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Discrete initial value problems and discrete parabolic potential theory

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§ 1. Introduction

In this paper, we shall study a discrete analogue of the initial value problems and the potential theory for the heat equation $\Delta u = \partial u/\partial t$, the potential theory established e.g. in Doob [1; 1. XV & XVII], Watson [4] and, in a more abstract form, in Maeda [3]. We choose an infinite network N and consider a "discrete cylinder" with base space N.

More precisely, let X be a countable infinite set of nodes, Y be a countable infinite set of arcs and K be the node-arc incidence function. We assume that the graph $\{X, Y, K\}$ is connected and locally finite and has no self-loop. Let r be a strictly positive real function on Y. We call the quartet $N = \{X, Y, K, r\}$ an infinite network (cf. [5], [6]). Next, let T be the set of all integers which will be regarded as the time space. For $s \in T$, put $T_s = \{t \in T;\ t \geq s\}$. We call $\{N, T\}$ (resp. $\{N, T_s\}$) the discrete cylinder (resp. discrete half-cylinder) with base N.

We set $\Xi = X \times T$ and denote by $L(\Xi)$ the set of all real functions on Ξ. For $u \in L(\Xi)$, we shall define the discrete (partial) derivatives du and ∂u and the Laplacian Δu. The operators d and Δ act on the variable $x \in X$ and ∂ on $t \in T$. The parabolic operator Π acting on $u \in L(\Xi)$ is defined by

$$\Pi u(\xi) = \Delta u(\xi) - \partial u(\xi), \ \xi = (x, t) \in \Xi.$$

Our initial value problems and potential theory will be discussed with respect to this operator Π.

For our study, we first recall in § 2 some properties of the 1-Green function of N relative to the equation $\Delta u = u$, and give some results on iterations of the 1-Green operators. In § 3, we consider superparabolic functions on a set in Ξ and give minimum principles. We study in § 4 an initial value problem on $\{N, T_s\}$. The existence and uniqueness of the parabolic Green function G_α of $\{N, T\}$ with pole at $\alpha \in \Xi$ will be studied in § 5. Solutions of an initial boundary value problem as well as the parabolic Green function of $\{N, T\}$ will be constructed by means of the iterations of the 1-Green operator of N. In case N has the harmonic Green function g_a with pole at $a \in X$, we have the following formula:

$$\sum_{t=s}^{\infty} G_\alpha(x, t) = g_a(x) \text{ with } \alpha = (a, s),$$

which has a continuous counterpart (cf. [1; 1.XVII.18]).

Discrete analogue of the Riesz decomposition theorem for nonnegative superparabolic functions will be proved in §6. We shall introduce the coparabolic operator Π^* in §7 and discuss the coparabolic Green function of $\{N, T\}$, and the duality between parabolic and coparabolic potentials.

§2. 1-Green function of N

First, we recall some results on the q-Green function of N discussed in [7], in case $q = 1$.

For notation and terminologies concerning the infinite network $N = \{X, Y, K, r\}$, we mainly follow [5], [6] and [7]: Denote by $L(X)$ (resp. $L^+(X)$) the set of all real (resp. nonnegative) functions on X and by $L_0(X)$ (resp. $L_0^+(X)$) the set of all real (resp. nonnegative) functions u on X with finite support $Su = \{x \in X; u(x) \neq 0\}$. For $u \in L(X)$, we define

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x),$$

$$D(u) = \sum_{y \in Y} r(y)[du(y)]^2,$$

$$\Delta_1 u(x) = \sum_{y \in Y} K(x, y)[du(y)] - u(x),$$

$$E_1(u) = D(u) + \sum_{x \in X} u(x)^2.$$

Let

$$\mathscr{E}(N; 1) = \{u \in L(X); E_1(u) < \infty\}$$

and for $u, v \in \mathscr{E}(N; 1)$,

$$E_1(u, v) = \sum_{y \in Y} r(y)[du(y)][dv(y)] + \sum_{x \in X} u(x)v(x).$$

Then $\mathscr{E}(N; 1)$ is a Hilbert space with respect to the inner product $E_1(u, v)$. For each $a \in X$, there exists a unique $\tilde{g}_a \in \mathscr{E}(N; 1)$ such that

$$u(a) = E_1(u, \tilde{g}_a) \quad \text{for every } u \in \mathscr{E}(N; 1).$$

We call $\tilde{g}_a$ the 1-Green function of N with pole at a. The following properties of $\tilde{g}_a$ are known ([7; Theorems 4.2, 4.3 and 4.5]):

(2.1) $$\tilde{g}_a(b) = \tilde{g}_b(a) \quad \text{for every } a, b \in X;$$

(2.2) $$\Delta_1 \tilde{g}_a(x) = -\varepsilon_a(x) \quad \text{on } X,$$

where ε_a is the characteristic function of the set $\{a\}$;

$$0 < \tilde{g}_a(x) \leq \tilde{g}_a(a) \quad \text{on } X; \tag{2.3}$$

$$\sum_{x \in X} \tilde{g}_a(x) \leq 1. \tag{2.4}$$

For $\mu \in L^+(X)$, the 1-Green potential $\tilde{G}\mu$ and 1-Green potential energy $\tilde{G}(\mu, \mu)$ of μ are defined by

$$\tilde{G}\mu(x) = \sum_{a \in X} \tilde{g}_a(x)\mu(a), \ \tilde{G}(\mu, \mu) = \sum_{x \in X} [\tilde{G}\mu(x)]\mu(x).$$

Lemma 2.1. ([7; Lemma 7.2 and Theorem 7.2]). *For $\mu \in L^+(X)$, $\tilde{G}\mu \in \mathscr{E}(N; 1)$ if and only if $\tilde{G}(\mu, \mu) < \infty$; and $\tilde{G}(\mu, \mu) = E_1(\tilde{G}\mu)$ in this case.*

For $u \in L(X)$ and $p > 0$, we put

$$\| u \|_p = (\sum_{x \in X} |u(x)|^p)^{1/p} \quad \text{and} \quad \| u \|_\infty = \sup\{|u(x)|; x \in X\}.$$

Note that $\| u \|_{p_2} \leq \| u \|_{p_1}$ if $p_1 \leq p_2 \leq \infty$.

