\beta)(\alpha^2+\beta^2)^k| \leq M$ for any $\beta \in U$. Therefore the lemma is proved when $\mu - 1/2 \in \boldsymbol{N} \cup \{0\}$.

2) Let $\mu \in \boldsymbol{N}$. It is well known that

$$\mathscr{H}^{(2)}_{-\mu}(z) = (-1)^{\mu}\{J_{\mu}(z) - (-1)^{\mu} N_{\mu}(z)\},$$

where $N_{\mu}$ is the Neumann function of order $\mu$. From the definition of $N_{\mu}$ and the same calculation as 1), we have the lemma.

LEMMA 3.12. *Let $a \in \boldsymbol{R}^p$ such that $\|a\| \neq 0$ and $\|a\|^{-1} a \in S$ or $-\|a\|^{-1} a \in S$, we have*

1) *If $(1-p)/2 > \nu > -p - q/2 + 3/2$ then*

$$\lim_{\delta \to +0} \delta^{(p+q-2)/2} \int_S |\langle \delta e_0 + ia, \xi \rangle^2 + \|a\|^2|^{\nu} d\xi = 0,$$

2) $$\lim_{\delta \to +0} \delta^{(p+q-2)/2} \int_S \langle \delta e_0 + ia, \xi \rangle^{p-1} [\langle \delta e_0 + ia, \xi \rangle^2 + \|a\|^2]^{-p-q/2+3/2} d\xi \neq 0,$$

*where $e_0 = (1, \cdots, 1) \in \boldsymbol{R}^p$.*

PROOF. For a positive number $\delta$, we set

$$I(\delta) = \int_S |\langle \delta e_0 + ia, \xi \rangle^2 + \|a\|^2|^{\nu} d\xi,$$

$$J(\delta) = \int_S \langle \delta e_0 + ia, \xi \rangle^{p-1} [\langle \delta e_0 + ia, \xi \rangle^2 + \|a\|^2]^{-p-q/2+3/2} d\xi.$$

If $\|a\|^{-1} a \in S$, then there exists an element $k(a)$ in $SO(p)$ such that $a = \|a\| K(a) e_1$. By the simple calculation, we have

$$I(\delta) = \int_{k(a)^{-1}S} |K(\delta; \xi; a)|^{\nu} d\xi,$$

$$J(\delta) = \int_{k(a)^{-1}S} (\langle \delta e_0, k(a)\xi \rangle + i\|a\| \langle e_1, \xi \rangle)^{p-1} K(\delta; \xi; a)^{-p-q/2+3/2} d\xi,$$

where

$$K(\delta; \xi; a) = \|a\|^2(1 - \langle e_1, \xi \rangle^2) + 2i\delta \|a\| \langle e_1, \xi \rangle \langle e_0, k(a)\xi \rangle + \langle \delta e_0, k(a)\xi \rangle^2.$$

Moreover, when $\|a\|^{-1} a \in S$, there exist real numbers $\rho_1$ $(0 \leq \rho_1 \leq \pi/2)$, $\rho_2$ $(\pi/2 \leq \rho_2 \leq \pi)$ and a compact set $C$ $(\subset [0, \pi]^{p-3} \times [0, 2\pi])$ such that

$$k(a)^{-1}S = \{(\xi(\theta));\, 0 \le \theta_1 \le \rho_1 \text{ or } \rho_2 \le \theta_1 \le \pi,\, (\theta_2, \cdots, \theta_{p-1}) \in C\}.$$

Of course, $\rho_1^2 + (\rho_2 - \pi)^2 \neq 0$. When $\rho_1 > 0$, we set

$$I'(\delta) = \int_0^{\rho_1} \int_C |K(\delta;\, \xi(\theta);\, a)|^{\nu} \omega_p(\theta) d\theta_1 \cdots d\theta_{p-1},$$

$$J'(\delta) = \int_0^{\rho_1} \int_C (\langle \delta e_0, k(a)\xi \rangle + i\|a\| \langle e_1, \xi \rangle)^{p-1} K(\delta;\, \xi;\, a)^{-p-q/2+3/2} d\xi.$$

