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Three Riemannian metrics on the moduli space of BPST-instantons over S^4

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The moduli space $\mathscr{M}$ of 1-instantons over unit 4-sphere S^4 with gauge group $SU(2)$ is known to be diffeomorphic to open 5-disk $\mathring{D}^5$. We shall define three natural Riemannian metrics on $\mathscr{M}$ and study their sectional curvatures.

1. Definition of three Riemannian symmetric tensors in a general case

Let (M, g) be a closed Riemannian 4-manifold, G a compact simple linear group, η a vector bundle associated to a principal G-bundle P_η over M. We denote $\Omega^p(\mathrm{ad}\,\eta) = \Gamma(\overset{p}{\wedge} T^*M \otimes \mathrm{ad}\,\eta)$. Then, the space $\mathscr{C}$ of connections on η is an affine space modelled by $\Omega^1(\mathrm{ad}\,\eta)$. The automorphism group $\mathscr{G}$ of P_η, which is called a gauge transformation group, operates on $\mathscr{C}$. Let A be a connection on η and d_A its covariant derivative. Then, we get a sequence

$$\Omega^0(\mathrm{ad}\,\eta) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\,\eta) \xrightarrow{d_A} \Omega^2(\mathrm{ad}\,\eta)\,.$$

And $d_A d_A(\varphi) = [F_A, \varphi]$, where $F_A = dA + A \wedge A \in \Omega^2(\mathrm{ad}\,\eta)$ is the curvature of A. A connection A is called self-dual if $*F_A = F_A$, where $*$ is the Hodge's star operator with respect to g. The space $\mathscr{S}$ of self-dual connections is invariant under the operation of $\mathscr{G}$. The operation of $\mathscr{G}^* = \mathscr{G}/Center\,(G)$ is free at a connection A if and only if A is an irreducible connection. Let $\mathscr{C}^*$ be the space of irreducible connections on η.

Now $\Omega^p(\mathrm{ad}\,\eta)$ has an L^2-innerproduct defined by

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \quad \text{with} \quad (\alpha, \beta) = -\mathrm{Tr}\,(\alpha \wedge *\beta)\,.$$

This innerproduct is invariant under the operation of $\mathscr{G}$. Note also that $\gamma^*(d_A\alpha) = d_{\gamma^*A}\gamma^*\alpha$ for $\gamma \in \mathscr{G}$.

Thus the following innerproducts on $\Omega^1(\mathrm{ad}\,\eta)$, identified with the tangent space of $\mathscr{C}^*$ at the class of an irreducible connection A, induce Riemannian symmetric tensors g_{J} (J = I, II and I–II) on $\mathscr{C}^*/\mathscr{G} = \mathscr{C}^*/\mathscr{G}^*$ with the projection $\rho\colon \mathscr{C}^* \to \mathscr{C}^*/\mathscr{G}$.

Type I: $\langle \alpha, \beta \rangle_{\mathrm{I}} = \langle \alpha^h, \beta^h \rangle \quad (= g_{\mathrm{I}}(\rho_*\alpha, \rho_*\beta)),$

where α^h is the orthogonal projection of α to the orthogonal complement Ker δ_A of $d_A\Omega^0(\mathrm{ad}\,\eta)$ in $\Omega^1(\mathrm{ad}\,\eta)$ and $\delta_A = -*d_A*$ is the adjoint operator.

Type II: $\langle\alpha, \beta\rangle_{\mathrm{II}} = \langle(d_A\alpha)^h, (d_A\beta)^h\rangle \quad (= g_{\mathrm{II}}(\rho_*\alpha, \rho_*\beta))$,
where $(d_A\alpha)^h$ is the orthogonal projection of $d_A\alpha$ ($\in \Omega^2(\mathrm{ad}\,\eta)$) to the orthogonal complement C of $d_Ad_A\Omega^0(\mathrm{ad}\,\eta)$ in the L^2-completion of $\Omega^2(\mathrm{ad}\,\eta)$. Note that Ker $\delta_A\delta_A$ is contained in C.