Lemma 2.2. *Let $\mu \in L^+(X)$.*

(i) *If $\tilde{G}\mu(x) \in L(X)$, then $\Delta_1 \tilde{G}\mu(x) = -\mu(x)$.*

(ii) *If $\| \mu \|_p < \infty$ with $1 \leq p \leq \infty$, then $\tilde{G}\mu \in L(X)$ and $\| \tilde{G}\mu \|_p \leq \| \mu \|_p$.*

(iii) *If $\| \mu \|_2 < \infty$, then $\tilde{G}\mu \in \mathscr{E}(N; 1)$ and*

$$2D(\tilde{G}\mu) + \| \tilde{G}\mu \|_2^2 \leq \| \mu \|_2^2. \tag{2.5}$$

Proof. (i) readily follows from (2.2). By (2.1) and (2.4), it is easy to see that (ii) holds. If $\mu \in L_0^+(X)$, then $\tilde{G}\mu \in \mathscr{E}(N; 1)$ by Lemma 2.1 and we have

$$D(\tilde{G}\mu) + \| \tilde{G}\mu \|_2^2 = E_1(\tilde{G}\mu) = \tilde{G}(\mu, \mu) \leq \frac{1}{2}(\| \tilde{G}\mu \|_2^2 + \| \mu \|_2^2),$$

which implies (2.5) for $\mu \in L_0^+(X)$. If $\| \mu \|_2 < \infty$, then choose $\mu_n \in L_0^+(X)$, $n = 1, 2, \ldots$, such that $\mu_n \uparrow \mu$. Then, $\tilde{G}\mu_n \uparrow \tilde{G}\mu$ and $D(\tilde{G}\mu) \leq \liminf_{n \to \infty} D(\tilde{G}\mu_n)$. Since each μ_n satisfies (2.5), it follows that $\tilde{G}\mu \in \mathscr{E}(N; 1)$ and (2.5) holds if $\| \mu \|_2 < \infty$. This completes the proof.

For $\mu \in L^+(X)$, we inductively define $\tilde{G}^{(n)}\mu$, $n = 0, 1, \ldots$, by $\tilde{G}^{(0)}\mu = \mu$ and $\tilde{G}^{(n+1)}\mu = \tilde{G}(\tilde{G}^{(n)}\mu)$. Then by the above lemma we have

Corollary 2.3. *Let $\mu \in L^+(X)$.*

(i) *If $\| \mu \|_p < \infty$ with $1 \leq p \leq \infty$, then $\tilde{G}^{(n)}\mu \in L(X)$ and $\| \tilde{G}^{(n)}\mu \|_p \leq \| \mu \|_p$ for all $n = 0, 1, \ldots$.*

(ii) *If $\| \mu \|_2 < \infty$, then $\tilde{G}^{(n)}\mu \in \mathscr{E}(N; 1)$ and*

$$2\{D(\tilde{G}\mu) + D(\tilde{G}^{(2)}\mu) + \cdots + D(\tilde{G}^{(n)}\mu)\} + \|\tilde{G}^{(n)}\mu\|_2^2 \leq \|\mu\|_2^2 \tag{2.6}$$

for all $n = 1, 2, \ldots$.

We establish

PROPOSITION 2.4. *Let* $\mu \in L^+(X)$ *and* $\|\mu\|_2 < \infty$. *Then* $D(\tilde{G}^{(n)}\mu) \to 0$ *and* $\|\tilde{G}^{(n)}\mu\|_2 \to 0$ *as* $n \to \infty$.

PROOF. By (2.6), we immediately deduce that $D(\tilde{G}^{(n)}\mu) \to 0$. By (ii) of Lemma 2.2, we see that $\{\|\tilde{G}^{(n)}\mu\|_2\}_n$ is nonincreasing. Let $A \equiv \lim_{n\to\infty} \|\tilde{G}^{(n)}\mu\|_2$.

For $u, v \in L^+(X)$, let $\langle u, v\rangle = \sum_{x\in X} u(x)v(x)$. Then, by (2.1), we see that $\langle \tilde{G}\mu, \nu\rangle = \langle \mu, \tilde{G}\nu\rangle$ for any $\mu, \nu \in L^+(X)$.

Let $\mu_n = \tilde{G}^{(n)}\mu$ for simplicity. Then, for any positive integers n and m, $\langle \mu_{n+2m}, \mu_n\rangle = \|\mu_{n+m}\|_2^2$ and

$$\begin{aligned}\langle \mu_{n+2m-1}, \mu_n\rangle = \langle \mu_{n+m}, \mu_{n+m-1}\rangle &= \tilde{G}(\mu_{n+m-1}, \mu_{n+m-1}) \\ &= E_1(\mu_{n+m}) \geq \|\mu_{n+m}\|_2^2\end{aligned}$$

by Lemma 2.1. Hence, we have

$$\begin{aligned}\|\mu_{n+2m} - \mu_n\|_2^2 &= \|\mu_{n+2m}\|_2^2 + \|\mu_n\|_2^2 - 2\langle \mu_{n+2m}, \mu_n\rangle \\ &= \|\mu_{n+2m}\|_2^2 + \|\mu_n\|_2^2 - 2\|\mu_{n+m}\|_2^2 \\ &\to A^2 + A^2 - 2A^2 = 0 \quad (n \to \infty)\end{aligned}$$

and

$$\begin{aligned}\|\mu_{n+2m-1} - \mu_n\|_2^2 &= \|\mu_{n+2m-1}\|_2^2 + \|\mu_n\|_2^2 - 2\langle \mu_{n+2m-1}, \mu_n\rangle \\ &\leq \|\mu_{n+2m-1}\|_2^2 + \|\mu_n\|_2^2 - 2\|\mu_{n+m}\|_2^2 \\ &\to A^2 + A^2 - 2A^2 = 0 \quad (n \to \infty).\end{aligned}$$

Therefore, $\{\mu_n\}$ is a Cauchy sequence in the norm $\|\cdot\|_2$, so that there is $\mu_0 \in L^+(X)$ with $\|\mu_0\|_2 < \infty$ such that $\|\mu_0 - \mu_n\|_2 \to 0$ $(n \to \infty)$. It then follows that $\mu_n(x) \to \mu_0(x)$ for every $x \in X$, so that $D(\mu_0) \leq \liminf_{n\to\infty} D(\mu_n) = 0$. Hence, $\mu_0 \equiv$ const., and since X is an infinite set and $\|\mu_0\|_2 < \infty$, it follows that $\mu_0 = 0$. Thus, $\|\mu_n\|_2 \to \|\mu_0\|_2 = 0$ $(n \to \infty)$.

