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Asymptotic expansion of the joint distribution of sample mean vector and sample covariance matrix from an elliptical population

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ABSTRACT. We consider the joint distribution of the sample mean vector and the sample covariance matrix based on the i.i.d. sample of size n. We give a basic lemma which can be used for deriving asymptotic expansions up to terms of $O(n^{-1})$ for the joint distribution of the sample mean vector and the sample covariance matrix. Using the lemma, we derive an asymptotic expansion for an elliptical population.

1. Introduction

Let $\bar{X}$ and S be the sample mean vector and the sample covariance matrix based on the i.i.d. sample of size n from a p dimensional probability distribution with mean vector μ and covariance matrix Ω. Let

$$Z = n^{1/2}\Omega^{-1/2}(S - \Omega)\Omega^{-1/2} \qquad \text{and} \qquad Y = n^{1/2}\Omega^{-1/2}(\bar{X} - \mu). \tag{1.1}$$

Then the limiting distribution of Z and Y is mutually independent normal. Wakaki [7] derived an asymptotic expansion for the joint distribution of Z and Y up to the order of $n^{-1/2}$ when the underlying distribution is an elliptical distribution. Unfortunately, the result included some miscalculations. The purposes of this paper are to correct them and to extend the result to an asymptotic expansion up to the order n^{-1}.

2. A basic lemma

In this section, we do not need the assumption that the underlying distribution is elliptical. For the validity of the following formal asymptotic expansion, we assume that the underlying distribution has a density function with respect to Lebesgue measure on R^p (see theorem 2 in Bhattacharya and Ghosh [1]).

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Let $X_1, X_2, \dots, X_n$ be the i.i.d. sample, and let

$$U_j = \Omega^{-1/2}(X_j - \mu), \qquad j = 1, 2, \dots, n. \tag{2.1}$$

Then

$$Y = n^{-1/2} \sum_{j=1}^n U_j \quad \text{and} \quad Z = n(n-1)^{-1}\{W - n^{-1/2}(YY' - I_p)\}, \tag{2.2}$$

where I_p is the identity matrix of order p and

$$W = n^{-1/2} \sum_{j=1}^n (U_j U_j' - I_p). \tag{2.3}$$

First we consider the joint distribution of W and Y.

For a $p \times p$ symmetric matrix $A = (a_{ij})$ and a $p \times 1$ vector $B = (b_j)$, we use the following notation.

$$\mathrm{Vec}\,(A|B) = (a_{11}, a_{22}, \dots, a_{pp}, a_{12}, a_{13}, \dots, a_{p-1,p}, b_1, b_2, \dots, b_p)'. \tag{2.4}$$

Let

$$V_j = \mathrm{Vec}\,(U_j U_j' - I_p | U_j), \qquad j = 1, 2, \dots, n. \tag{2.5}$$

Then

$$\xi = \mathrm{Vec}\,(W|Y) = n^{-1/2} \sum_{j=1}^n V_j. \tag{2.6}$$

Therefore our problem can be reduced to deriving an asymptotic expansion for the distribution of the sample mean vector of $V_1, V_2, \dots, V_n$.

Let $\phi(t)$ be the characteristic function of ξ. If $E[\|V_1\|^5] < \infty$, where $\|*\|$ is the Euclidean norm, then $\phi(t)$ can be expanded as

$$\begin{aligned} \phi(t) &= \mathrm{E}[\exp(in^{-1/2} V_1' t)]^n \\ &= \{1 - (1/2)n^{-1}\mathrm{E}[(V_1't)^2] - (i/6)n^{-3/2}\mathrm{E}[(V_1't)^3] \\ &\quad + (1/24)n^{-2}\mathrm{E}[(V_1't)^4] + \mathrm{O}(n^{-5/2})\}^n \\ &= \exp[n \log\{1 - (1/2)n^{-1}\mathrm{E}[(V_1't)^2] - (i/6)n^{-3/2}\mathrm{E}[(V_1't)^3] \\ &\quad + (1/24)n^{-2}\mathrm{E}[(V_1't)^4] + \mathrm{O}(n^{-5/2})\}] \\ &= \exp\{-(1/2)t'C_q t\}[1 - (i/6)n^{-1/2}\mathrm{E}[(V_1't)^3] \\ &\quad + (1/24)n^{-1}\{\mathrm{E}[(V_1't)^4] - 3(t'C_q t)^2\} \\ &\quad - (1/72)n^{-1}\mathrm{E}[(V_1't)^3]^2 + \mathrm{O}(n^{-3/2})], \end{aligned} \tag{2.7}$$

where C_q is the $q \times q$ covariance matrix of V_1 with $q = p(p+3)/2$. For any nonrandom vector a, $\mathrm{Prob}\,(a'V_1 = 0) = 0$ since X_1 has a density function. This shows that C_q is nonsingular. Inverting $\phi(t)$, the density function of ξ

can be expressed as

$$f_W(\xi) = (2\pi)^{-q}|C_q|^{-1/2} \int \exp\{-i\xi' t - (1/2)t' C_q t\} \cdot \{1 + n^{-1/2} g_1(t) + n^{-1} g_2(t)\}(dt) + O(n^{-3/2}), \tag{2.8}$$

where

$$g_1(t) = \mathrm{E}[-(i/6)(V_1' t)^3] \tag{2.9}$$

and

$$g_2(t) = \mathrm{E}[(1/24)(V_1' t)^4 - (1/8)(t' C_q t)^2 - (1/72)(V_1' t)^3 (V_2' t)^3]. \tag{2.10}$$