Type I–II: $\langle\alpha, \beta\rangle_{\mathrm{I-II}} = \langle d_A(\alpha^h), d_A(\beta^h)\rangle \quad (= g_{\mathrm{I-II}}(\rho_*\alpha, \rho_*\beta))$.

The constant multiple of the L^2-innerproduct gives the same constant multiple to the Riemannian symmetric tensors. Note also that g_{II} is independent of the conformal change of g and g_{I} is changed to c^2g_{I} if we change g to c^2g.

Type I always gives a Riemannian metric on $\mathscr{C}^*/\mathscr{G}$ but type I–II and type II may have a direction of zero length. Moreover g_{II} might have some difficulty with regularity and type I–II is introduced by Ryoichi Kobayashi.

When $\mathscr{M}^* = (\mathscr{C}^* \cap \mathscr{S})/\mathscr{G}$ is a submanifold of $\mathscr{C}^*/\mathscr{G}$, these Riemannian symmetric tensors induce those on $\mathscr{M}^*$. This is the case for a generic Riemannian metric on M when $G = SU(2)$ [4]. Since a self-dual connection is of class C^∞, $\mathscr{M}^*$ is independent of the choice of Sobolev completions.

2. Case of 1-instantons over S^4

The stereographic projection of S^4-{*North pole*} onto $\mathbf{R}^4$ induces a conformally flat metric

$$ds^2 = (4/(1 + |x|^2)^2)|dx|^2 .$$

Since the Hodge's star operator on the 2-forms over 4-manifolds is conformally invariant, a modification [3] of a BPST solution [2],

$$A = \frac{\mathrm{Im}\,\{(1 + \lambda^2|a|^2)x\,d\bar{x} + (\lambda^2 - 1)a\,d\bar{x}\}}{|x - a|^2 + \lambda^2|a\bar{x} + 1|^2} \quad (0 < \lambda \leqq 1, a \in \mathbf{H} \cong \mathbf{R}^4),$$

gives a self-dual connection on η over S^4 with gauge group $SU(2)$ and $c_2(\eta) = -1$. In fact.

$$F = \frac{\{\lambda^2(1 + |a|^2) + (\lambda^2 - 1)(1 + \lambda^2|a|^2)(a\bar{x} + x\bar{a} - \bar{a}x - \bar{x}a)\}\,dx \wedge d\bar{x}}{(|x - a|^2 + \lambda^2|a\bar{x} + 1|^2)^2}$$

In case $\lambda = 1$, A is the central element $\mathrm{Im}\,(x\,d\bar{x})/(1 + |x|^2)$ independent of a. A $SU(2)$-equivariant coordinate system is given by (a, λ) with $\lambda \in (0, 1)$ for an open dense subset of $\mathscr{M} = \mathscr{M}^* \cong \mathring{D}^5 \cong (\mathbf{H} \cup \{\infty\}) \times (0, 1]/(\mathbf{H} \cup \{\infty\}) \times 1$. Here, $x = (x_1, x_2, x_3, x_4)$ is identified with $x = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$ of the field

$\mathbf{H}$ of quaternion numbers. Note that $\bar{x} = x_1 - x_2\boldsymbol{i} - x_3\boldsymbol{j} - x_4\boldsymbol{k}$, $dx = dx_1 + \boldsymbol{i}\,dx_2 + \boldsymbol{j}\,dx_3 + \boldsymbol{k}\,dx_4$ and $|x|^2 = x\bar{x}$, $\operatorname{Im}\mathbf{H} = \{x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}\}$ is also identified with the Lie algebra of $SU(2)$. The innerproduct of $\operatorname{Im}\mathbf{H}$ is given by $(x, y) = 2\operatorname{Re}(x\bar{y})$ in this identification.

Now we present an expression of the metrics on $\mathcal{M}$ in the coordinate system (a, λ) with $\lambda \in (0, 1)$ and discuss some properties.