PROPOSITION 2.5. *If* $1 < p < \infty$, *then*

$$\lim_{n\to\infty} \|\tilde{G}^{(n)}\mu\|_p = 0 \text{ for any } \mu \in L^+(X) \text{ with } \|\mu\|_p < \infty. \tag{2.7}$$

PROOF. Let $\mu \in L^+(X)$ and $\|\mu\|_p < \infty$. For $\varepsilon > 0$, choose $\mu' \in L_0^+(X)$ such that $\mu' \leq \mu$ and $\|\mu - \mu'\|_p < \varepsilon$. Then, $\|\tilde{G}^{(n)}\mu'\|_2 \to 0$ $(n \to \infty)$ by the above proposition. If $2 < p < \infty$, then

$$\| \tilde{G}^{(n)} \mu' \|_p \leq \| \tilde{G}^{(n)} \mu' \|_2 \longrightarrow 0 \ (n \longrightarrow \infty).$$

If $1 < p < 2$, then using Hölder's inequality and Corollary 2.3 (i), we have

$$\| \tilde{G}^{(n)} \mu' \|_p^p \leq \| \tilde{G}^{(n)} \mu' \|_1^{2-p} \cdot \| \tilde{G}^{(n)} \mu' \|_2^{2(p-1)} \leq \| \mu' \|_1^{2-p} \cdot \| \tilde{G}^{(n)} \mu' \|_2^{2(p-1)} \longrightarrow 0 \quad (n \to \infty).$$

Hence, again by Corollary 2.3 (i),

$$\text{limsup}_{n \to \infty} \| \tilde{G}^{(n)} \mu \|_p \leq \text{limsup}_{n \to \infty} \| \tilde{G}^{(n)} (\mu - \mu') \|_p \leq \| \mu - \mu' \|_p < \varepsilon$$

if $1 < p < \infty$, which completes the proof.

REMARK 2.6. In case $p = 1$ or $p = \infty$, (2.7) does not hold in general; in fact if $\tilde{G}1(x) = \sum_{a \in X} \tilde{g}_a(x) = 1$ for all $x \in X$ (see [7; §5] as to when this occurs), then $\| \tilde{G}^{(n)} \mu \|_1 = \| \mu \|_1$ for any $\mu \in L^+(X)$ and n, and $\| \tilde{G}^{(n)} 1 \|_\infty = 1$ for all n.

For $f \in L(X)$ and $n = 0, 1, \ldots$, we define $\tilde{G}^{(n)} f = \tilde{G}^{(n)} f^+ - \tilde{G}^{(n)} f^-$ whenever $\tilde{G}^{(n)} f^+$, $\tilde{G}^{(n)} f^- \in L(X)$. By Corollary 2.3, $\tilde{G}^{(n)} f$ is defined for each n if f is bounded.

§ 3. Superparabolic functions and minimum principle

Now let T be the set of all integers. Given $s \in T$, let

$$T_s = \{t \in T;\ t \geq s\},\ T_s^\circ = \{t \in T;\ t > s\} \text{ and } T_s^* = \{t \in T;\ t \leq s\}.$$

We write

$$\Xi = X \times T,\ \Xi_s = X \times T_s,\ \Xi_s^\circ = X \times T_s^\circ \text{ and } \Xi_s^* = X \times T_s^*.$$

We call $\{N, T\}$ (resp. $\{N, T_s\}$) the *discrete cylinder* (resp. *discrete half-cylinder*) with base N. For the set Ξ, we define $L(\Xi)$, $L^+(\Xi)$, $L_0(\Xi)$ and $L_0^+(\Xi)$ in the same manner as $L(X)$, $L^+(X)$, $L_0(X)$ and $L_0^+(X)$.

For $u \in L(\Xi)$, we set

$$du(y, t) = -r(y)^{-1} \sum_{x \in X} K(x, y) u(x, t),$$

$$\partial u(x, t) = u(x, t) - u(x, t-1),$$

$$\Delta u(x, t) = \sum_{y \in Y} K(x, y)[du(y, t)],$$

$$\Pi u(\xi) = \Delta u(\xi) - \partial u(\xi).$$

Note that

(3.1) $$\Pi u(\cdot, t) = \Delta_1 u(\cdot, t) + u(\cdot, t-1).$$

Thus, $\Pi u(\xi)$ can be also defined for $u \in L(\Xi_s)$ and $\xi \in \Xi_s^\circ$.

We say that a function $u \in L(\Xi)$ is *superparabolic* (resp. *parabolic*) on a set Ω if $\Pi u(\xi) \le 0$ (resp. $\Pi u(\xi) = 0$) on Ω. Denote by $SPR(N, T)$ (resp. $PR(N, T)$) the set of all superparabolic (resp. parabolic) functions on Ξ. If u_1 and u_2 are superparabolic on $\Omega \subset \Xi$ and if c is a positive number, then $u_1 + u_2$ and cu_1 are superparabolic on Ω. A function u is said to be *subparabolic* on Ω if $-u$ is superparabolic on Ω.

In order to rewrite the parabolic operator in a more geometric form, let us define $\rho(\alpha)$ and $\rho(\xi, \alpha)$ for $\alpha = (a, s)$ and $\xi = (x, t)$ by

$$\rho(\alpha) = 1 + \textstyle\sum_{y \in Y} r(y)^{-1} |K(a, y)|,$$

$$\rho(\xi, \alpha) = \textstyle\sum_{y \in Y} r(y)^{-1} |K(x, y)K(a, y)| \quad \text{if } t = s \text{ and } \xi \neq \alpha,$$

$$\rho(\alpha^-, \alpha) = 1, \text{ where } \alpha^- = (a, s-1),$$

$$\rho(\xi, \alpha) = 0 \text{ for any other pair } (\xi, \alpha).$$

Then $\sum_{\xi \in \Xi} \rho(\xi, \alpha) = \rho(\alpha)$ and

$$\Pi u(\alpha) = -\rho(\alpha)u(\alpha) + \textstyle\sum_{\xi \in \Xi} \rho(\xi, \alpha)u(\xi). \tag{3.2}$$

For each $u \in L(\Xi)$ and $\alpha \in \Xi$, define a discrete analogue of the Poisson integral of u by

$$P_u(\alpha) = \rho(\alpha)^{-1} \textstyle\sum_{\xi \in \Xi} \rho(\xi, \alpha)u(\xi).$$

Then, by (3.2), $\Pi u(\alpha) \le 0$ (resp. $\Pi u(\alpha) = 0$) if and only if $P_u(\alpha) \le u(\alpha)$ (resp. $P_u(\alpha) = u(\alpha)$). From this, we see that if u_1 and u_2 are superparabolic on $\Omega \subset \Xi$, then so is $\min(u_1, u_2)$. For $\alpha \in \Xi$, put $\Xi(\alpha) = \{\alpha\} \cup \{\xi \in \Xi; \rho(\xi, \alpha) \neq 0\}$.