Since

$$-i\xi' t - (1/2)t' C_q t = -(1/2)(t + iC_q^{-1}\xi)' C_q (t + iC_q^{-1}\xi) - (1/2)\xi' C_q^{-1}\xi, \tag{2.11}$$

$f_W(\xi)$ can be expressed as follows.

$$f_W(\xi) = (2\pi)^{-q/2}|C_q|^{-1/2} \exp\{-(1/2)\xi' C_q^{-1}\xi\} \cdot \{1 + n^{-1/2}\mathrm{E_T}[g_1(T)] + n^{-1}\mathrm{E_T}[g_2(T)]\} + O(n^{-3/2}), \tag{2.12}$$

where $\mathrm{E_T}$ means the expectation with respect to T when T is distributed as q dimensional normal distribution with mean vector $-iC_q^{-1}\xi$ and covariance matrix C_q^{-1}. Calculating the expectation $\mathrm{E_T}$, we obtain the following lemma.

LEMMA. *Let ξ be given by* (2.6). *Assume that* $\mathrm{E}[\|X_1\|^{10}] < \infty$ *and that X_1 has a density function with respect to Lebesgue measure on R^p then the density function of ξ can be expanded as*

$$f_W(\xi) = (2\pi)^{-q/2}|C_q|^{-1/2} \exp\{-(1/2)\xi' C_q^{-1}\xi\} \cdot \{1 + n^{-1/2} q_1(\xi) + n^{-1} q_2(\xi)\} + O(n^{-3/2}), \tag{2.13}$$

where

$$q_1(\xi) = (1/6)\mathrm{E_V}[(V_1' C_q^{-1}\xi)^3 - 3(V_1' C_q^{-1}\xi)(V_1' C_q^{-1} V_1)] \tag{2.14}$$

and

$$\begin{aligned} q_2(\xi) = {} & (1/2)\{q_1(\xi)\}^2 - (1/8)\{q(q+2) - 2(q+2)(\xi' C_q^{-1}\xi) + (\xi' C_q^{-1}\xi)^2\} \\ & + (1/24)\mathrm{E_V}[-3(V_1' C_q^{-1}\xi)^2 (V_1' C_q^{-1} V_2)(V_2' C_q^{-1}\xi)^2 \\ & + 6(V_1' C_q^{-1}\xi)^2 (V_1' C_q^{-1} V_2)(V_2' C_q^{-1} V_2) \\ & - 3(V_1' C_q^{-1} V_1)(V_1' C_q^{-1} V_2)(V_2' C_q^{-1} V_2) \end{aligned} \tag{2.15}$$

$$- 2(V_1'C_q^{-1}V_2)^3 + 6(V_1'C_q^{-1}V_2)^2(V_1'C_q^{-1}\xi)(V_2'C_q^{-1}\xi)$$
$$+ (V_1'C_q^{-1}\xi)^4 - 6(V_1'C_q^{-1}\xi)^2(V_1'C_q^{-1}V_1) + 3(V_1'C_q^{-1}V_1)^2].$$

Here V_1 and V_2 are given by (2.5) *and* E_{V} *means the expectation with respect to the distribution of* V_1 *and* V_2.

The above lemma may be useful for deriving asymptotic expansions of the distributions or the moments of some functions of the sample mean vector and the sample covariance matrix. Here we note that some works have been done for these distribution problems. Fujikoshi [3] derived asymptotic expansions of the distribution of a multivariate Student's t statistic defined by $u = n^{1/2}S^{-1/2}(\bar{X} - \mu)$ as well as Hotelling's T^2-statistic under nonnormality. Kano [4] also derived an asymptotic expansion for the distribution of Hotelling's T^2-statistic under a general distribution, independently with Fujikoshi. He derived a formula of Edgeworth expansion of the distribution of a multivariate statistic corresponding to ξ, with using multivariate Hermite polynomials (cf. Appendix of Takemura and Takeuchi [6]).