We put $x = \delta^{1/2} \cot\theta_1$. Then, by the simple calculation,

$$I'(\delta) = \delta^{\nu+p/2-1/2} \int_{d(\delta)}^{\infty} \int_C |K_1(\delta;\, x,\, \xi';\, a)|^{\nu} (\delta + x^2)^{-\nu-p/2} dx d\xi',$$

$$J'(\delta) = \delta^{-d} \int_{d(\delta)}^{\infty} \int_C K_2(\delta;\, x,\, \xi';\, a)^{p-1} K_1(\delta;\, x,\, \xi';\, a)^{\kappa} (\delta + x^2)^{q/2-1} d\xi' dx,$$

where

$K_1(\delta;\, x,\, \xi';\, a) = \|a\|^2 + 2i\|a\| x(a'x + \delta^{1/2}\langle e', \xi' \rangle) + \delta(a'x + \delta^{1/2}\langle e', \xi \rangle)^2$,

$K_2(\delta;\, x,\, \xi';\, a) = a'\delta x + \delta\langle e', \xi' \rangle + i\|a\| x$, $d(\delta) = \delta^{1/2}\cot\rho_1$,

$d = (p+q-2)/2$, $\kappa = -p - q/2 + 3/2$, $a' = \|a\|^{-1} \sum a_j$, $e' = k(a)^{-1} e_0 - a' e_1$,

$\xi' = \xi'(\theta') = (\xi(\theta) - \cos\theta_1 e_1)(\sin\theta_1)^{-1}$ and

$d\xi' = 2^{-1}\pi^{-p/2}\Gamma(p/2)(\sin\theta_2)^{p-3} \cdots \sin\theta_{p-2} d\theta_2 \cdots d\theta_{p-1}$.

Hence if $-\nu - p/2 > -1/2$ then

$$\lim_{\delta \to +0} \delta^{-\nu-p/2+1/2} I'(\delta) = \tilde{c}\, \|a\|^{\nu} \int_0^{\infty} x^{-2\nu-p} (\|a\| + 2ia'x^2)^{\nu} dx,$$

$$\lim_{\delta \to +0} \delta^{(p+q-2)/2} J'(\delta) = i^{p-1} \tilde{c}\, \|a\|^{(-q+1)/2} \int_0^{\infty} x^{p+q-3} (\|a\| + 2ia'x^2)^{\kappa} dx,$$

where $\tilde{c} = \displaystyle\int_C d\xi'$. When $\rho_2 < \pi$, we set

$$I''(\delta) = \int_{\rho_2}^{\pi} \int_C |K(\delta;\, \xi(\theta);\, a)|^{\nu} \omega_p(\theta) d\theta_1 \cdots d\theta_{p-1},$$

$$J''(\delta) = \int_{\rho_2}^{\pi} \int_C (\langle \delta e_0, k(a)\xi \rangle + i\|a\| \langle e_1, \xi \rangle)^{p-1} K(\delta;\, \xi;\, a)^{-p-q/2+3/2} d\xi.$$

Then, by the same calculation as $I'(\delta)$ and $J'(\delta)$, if $-\nu - p/2 > -1/2$ we obtain

$$\lim_{\delta\to+0} \delta^{-\nu-p/2+1/2} I''(\delta) = \tilde{c}\,\|a\|^{\nu} \int_{-\infty}^{0} (-x)^{-2\nu-p}(\|a\| + 2ia'x^2)^{\nu} dx,$$

$$\lim_{\delta\to+0} \delta^{(p+q-2)/2} J''(\delta) = i^{p-1}\tilde{c}\,\|a\|^{(-q+1)/2} \int_{-\infty}^{0} (-x)^{p+q-3}(\|a\| + 2ia'x^2)^{\kappa} dx.$$

Hence if $(1-p)/2 > \nu > \kappa = -p - q/2 + 3/2$ then $\lim_{\delta\to+0} \delta^{(p+q-2)/2} I(\delta) = 0.$ Therefore, if $\|a\|^{-1}a \in S$, we have 1) of the lemma. When $-\|a\|^{-1}a \in S$, we obtain 1) of the lemma by the same proof. Moreover, $\tilde{c} \neq 0$ and

$$\int_{-\infty}^{\infty} |x|^{p+q-3}(\|a\| + 2ia'x^2)^{-p-q/2+3/2} dx$$

$$= \frac{\Gamma((p+q-2)/2)\Gamma((p-1)/2)}{\Gamma((2p+q-3)/2)} \left[\frac{2ia'}{\|a\|}\right]^{-(p+q-2)/2} \neq 0,$$

since $a' = \|a\|^{-1}\sum a_j > 0$. Hence we have 2) of the lemma when $\|a\|^{-1}a \in S$. But when $-\|a\|^{-1}a \in S$ we have the same. Therefore the lemma is proved.