We prepare

LEMMA 3.1. *Assume that $\Pi u(\alpha) \le 0$ and $u(\alpha) = \min\{u(\xi); \xi \in \Xi(\alpha)\}$. Then $u(\xi) = u(\alpha)$ on $\Xi(\alpha)$.*

PROOF. Since $u(\xi) \ge u(\alpha)$ on $\Xi(\alpha)$ and $\Pi u(\alpha) \le 0$, by (3.2) we have

$$\rho(\alpha)u(\alpha) \ge \textstyle\sum_{\xi \in \Xi(\alpha)} \rho(\xi, \alpha)u(\xi) \ge u(\alpha) \sum_{\xi \in \Xi} \rho(\xi, \alpha) = u(\alpha)\rho(\alpha),$$

and hence $u(\xi) = u(\alpha)$ on $\Xi(\alpha)$.

By this lemma, we obtain the followng minimum principle:

THEOREM 3.2. *Let $s < s'$ $(s, s' \in T)$ and let $\Omega = \Xi_s \cap \Xi_{s'}^*$* and *$\Omega^\circ = \Xi_s^\circ \cap \Xi_{s'}^*$.*
(i) *If u is superparabolic on Ω° and if u attains its minimum on Ω at $\alpha = (a, s')$, then $u(\xi) = u(\alpha)$ for every $\xi \in \Omega$.*
(ii) *Let Ω' be a finite subset of Ω°. If u is superparabolic on Ω' and $u(\xi) \ge 0$ on*

$\Omega - \Omega'$, *then* $u(\xi) \geq 0$ *on* Ω.

COROLLARY 3.3. *Let* $s \in T$ *and suppose* u *is superparabolic on* Ξ_s°. *If* u *satisfies the following two conditions, then* $u \geq 0$ *on* Ξ_s:

(a) $u(x, s) \geq 0$ *for all* $x \in X$;

(*b*) *there are* $f \in L^+(\Xi)$ *and* $p < \infty$ *such that* $\|f(\cdot, t)\|_p < \infty$ *for all* $t \in T_s^\circ$ *and* $u \geq -f$ *on* Ξ_s°.

PROOF. Let $s' > s$ be arbitrarily fixed and let Ω be as in the above theorem. Since $\|f(\cdot, t)\|_p < \infty$ for $s < t \leq s'$, given $\varepsilon > 0$, there is a finite set $\Omega' \subset \Omega^\circ$ such that $f(\xi) < \varepsilon$ for $\xi \in \Omega^\circ - \Omega'$. Then, $u + \varepsilon > 0$ on $\Omega - \Omega'$. Since $u + \varepsilon$ is superparabolic on Ω°, (ii) of the above theorem implies that $u + \varepsilon \geq 0$ on $\Omega = \Xi_s \cap \Xi_{s'}^*$. By the arbitrariness of $\varepsilon > 0$ and $s' > s$, we see that $u \geq 0$ on Ξ_s.

§4. Initial value problem for the discrete half-cylinder

The initial value problem on Ξ_s may be formulated as follows:
[IP: f]$_s$: Given $f \in L(X)$, find $u \in L(\Xi_s)$ satisfying

$$\begin{cases} u(x, s) = f(x) \text{ for all } x \in X, \\ \Pi u(\xi) = 0 \text{ for all } \xi \in \Xi_s^\circ, \text{ namely } u \text{ is parabolic on } \Xi_s^\circ. \end{cases}$$

By translation, it suffices to consider the case $s = 0$. We simply write [IP: f] for the problem [IP: f]$_0$.

Given a bounded $f \in L(X)$ and $m \in T$, we set

$$U_f^{(m)}(x, t) = \begin{cases} 0, & \text{if } t < m \\ [\tilde{G}^{(t-m)} f](x), & \text{if } t \geq m. \end{cases} \tag{4.1}$$

By Corollary 2.3, Lemma 2.2 (i) and (3.1), we immediately obtain

LEMMA 4.1. *If* $f \in L(X)$ *is bounded, then* $U_f^{(m)}(\cdot, m) = f$, $|U_f^{(m)}(\xi)| \leq \|f\|_\infty$ *for any* $m \in T$ *and* $\xi \in \Xi$, *and*

$$\Pi U_f^{(m)}(x, t) = \begin{cases} 0, & \text{if } t \neq m, \\ \Delta_1 f(x), & \text{if } t = m. \end{cases}$$

Thus, together with Corollary 3.3 and Proposition 2.5, we obtain

THEOREM 4.2. *If* $f \in L(X)$ *is bounded, then* $u = U_f^{(0)}$ *(restricted to* Ξ_0*) is a bounded solution of the problem* [IP: f]. *Furthermore, it has the following properties*:

(i) $|u(\xi)| \leq \|f\|_\infty$ *for all* $\xi \in \Xi_0$.

(ii) *If* $\|f\|_p < \infty$ *with* $1 \leq p < \infty$, *then* u *is the unique solution of* [IP: f] *satisfying* $\|u(\cdot, t)\|_p < \infty$ *for all* $t \in T_0^\circ$.

(iii) *If* $\|f\|_p < \infty$ *with* $1 < p < \infty$, *then* $\lim_{t\to\infty} \|u(\cdot, t)\|_p = 0$.