TYPE I (Doi-Matsumoto-Matumoto [3], cf. Groisser-Parker [5]): Since $\delta_A(\partial A/\partial\lambda) = 0$ and $(\partial A/\partial a_\nu)^h = \partial A/\partial a_\nu + ((1 - \lambda^2)^2/(1 + \lambda^2))d_A(A_\nu)$ with $(\lambda^2 + |x|^2)A_\nu = \operatorname{Im}(x)$, $-\operatorname{Im}(\boldsymbol{i}\bar{x})$, $-\operatorname{Im}(\boldsymbol{j}\bar{x})$ and $-\operatorname{Im}(\boldsymbol{k}\bar{x})$ ($\nu = 1, 2, 3$ and 4 respectively) as is proved in [3], we have

$$ds^2 = \frac{16\pi^2}{5}\left[\left(\frac{1 - \lambda^2}{1 + \lambda^2}\right)^2 \{5 - \lambda^4 F(4, 3, 6; 1 - \lambda^2)\}\frac{|da|^2}{(1 + |a|^2)^2} + \lambda^2 F(4, 3, 6; 1 - \lambda^2)\, d\lambda^2\right]$$

where $F(4, 3, 6; 1 - \xi) = 10\{1/\xi + 12/(1 - \xi)^2 + 6(1 + \xi)\log\xi/(1 - \xi)^3\}/(1 - \xi)^2$ is a hypergeometric function. We have proved in [3] that

$$3/16\pi^2 < \text{sectional curvature} \leqq 5/16\pi^2$$

and the maximum is taken in all the planes at the center ($\lambda = 1$). From the above we see easily that the metric is asymptotically

$$ds^2 \sim 16\pi^2\{(1 - 6\lambda^2)|da|^2/(1 + |a|^2)^2 + 2(12\lambda^2 \log\lambda + 14\lambda^2 + 1)\, d\lambda^2\}$$

as $\lambda \to 0$. This implies that the metric extends to $\bar{\mathcal{M}}$ in C^1 sense and $\partial\mathcal{M}$ is isomorphic to 4-sphere of radius 2π but the metric on $\bar{\mathcal{M}}$ cannot be of class C^2 (cf. [6]).

TYPE II (cf. [8]):
Since $d_A(\partial A/\partial t) = \partial F/\partial t$ and $\delta_A\delta_A(\partial F/\partial t) = 0$ for $t = \lambda$ and $t = a_\nu$ with $\nu = 1, 2, 3$ and 4, we have

$$ds^2 = \frac{32\pi^2}{5}\left[\frac{(1 - \lambda^2)^2}{\lambda^2}\frac{|da|^2}{(1 + |a|^2)^2} + \frac{d\lambda^2}{\lambda^2}\right].$$

This metric has a constant negative sectional curvature $-5/32\pi^2$. So, the metric is hyperbolic and complete.

TYPE I–II:
By calculating $((1 - \lambda^2)^4/(1 + \lambda^2)^2)\langle d_A(A_\nu), d_A(A_\nu)\rangle$, we get

$$ds^2 = \frac{32\pi^2}{5}\left[\frac{(1 - \lambda^2)^2}{\lambda^2}\left(1 + \frac{(1 - \lambda^2)^2}{2(1 + \lambda^2)^2}\right)\frac{|da|^2}{(1 + |a|^2)^2} + \frac{d\lambda^2}{\lambda^2}\right].$$

The sectional curvature K is a convex combination of $K\mathrm{I}$ and $K\mathrm{II}$, where $K\mathrm{I} = K(\partial/\partial a_\mu, \partial/\partial\lambda) = (-5/32\pi^2)\{(9\lambda^{12} + 30\lambda^{10} + 183\lambda^8 + 196\lambda^6 + 183\lambda^4 + 30\lambda^2 + 9)/DEN\}$, $K\mathrm{II} = K(\partial/\partial a_\mu, \partial/\partial a_\nu) = (-5/32\pi^2)\{(9\lambda^{12} + 66\lambda^{10} + 167\lambda^8 + 156\lambda^6 + 167\lambda^4 + 66\lambda^2 + 9)/DEN\}$ and $DEN = 9\lambda^{12} + 30\lambda^{10} + 55\lambda^8 + 68\lambda^6 + 55\lambda^4 + 30\lambda^2 + 9$. Hence, K is negative everywhere and $K \to -25/64\pi^2$ $(\lambda \to 1)$, $K \to -5/32\pi^2$ $(\lambda \to 0)$. In consequence the metric is complete.

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