In the proof of Proposition 3.8, we use the following notation. For each $w = (w_1, \cdots, w_q) \in \boldsymbol{C}^q$, we set $\gamma(w) = (\sum_{1\le j\le q} w_j^2)^{1/2}$. Here $z^{1/2} = |z|^{1/2} e^{(i\mathrm{Arg}z)/2}$ for each $z \in \boldsymbol{C}$, where Arg$z$ is the principal value of arg$z$. Then the notation $\gamma$ is an extension of the notation $\|\ \ \|$ in §2.

PROOF OF PROPOSITION 3.8. For a positive number $\delta$, we put $z(\delta) = (\delta + ia_1, \cdots, \delta + ia_p)$ and $w_0 = (ib_1, \cdots, ib_q)$. Then $(z(\delta), w_0) \in D_1$. It is well known that

$$\int_{S^{q-1}} e^{i\langle w,\eta\rangle} d\eta = 2^{(q-2)/2}\, \Gamma(q/2)\gamma(w)^{-(q-2)/2}\, J_{(q-2)/2}(\gamma(w)).$$

Since $\gamma(\lambda \mathrm{ch}(t-i\mu)b) = \lambda\mathrm{ch}(t-i\mu)\|b\| = \mathrm{ch}(t-i\mu)\gamma(\lambda b)$ *for any* $t \ge 0$ and $b \in \boldsymbol{R}^q$, we have

$$\varphi_1(z(\delta), w_0) = c_0\ (\lambda\|b\|)^{-(q-2)/2} \times$$

$$\int_0^{\infty}\int_S e^{-\lambda\langle\delta e_0 + ia,\xi\rangle \mathrm{sh}(t-i\mu)}\, J_{(q-2)/2}(\lambda\|b\|\mathrm{ch}(t-i\mu))\Delta(t-i\mu;\, q/2+1,\, p)\, d\xi dt,$$

where $c_0 = i^{p+q-2}\, 2^{(q-2)/2}\, \Gamma(q/2)$. Set

$$I_1(\delta) = c_2' \int_S \langle\delta e_0 + ia,\, \xi\rangle^{p-1}\, [\langle\delta e_0 + ia,\, \xi\rangle + \|b\|^2]^{-p-q/2+3/2}\, d\xi,$$

$$I_2(\delta)=\int_S L(\delta;\xi;a,b)d\xi \text{ and}$$

$$I_3(\delta)=-\int_S\int_0^1 L_1(\delta;\xi;a,b)dxd\xi$$

$$+i\int_S\int_0^\mu [e^{\beta \mathrm{sh} it}\Delta(-it;q/2+1,p)J_{(q-2)/2}(\alpha \mathrm{ch}(-it))]_{\substack{\alpha=\lambda\|b\| \\ \beta=\lambda\langle\delta e_0+ia,\xi\rangle}} dtd\xi,$$

where $c_2'=c_2(p-1,(q-1)/2)\lambda^{-p-q+2}$,

$$L(\delta;\xi;a,b)=[(\partial/\partial\beta)^{p-1}\{(\alpha^2+\beta^2)^{(-q+1)/4}\mathscr{H}^{(2)}_{(-q+1)/2}((\alpha^2+\beta^2)^{1/2})\}$$
$$-c_2(p-1,(q-1)/2)\beta^{p-1}(\alpha^2+\beta^2)^{-p-q/2+3/2}]_{\substack{\alpha=\lambda\|b\| \\ \beta=\lambda\langle\delta e_0+ia,\xi\rangle}}$$

and

$$L_1(\delta;\xi;a,b)=$$
$$[e^{-\beta(x^2-1)^{1/2}}x^{q/2}(x^2-1)^{(p-2)/2}J_{(q-2)/2}(\alpha x)]_{\substack{\alpha=\lambda\|b\| \\ \beta=\lambda\langle\delta e_0+ia,\xi\rangle}}.$$

Then from Lemma 3.10, it is easily seen that

$$\varphi_1(z(\delta),w_0)=c_0\|\lambda b\|^{(-q+2)/2}\{c_1(I_1(\delta)+I_2(\delta))+I_3(\delta)\}.$$