Next, we consider the class

$$HB(N; 1) = \{h \in L(X); h \text{ is bounded and } \Delta_1 h = 0 \text{ on } X\}.$$

A function u on Ξ_0 will be called *time-locally bounded* if $u(\cdot, t)$ is bounded for every $t \in T_0$. The following theorem determines the set of all time-locally bounded solutions of [IP: f]:

THEOREM 4.3. *Let* $f \in L(X)$ *be bounded.*
(i) *If* $\{h_m\}_{m=1}^{\infty}$ *is a sequence of functions in* $HB(N; 1)$, *then*

$$u = U_f^{(0)} + \sum_{m=1}^{\infty} U_{h_m}^{(m)} \tag{4.2}$$

gives a time-locally bounded solution of [IP: f]. *If, in addition,* $\sum_{m=1}^{\infty} \|h_m\|_\infty < \infty$, *then it is a bounded solution.*
(ii) *Conversely, any time-locally bounded solution of* [IP: f] *can be expressed in the form* (4.2) *on* Ξ_0 *with* $h_m \in HB(N; 1)$, $m = 1, 2, \dots$.

PROOF. (i) For each $t \in T_0$, $U_{h_m}^{(m)}(\cdot, t) = 0$ for $m > t$, so that the right hand side of (4.2) is in fact a finite sum at each point of Ξ_0 and $u(\cdot, t)$ is bounded, i.e., u is time-locally bounded. By Lemma 4.1, $\Pi U_{h_m}^{(m)} = 0$ on Ξ_0° for each $m \geq 1$. Hence u is a solution of [IP; f]. If $\sum_{m=1}^{\infty} \|h_m\|_\infty < \infty$, then

$$|u(\xi)| \leq \|f\|_\infty + \sum_{m=1}^{\infty} \|h_m\|_\infty < \infty \qquad \text{for any } \xi \in \Xi_0.$$

(ii) Let u be any time-locally bounded solution of [IP: f]. We inductively define $h_m \in L(X)$, $m = 1, 2, \dots$, by

$$\begin{cases} h_1(x) = u(x, 1) - U_f^{(0)}(x, 1), \\ h_m(x) = u(x, m) - U_f^0(x, m) - \sum_{j=1}^{m-1} U_{h_j}^{(j)}(x, m), \ m = 2, 3, \dots. \end{cases} \tag{4.3}$$

Then, each h_m is bounded on X, and by (3.1) and Lemma 4.1, we have

$$\begin{aligned} \Delta_1 h_m &= \Delta_1\{u(\cdot, m) - U_f^{(0)}(\cdot, m) - \textstyle\sum_{j=1}^{m-1} U_{h_j}^{(j)}(\cdot, m)\} \\ &= \{\Pi u(\cdot, m) - \Pi U_f^{(0)}(\cdot, m) - \textstyle\sum_{j=1}^{m-1} \Pi U_{h_j}^{(j)}(\cdot, m)\} \\ &\quad - \{u(\cdot, m-1) - U_f^{(0)}(\cdot, m-1) - \textstyle\sum_{j=1}^{m-1} U_{h_j}^{(j)}(\cdot, m-1)\} \\ &= \begin{cases} -u(\cdot, 0) + U_f^{(0)}(\cdot, 0) = -f + f = 0, & \text{if } m = 1, \\ -h_{m-1} + U_{h_{m-1}}^{(m-1)}(\cdot, m-1) = -h_{m-1} + h_{m-1} = 0, & \text{if } m \geq 2. \end{cases} \end{aligned}$$

Hence, $h_m \in HB(N; 1)$ for all $m = 1, 2, \dots$. Since $U_{h_m}^{(m)}(\cdot, t) = 0$ for $m > t$ and $U_{h_m}^{(m)}(\cdot, m) = h_m$, (4.3) implies (4.2).

THEOREM 4.4. (i) *If* $\tilde{G}1 = 1$ (*i.e.*, $\sum_{a\in X} \tilde{g}_a(x) = 1$ *for all* $x \in X$), *then* u

$= U_f^{(0)}$ *gives the unique bounded solution of* [IP: f] *for any bounded* $f \in L(X)$.
(ii) *If* $\tilde{G}1 \neq 1$, *then the linear space of bounded solutions of* [IP: 0] *is infinite dimensional.*

PROOF. (i) We know ([7; Theorem 5.3]) that $\tilde{G}1 = 1$ if and only if $HB(N; 1) = \{0\}$. Therefore, (ii) of the above theorem implies that $U_f^{(0)}$ gives the unique (time-locally) bounded solution of [IP: f].
(ii) If $\tilde{G}1 \neq 1$, then $\tilde{h} = 1 - \tilde{G}1 \in HB(N; 1)$ and $\tilde{h} \neq 0$. Then $\{U_{\tilde{h}}^{(m)}\}_{m=1}^{\infty}$ provides a linearly independent infinite set of bounded solutions of [IP: 0].

REMARK 4.5. The condition $\sum_{m=1}^{\infty} \|h_m\|_\infty < \infty$ in Theorem 4.3 (i) is by no means a necessary condition for (4.2) to be bounded, even if $h_m \geq 0$ for all m. For example, we see that $u = \sum_{m=1}^{\infty} U_{\tilde{h}}^{(m)}$ gives a bounded solution of [IP: 0] (in fact, $u(\cdot, t) = 1 - \tilde{G}^{(t)}1$ for $t \in T_0$).

§5. Parabolic Green function of the discrete cylinder

Given $\alpha \in \Xi$, a function $G_\alpha \in L(\Xi)$ is called the *parabolic Green function* of $\{N, T\}$ with pole at α if it satisfies the following three conditions:

(G.1) $G_\alpha(\xi) \geq 0$ for all $\xi \in \Xi$;

(G.2) $\Pi G_\alpha(\xi) = -\varepsilon_\alpha(\xi)$ on Ξ;

(G.3) If $u \in L(\Xi)$ satisfies conditions
(i) $u(\xi) \geq 0$ for all $\xi \in \Xi$, and
(ii) $\Pi u(\xi) \leq -\varepsilon_\alpha(\xi)$ on Ξ,
then $u(\xi) \geq G_\alpha(\xi)$ on Ξ.

The uniqueness of the parabolic Green function G_α is assured by condition (G.3).

THEOREM 5.1. *The parabolic Green function of* $\{N, T\}$ *with pole at* $\alpha = (a, s) \in \Xi$ *always exists*; *in fact it is given by* $G_\alpha = U_{\tilde{g}_a}^{(s)}$, *namely*

$$G_\alpha(x, t) = 0 \textit{ if } t < s,$$

$$G_\alpha(x, t) = [\tilde{G}^{(t-s)}\tilde{g}_a](x) \textit{ if } t \geq s.$$

PROOF. Condition (G.1) is clear. We see that (G.2) holds by Lemma 4.1 and (2.2). To show (G.3), let $u \in L(\Xi)$ satisfy conditions (i) and (ii) in (G.3). Let $v(\xi) = u(\xi) - G_\alpha(\xi)$. Then $v(\xi) \geq 0$ for $\xi = (x, t)$ with $t < s$, $v(\xi) \geq -G_\alpha(\xi)$ and $\Pi v(\xi) \leq 0$ on Ξ. Since $\|\tilde{g}_a\|_1 < \infty$, $\|G_\alpha(\cdot, t)\|_1 < \infty$ for all $t \geq s$ by Corollary 2.3. Thus, by Corollary 3.3, we see that $v \geq 0$ on Ξ.

By (2.3), (2,4), Corollary 2.3 and Propositions 2.4 and 2.5, we obtain

THEOREM 5.2. *The parabolic Green function* $G_\alpha(\xi)$, $\alpha = (a, s)$, *has the following properties*:

(G.4) $G_\alpha(x, t) > 0$ *if* $t \geq s$.

(G.5) $\|G_\alpha(\cdot, t)\|_p \leq 1$ *for any* $t \in T$ $(1 \leq p \leq \infty)$, *in particular* $G_\alpha(\xi) \leq 1$ *for every* $\xi \in \Xi$.

(G.6) $G_\alpha(\cdot, t) \in \mathscr{E}(N: 1)$ *for any* $t \in T$ *and* $\lim_{t \to \infty} E_1(G_\alpha(\cdot, t)) = 0$.

(G.7) $\lim_{t \to \infty} \|G_\alpha(\cdot, t)\|_p = 0$ for $p > 1$.

REMARK 5.3. (G.7) does not hold for $p = 1$ in general; see Remark 2.6.

We say that a function $g_a \in L(X)$ is the harmonic Green function of N with pole at $a \in X$ if

$$\Delta g_a(x) = -\varepsilon_a(x) \text{ on } X \text{ and } g_a \in D_0(N).$$

Here $D_0(N)$ is the closure of $L_0(X)$ in $D(N) = \{u \in L(X); D(u) < \infty\}$ with respect to the norm $[D(u) + u(x_0)^2]^{1/2}$ $(x_0 \in X)$. The harmonic Green function exists if and only if N is of hyperbolic type, i.e., $D_0(N) \neq D(N)$, or equivalently $1 \notin D(N)$ (cf. [5], [6]).

Now we show a fundamental formula expressing the harmonic Green function of N with pole at $a \in X$ by the parabolic Green function of $\{N, T\}$ with pole at $\alpha = (a, s) \in \Xi$.

THEOREM 5.4. *Assume that N is of hyperbolic type. Then*

$$g_a(x) = \sum_{t=s}^{\infty} G_\alpha(x, t) \quad \text{for } \alpha = (a, s).$$

PROOF. For $m \in T$ with $m > s$, put $v_m(x) = \sum_{t=s}^{m} G_\alpha(x, t)$ and $h_m = g_a - v_m$. Then

$$\begin{aligned} \Delta v_m &= \Delta_1 v_m + v_m = \sum_{t=s}^{m} \Delta_1[\tilde{G}^{(t-s)}\tilde{g}_a] + v_m \\ &= -\sum_{t=s+1}^{m} [\tilde{G}^{(t-s-1)}\tilde{g}_a] - \varepsilon_a + v_m = [\tilde{G}^{(m-s)}\tilde{g}_a] - \varepsilon_a, \end{aligned}$$

so that

$$\Delta h_m = \Delta g_a - \Delta v_m = -[\tilde{G}^{(m-s)}\tilde{g}_a] \leq 0$$

on X. Hence h_m is superharmonic on X and $h_m \geq -v_m$. By Corollary 2.3, $\|v_m\|_1 < \infty$. Hence, by an argument similar to the proof of Corollary 3.3, together with the minimum principle ([6; Lemma 2.1]), we conclude that $h_m \geq 0$ on X, i.e., $v_m \leq g_a$ on X. It follows that v_m converges to $v = \sum_{t=s}^{\infty} G_\alpha(\cdot, t)$, $v \leq g_a$ on X, v is a nonnegative superharmonic function and $\Delta v = -\varepsilon_a$ on X. By the Riesz decomposition theorem ([6; Theorem 5.1]), we conclude that $v = g_a$.

§6. Riesz decomposition theorem

For $u \in L(\Xi)$ and $\alpha \in \Xi$, let us define $\tau_\alpha u \in L(\Xi)$ by

$$\tau_\alpha u(\xi) = u(\xi) \text{ for } \xi \neq \alpha \text{ and } \tau_\alpha u(\alpha) = P_u(\alpha).$$

By (3.2), $\Pi(\tau_\alpha u)(\alpha) = 0$. If $u \in SPR(N, T)$, then $\tau_\alpha u \in SPR(N, T)$ and $\tau_\alpha u \leq u$ on Ξ.

As in the continuous case, we obtain the following lemma and its corollaries:

LEMMA 6.1. *If $\mathscr{P}$ is a Perron's family, namely if $\mathscr{P}$ is a nonempty subset of $SPR(N, T)$ satisfying the following three conditions*:

(P.1) *If $u_1, u_2 \in \mathscr{P}$, then $\min\{u_1, u_2\} \in \mathscr{P}$;*
(P.2) *$\tau_\alpha u \in \mathscr{P}$ for every $u \in \mathscr{P}$ and $\alpha \in \Xi$;*
(P.3) *$\{u(\xi); u \in \mathscr{P}\}$ is bounded below at each point $\xi \in \Xi$,*

then its lower envelope: $(\inf \mathscr{P})(\xi) = \inf\{u(\xi); u \in \mathscr{P}\}$ *is parabolic on Ξ.*

PROOF. Let $\tilde{u} = \inf \mathscr{P}$. Then $\tilde{u} \in L(\Xi)$ by (P.3) and $\tilde{u} \leq u$ on Ξ for every $u \in \mathscr{P}$. We show that $\Pi\tilde{u}(\alpha) = 0$ for any $\alpha \in \Xi$. By (P.1) and (P.2), we can choose a sequence $\{u_n\}$ in $\mathscr{P}$ such that $u_n(\xi) \to \tilde{u}(\xi)$ as $n \to \infty$ for all $\xi \in \Xi(\alpha)$ and $P_{u_n}(\alpha) = u_n(\alpha)$ for all n. Then $P_{\tilde{u}}(\alpha) = \tilde{u}(\alpha)$, i.e., $\Pi\tilde{u}(\alpha) = 0$.