Indeed, if $\mathrm{Re}\beta>|\mathrm{Im}\alpha|$, $\mathrm{Re}\, e^{-\mu}(-\beta\pm i\alpha)<0$ and $|\mu|<\pi$, we have

$$I(\alpha,\beta)=\int_0^\infty e^{-\beta \mathrm{sh}(t-i\mu)}(\mathrm{ch}(t-i\mu))^{\nu+1}(\mathrm{sh}(t-i\mu))^{\nu'}\ J_\nu\ (\alpha \mathrm{ch}(t-i\mu))dt$$

$$=\int_0^\infty e^{-\beta \mathrm{sh} t}(\mathrm{ch}(t))^{\nu+1}(\mathrm{sh}(t))^{\nu'}J_\nu(\alpha \mathrm{ch}(t))dt$$

$$+i\int_0^\mu e^{\beta \mathrm{sh} it}(\mathrm{ch}(-it))^{\nu+1}(\mathrm{sh}(-it))^{\nu'}J_\nu(\alpha \mathrm{ch}(-it))dt,$$

from Cauchy's integral formula. Hence, from Lemma 3.10,

$$I(\alpha,\beta)=c_1(\nu',\nu)(\partial/\partial\beta)^{\nu'}\{\alpha^\nu(\alpha^2+\beta^2)^{-(2\nu+1)/4}\mathscr{H}^{(2)}_{-\nu-1/2}((\alpha^2+\beta^2)^{1/2})\}$$

$$-\int_0^1 e^{-\beta(x^2-1)^{1/2}}x^\nu(x^2-1)^{(\nu'-1)/2}J_\nu(\alpha x)\,dx$$

$$+i\int_0^\mu e^{\beta \mathrm{sh} it}(\mathrm{ch}(-it))^{\nu+1}(\mathrm{sh}(-it))^{\nu'}J_\nu(\alpha \mathrm{ch}(-it))dt.$$

First, from Lemma 3.12 2), we have $\lim_{\delta\to+0}\delta^{(p+q-2)/2}I_1(\delta)\neq 0$. Secondly, from Lemma 3.11, we have 3.11, we have

$$|\delta^{(p+q-2)/2}I_2(\delta)| \leq M'\delta^{(p+q-2)/2}\int_S |\langle \delta e_0 + ia, \xi\rangle^2 + \|b\|^2|^{-p-q/2+5/2}d\xi,$$

where $M' = M|\lambda|^{-2p-q+5}$ (see Lemma 3.11 for $M$). Hence from Lemma 3.12 1), we have $\lim_{\delta\to+0} \delta^{(p+q-2)/2}I_2(\delta) = 0$, if $\|a\| = \|b\| \neq 0$ and $\|a\|^{-1}a \in \pm S$. Finally, since $\lim_{\delta\to+0} I_3(\delta)$ exists, we have $\lim_{\delta\to+0} \delta^{(p+q-2)/2} I_3(\delta) = 0$. Therefore

$$\lim_{\delta\to+0} \delta^{(p+q-2)/2}\varphi_1(z(\delta), w_0) \neq 0.$$

Since $(z(\delta), w_0) \to (ia_1, \cdots, ia_p, ib_1, \cdots, ib_q)$ if $\delta \to +0$, Proposition 3.8 is proved.

Now, we have the following proposition from Corollary 3.9.

PROPOSITION 3.13. *$S.S\ g$ coincides with the following set $A$;*

$$A = \{(x, y; i(a, b)\infty); \|a\| = \|b\| = 2^{-1/2},\ a_jx_k = a_kx_j,\ b_my_n = y_mb_n,$$
$$b_mx_j = -a_jy_m \text{ for any } 1 \leq j \leq p,\ 1 \leq k \leq p,\ 1 \leq m \leq q,\ 1 \leq n \leq q\}.$$