COROLLARY 6.2. *If $u \in SPR(N, T)$ has a subparabolic minorant, then u has the greatest parabolic minorant* GPM(u), *which is equal to the greatest subparabolic minorant of u.*

COROLLARY 6.3. *Let $f \in L^+(\Xi)$. If there exists $v \in SPR(N, T)$ such that $v \geq f$ on Ξ, then the reduction function*

$$Rf(\xi) = \inf\{u(\xi); u \in SPR(N, T) \textit{ and } u \geq f \textit{ on } \Xi\}$$

is superparabolic on Ξ and parabolic on the set $\{\xi \in \Xi; \Pi f(\xi) \geq 0\}$; in particular, it is parabolic on the set $\{\xi \in \Xi; f(\xi) = 0\}$.

In order to obtain a discrete analogue of the Riesz decomposition theorem, we introduce parabolic Green potentials.

For $\nu \in L^+(\Xi)$, its *parabolic Green potential* $G\nu$ is defined by

$$G\nu(\xi) = \textstyle\sum_{\alpha \in \Xi} G_\alpha(\xi)\nu(\alpha).$$

Let

$$M(G) = \{\nu \in L^+(\Xi); G\nu \in L(\Xi)\}.$$

It follows from (G.5) that

$$L_0^+(\Xi) \subset \{v \in L^+(\Xi);\, v(\Xi) < \infty\} \subset M(G),$$

where $v(\Xi) = \sum_{\xi \in \Xi} v(\xi)$. If $v \in M(G)$, then $Gv \in SPR(N, T)$ and $\Pi(Gv) = -v$ on Ξ by (G.2).

LEMMA 6.4. *If* $v \in L_0^+(\Xi)$, *then* $\mathrm{GPM}(Gv) = 0$.

PROOF. Put $u = \mathrm{GPM}(Gv)$. There is $s \in T$ such that $v = 0$ on Ξ_s^*. Clearly, $u \geq 0$ on Ξ and $u = 0$ on Ξ_s^*. By (G.5), $\|Gv(\cdot, t)\|_p < \infty$ for any $t \in T_s^\circ$ $(p < \infty)$. Since $-u \geq -Gv$, Corollary 3.3 implies that $-u \geq 0$.

Now we prove the Riesz decomposition theorem:

THEOREM 6.5. *Let* $u \in SPR(N, T)$ *and assume* u *has a subparabolic minorant. Let* $v = -\Pi u \geq 0$. *Then* $v \in M(G)$ *and* u *can be decomposed in the form*: $u = Gv + \mathrm{GPM}(u)$.

PROOF. Let $\{\Xi_n\}$ be an exhaustion of Ξ by finite sets. Define v_n by $v_n = v$ on Ξ_n and $v_n = 0$ on $\Xi - \Xi_n$. For each n, $h_n = u - Gv_n$ is superparabolic on Ξ and parabolic on Ξ_n. Let $h = \mathrm{GPM}(u)$. Since $h - h_n \leq u - h_n = Gv_n$ and $h - h_n$ is subparabolic, we have $h \leq h_n$ by Lemma 6.4, namely $Gv_n \leq u - h$. Since $Gv_n \uparrow Gv$ $(n \to \infty)$, it follows that $v \in M(G)$ and h_n decreases to a parabolic function $h_0 \geq h$. Then $h_0 = u - Gv \leq u$, and hence $h_0 = h = \mathrm{GPM}(u)$ and $u = Gv + \mathrm{GPM}(u)$.

COROLLARY 6.6. *Let* $v \in SPR(N, T) \cap L^+(\Xi)$. *Then* v *is a parabolic Green potential if and only if* $\mathrm{GPM}(v) = 0$.

THEOREM 6.7. *If* $u \in L_0(\Xi)$, *then* $u(\xi) = -\sum_{\alpha \in \Xi} G_\alpha(\xi)[\Pi u(\alpha)]$.

PROOF. Let $\mu = \max\{\Pi u, 0\}$ and $v = \max\{-\Pi u, 0\}$. Then μ, $v \in L_0^+(\Xi)$ and $\Pi u = \mu - v$. Put $h = u - Gv + G\mu$ and $\Xi' = \{\xi \in \Xi;\, u(\xi) \neq 0\}$. Then h is parabolic on Ξ and $-Gv \leq h \leq G\mu$ on $\Xi - \Xi'$. By Theorem 3.2 (ii), $-Gv \leq h \leq G\mu$ on Ξ. It follows from Lemma 6.4 that $h = 0$, i.e., $u = Gv - G\mu$.

COROLLARY 6.8. *If* $f \in L_0^+(X)$, *then the reduction function* Rf *is a parabolic Green potential.*

PROOF. By the above theorem, $f \leq Gv$ with $v = \max\{-\Pi f, 0\}$. Hence, $0 \leq Rf \leq Gv$. By Corollaries 6.3 and 6.6, we see that Rf is a prarabolic Green potential.

As another application of the Riesz decomposition theorem, we shall prove the following domination principle by the same argument as in [2; Proposition 2.5]:

THEOREM 6.9. *Let $\mu \in M(G)$ and $v \in SPR(N, T) \cap L^+(\Xi)$. If $G\mu(\xi) \leq v(\xi)$ on the support $S\mu$ of μ, then the same inequality holds on Ξ.*

PROOF. Let $f(\xi) = \min\{0, v(\xi) - G\mu(\xi)\}$ and $\Xi' = \Xi - S\mu$. Then $v(\xi) - G\mu(\xi) \geq 0$ on $\Xi - \Xi'$ and $v - G\mu$ is superparabolic on Ξ'. Using (3.2), we easily see that f is superparabolic on Ξ. Obviously $f(\xi) \geq -G\mu(\xi)$ on Ξ. It follows from Corollary 6.6 that $f(\xi) \geq 0$ on Ξ, namely $G\mu(\xi) \leq v(\xi)$ on Ξ.

§7. Coparabolic operator and duality

As in the continuous case, we define the coparabolic operator Π^* on $L(\Xi)$ by

$$\Pi^* u(x, t) = \Delta u(x, t) + \partial u(x, t+1).$$

Similarly to (3.1), we have

$$\Pi^* u(\cdot, t) = \Delta_1 u(\cdot, t) + u(\cdot, t+1).$$

We say that a function $u \in L(\Xi)$ is *cosuperparabolic* (resp. *coparabolic*) on a set Ω if $\Pi^* u(\xi) \leq 0$ (resp. $\Pi^* u(\xi) = 0$) on Ω. Denote by $SPR^*(N, T)$ (resp. $PR^*(N, T)$) the set of all cosuperparabolic (resp. coparabolic) functions on Ξ.

By the interchange of the order of summation, we easily obtain the following discrete analogue of [3; Proposition 1.1]:

THEOREM 7.1. *Let $u, v \in L(\Xi)$. If u or v belongs to $L_0(\Xi)$, then the following equality holds:*

$$\textstyle\sum_{\xi \in \Xi} u(\xi) \Pi^* v(\xi) = \sum_{\xi \in \Xi} v(\xi) \Pi u(\xi).$$

COROLLARY 7.2. *A function $u \in L(\Xi)$ is parabolic (resp. superparabolic) on Ξ if and only if*

$$\textstyle\sum_{\xi \in \Xi} u(\xi) \Pi^* v(\xi) = 0 \quad (\textit{resp.} \leq 0) \quad \textit{for all } v \in L_0^+(\Xi).$$

A function $v \in L(\Xi)$ is coparabolic (resp. cosuperparabolic) on Ξ if and only if

$$\textstyle\sum_{\xi \in \Xi} v(\xi) \Pi u(\xi) = 0 \ (\textit{resp.} \leq 0) \ \textit{ for all } u \in L_0^+(\Xi).$$

COROLLARY 7.3. *For any $u \in L_0(\Xi)$,*

$$\textstyle\sum_{\xi \in \Xi} \Pi u(\xi) = 0 \quad \textit{and} \quad \sum_{\xi \in \Xi} \Pi^* u(\xi) = 0.$$

We obtain the dual statements of the results in §§3–6 with respect to the operator Π^* or cosuperparabolic functions. As to the Green function with respect to Π^*, we have

THEOREM 7.4. *Let* $u^*(\xi) = G_\xi(\alpha)$. *Then* u^* *has the following properties*:

(G*.1) $u^*(\xi) \geq 0$ *for all* $\xi \in \Xi$;

(G*.2) $\Pi^* u^*(\xi) = -\varepsilon_\alpha(\xi)$ *on* Ξ;

(G*.3) *If* $v \in L(\Xi)$ *satisfies conditions*
 (i) $v(\xi) \geq 0$ *on* Ξ,
 (ii) $\Pi^* v(\xi) \leq -\varepsilon_\alpha(\xi)$ *on* Ξ,
then $v(\xi) \geq u^*(\xi)$ *on* Ξ.

In view of this theorem, we call $G_\alpha^*(\xi) = G_\xi(\alpha)$ the *coparabolic Green function* of $\{N, T\}$ with pole at α. For $v \in L^+(\Xi)$, the *coparabolic Green potential* G^*v is defined by

$$G^*v(\xi) = \sum_{\alpha \in \Xi} G_\alpha^*(\xi) v(\alpha) = \sum_{\alpha \in \Xi} G_\xi(\alpha) v(\alpha).$$

Let $M(G^*) = \{v \in L^+(\Xi); G^*v \in L(\Xi)\}$. If $\mu \in M(G)$ and $v \in M(G^*)$, then

$$\sum_{\xi \in \Xi} G\mu(\xi) v(\xi) = \sum_{\alpha \in \Xi} \mu(\alpha) G^*v(\alpha). \tag{7.1}$$

The reduction operator R^*f for $f \in L^+(\Xi)$ is defined by

$$R^*f(\xi) = \inf\{u(\xi);\, u \in SPR^*(N, T) \text{ and } u \geq f \text{ on } \Xi\}.$$

If $f \in L_0^+(\Xi)$, then R^*f is a coparabolic Green potential by the dual statement of Corollary 6.8, namely, $R^*f = G^*\lambda_f^*$ with $\lambda_f^* \in L_0^+(\Xi)$.

LEMMA 7.5. *Let* $v \in SPR^*(N, T) \cap L^*(\Xi)$ *and* $\mu \in M(G)$. *If* $\{f_n\}$ *is a sequence of functions in* $L_0^+(\Xi)$ *which increases to* v, *then*

$$\sum_{\alpha \in \Xi} v(\alpha)\mu(\alpha) = \lim_{n \to \infty} \sum_{\xi \in \Xi} G\mu(\xi) \lambda_{f_n}^*(\xi).$$

PROOF. Since $f_n \leq R^*f_n \leq v$, we see that $R^*f_n \uparrow v$ on Ξ. Hence, using (7.1) we have

$$\begin{aligned}\sum_{\alpha \in \Xi} v(\alpha)\mu(\alpha) &= \lim_{n \to \infty} \sum_{\alpha \in \Xi} R^*f_n(\alpha)\mu(\alpha)\\ &= \lim_{n \to \infty} \sum_{\alpha \in \Xi} G^*\lambda_{f_n}^*(\alpha)\mu(\alpha) = \lim_{n \to \infty} \sum_{\xi \in \Xi} G\mu(\xi)\lambda_{f_n}^*(\xi).\end{aligned}$$

From this lemma, we immediately obtain

THEOREM 7.6. (cf. [3; Lemma 1.3]). *Let* $\mu_1, \mu_2 \in M(G)$. *If* $G\mu_1 \leq G\mu_2$ *on* Ξ, *then*

$$\sum_{\alpha \in \Xi} v(\alpha)\mu_1(\alpha) \leq \sum_{\alpha \in \Xi} v(\alpha)\mu_2(\alpha)$$

for any $v \in SPR^*(N, T) \cap L^+(\Xi)$; *in particular*

$$\mu_1(\Xi) \leq \mu_2(\Xi).$$

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