PROOF. Thanks to Sato's theorem, we have $S.S\ g \subset A$. Put $A_0 = A \cap \{x = y = 0\}$ and $A_1 = A \cap \{x \neq 0 \text{ or } y \neq 0\}$. First we prove that $S.S\ g \cap A_0 \neq \phi$. Indeed, from the remark of the singular spectrum of $g_\varepsilon$, we have $S.S\ g_\varepsilon \cap \{x = y = 0\} \subset \tilde{\Gamma}_\varepsilon(0, 0)$ for each $\varepsilon$ and $S.S\ g \subset S.S\ g_\varepsilon$. But from the definition of $\tilde{\Gamma}_\varepsilon$, $(0, 0; i(a, b)\infty) \notin \tilde{\Gamma}_\varepsilon$, if $\varepsilon \neq (1, \cdots, 1)$, $\|a\| = \|b\| = 2^{-1/2}$ and $a_j > 0$ (for any $1 \leq j \leq p$). Thus we have $S.S\ g \cap A_0 \neq \phi$ from Corollary 3.9. We recall the Lie group $G_0 = SO_0(p, q)$ and it's natural action on $\boldsymbol{R}^{p+q}$. This action induces the action on $\sqrt{-1}\ S^*\boldsymbol{R}^{p+q}$, naturally. It is easily seen that $A_0$ is $G_0$- stable under this induced action of $G_0$. Moreover $A_0$ is $G_0$-transitive.
Hence $S.S\ g \cap A_0 = A_0$. In fact, if $p \in A_0$ and $p \notin S.S\ g \cap A_0$, then for $p_0 \in S.S\ g \cap A_0$ $(\neq \phi)$ there exists $k \in G_0$ such that $p = kp_0$, because $A_0$ is $G_0$-transitive. But, since $S.S\ g$ is $G_0$-stable, $p \in S.S\ g \cap A_0$. This contradicts to $p \notin S.S\ g \cap A_0$. Thus $S.S\ g \cap A_0 = A_0$.

On the other hand, since the differential operator $P = \sum(\partial/\partial x_j)^2 - \sum(\partial/\partial y_k)^2$ is simply characteristic, it is well known that the singular spectrum propagates along the bicharacteristic curve of the Hamiltonian vector field $H_{\sigma(P)}$, where $\sigma(P)$ is the principal symbol of the differential operator $P$ (see [6]). Thus $S.S\ g \cap A_1 = A_1$. In fact, it is easily seen that the bicharacteristic curve through the point $(a, b; i(c, d)\infty) \in \sqrt{-1}\ S^*\boldsymbol{R}^{p+q}$ is

$$\gamma(t; a, b, c, d) = (c_1t + a_1, \cdots, c_pt + a_p, -d_1t + b_1, \cdots, -d_qt + b_p; i(c, d)\infty).$$

Hence $A_1 \subset S.S\ g$, since for any $(x, y; i(a, b)\infty) \in A_1$ $\gamma(t; 0, 0, a, b)$ through the point $(0, 0; i(a, b)\infty) \in A_0$. Thus $S.S\ g = A$, since $A = A_0 \cup A_1$. Therefore the proposition is proved.

We recall the Lie group $G = O(p, q)$. Then we have

PROPOSITION 3.14. *$f$ and $g$ are both $G$-invariant,*

PROOF.

Let $k_1 = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}$, $k_2 = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & & -1 \end{bmatrix}$. Then $k_j \in G$ and $G = G_0 \cup k_1 G_0 \cup k_2 G_0 \cup k_1 k_2 G_0$. Hence it is sufficient to prove that $f^{k_j} = f$ and $g^{k_j} = g$ $(j = 1, 2)$. The proof of the $k_j$-invariance of $f$ is as the same proof of $f_0$ in Proposition 2.6. Since

$$\psi_\varepsilon(-z_1, z_2, \cdots, z_p, w) = -\psi_{(-\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)}(z, w)$$

for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$, we have $g^{k_1} = -[(U_1^{(-\varepsilon_1)} \cap U_2^{(\varepsilon_2)} \cap \cdots \cap U_p^{(\varepsilon_p)} \cap V_1^{(1)} \cap \cdots \cap V_q^{(1)}; -\varphi_{(-\varepsilon_1 \cdots \varepsilon_p)})] = g$. Since $\varphi_\varepsilon(z, w)$ is $k_2$-invariant, we have $g^{k_2} = g$. Therefore the proposition is proved.

Finally, we have the following theorem.

THEOREM 3.15. *If $p \geq 2$ and $q \geq 2$ then*

$$\mathscr{B}_\nu^G(\boldsymbol{R}^{p+q}) = \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{p+q}) = \langle f \rangle \oplus \langle g \rangle.$$

PROOF. It is clear that $f$ and $g$ are linearly independent from Proposition 3.13 and $S.S\ f = \phi$. Therefore, from the Cerezo's result; $\dim \mathscr{B}_\nu^G(\boldsymbol{R}^{p+q}) = \dim \mathscr{B}_\nu^{G_0}(\boldsymbol{R}^{p+q}) = 2$ $(p \geq 2,\ q \geq 2)$ and Proposition 3.7, we have the theorem.

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