3. Asymptotic expansion for the distribution of Z and Y in elliptical case

If the underlying distributon is elliptical we can evaluate the expectations involved in the lemma in Section 2. Suppose that the underlying distribution is elliptical with characteristic function $\exp(i\mu' t)\psi(t'\Gamma t)$, then the covariance matrix is $-2\psi'(0)\Gamma$. Hence the characteristic function of $U_1 = \Omega^{-1}(X_1 - \mu)$ is

$$c(t) = \psi[-t't/\{2\psi'(0)\}]. \tag{3.1}$$

Let $U = (u_1, u_2, \dots, u_p)'$ and $t = (t_1, t_2, \dots, t_p)'$, then

$$\mathrm{E}[u_i u_j \dots u_k] = (\partial/\partial t_i)(\partial/\partial t_j)\dots(\partial/\partial t_k)c(t)|_{t=0}. \tag{3.2}$$

Calculating the derivatives to the 8-th order, we obtain

$$\begin{aligned} \mathrm{E}[u_i u_j] &= \langle ij \rangle, \\ \mathrm{E}[u_i u_j u_k u_l] &= \kappa_2 \langle ijkl \rangle, \\ \mathrm{E}[u_i u_j u_k u_l u_m u_n] &= \kappa_3 \langle ijklmn \rangle, \\ \mathrm{E}[u_i u_j u_k u_l u_m u_n u_o u_p] &= \kappa_4 \langle ijklmnop \rangle, \end{aligned} \tag{3.3}$$

where

$$\kappa_j = \psi^{(j)}(0)/\{\psi'(0)\}^j, \qquad j = 1, 2, \dots \tag{3.4}$$

and the notation $\langle * \rangle$ means

$$
(3.5)\quad
\begin{aligned}
\langle ij\rangle &= \delta_{ij} \qquad \text{(the Kroneker's delta)},\\
\langle ijkl\rangle &= \langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle,\\
\langle ijklmn\rangle &= \langle ij\rangle\langle klmn\rangle + \langle ik\rangle\langle jlmn\rangle + \cdots + \langle in\rangle\langle jklm\rangle,\\
\langle ijklmnop\rangle &= \langle ij\rangle\langle klmnop\rangle + \langle ik\rangle\langle jlmnop\rangle + \cdots + \langle ip\rangle\langle jklmno\rangle.
\end{aligned}
$$

Let C_q be partitioned to 3×3 blocks as $[C_{jk}]$ where the size of C_{11}, C_{22} and C_{33} are $p \times p$, $p(p-1)/2$ and $p \times p$, respectively. Then, using the formulas (3.3) we obtain $C_{12} = O$, $C_{13} = O$, $C_{23} = O$,

$$
(3.6)\quad C_{11} = 2\kappa_2 I_p + (\kappa_2 - 1)G_p, \qquad C_{22} = \kappa_2 I_{p(p-1)/2} \qquad \text{and} \qquad C_{33} = I_p,
$$

where G_p is a $p \times p$ matrix whose all elements are equal to 1. Let $J_p = I_p - p^{-1}G_p$, then J_p and $p^{-1}G_p$ are idempotent, $J_pG_p = O$ and

$$
(3.7)\qquad C_{11} = 2\kappa_2 J_p + (p\kappa_2 + 2\kappa_2 - p)p^{-1}G_p.
$$

Therefore

$$
(3.8)\qquad C_{11}^{-1} = uJ_p + vp^{-1}G_p \qquad \text{and} \qquad C_{22}^{-1} = 2uI_{p(p-1)/2},
$$

where

$$
(3.9)\qquad u = (2\kappa_2)^{-1} \qquad \text{and} \qquad v = (p\kappa_2 + 2\kappa_2 - p)^{-1}.
$$

Since

$$
(3.10)\quad \mathrm{Vec}\,(A|B)'C_q^{-1}\,\mathrm{Vec}\,(C|D) = u\,\mathrm{tr}\,(AC) + p^{-1}(v-u)\,\mathrm{tr}\,(A)\,\mathrm{tr}\,(C) + B'D,
$$

$$
\begin{aligned}
(3.11)\quad V_1'C_q^{-1}V_2 &= u(U_1'U_2)^2 + p^{-1}(v-u)(U_1'U_1)(U_2'U_2)\\
&\quad - v(U_1'U_1 + U_2'U_2) + U_1'U_2 + pv = a(U_1, U_2) \quad \text{(say)}.
\end{aligned}
$$

From (2.2),

$$
(3.12)\qquad W = (1 - n^{-1})Z + n^{-1/2}(YY' - I_p),
$$

$$
\begin{aligned}
(3.13)\qquad \xi'C_q^{-1}\xi &= u\,\mathrm{tr}\,(W^2) + p^{-1}(v-u)\,\mathrm{tr}^2(W) + Y'Y\\
&= r_0(Z, Y) + n^{-1/2}b_1(Z, Y) + n^{-1}b_2(Z, Y) + \mathrm{O}(n^{-3/2}),
\end{aligned}
$$

and

$$
(3.14)\qquad V_1'C_q^{-1}\xi = c(U_1, Z, Y) + n^{-1/2}d(U_1, Y) + \mathrm{O}(n^{-1}),
$$

where

$$
\begin{aligned}
r_0(Z, Y) &= u\,\mathrm{tr}\,(Z^2) + p^{-1}(v-u)\,\mathrm{tr}^2(Z) + Y'Y,\\
(3.15)\qquad b_1(Z, Y) &= 2\{u(Y'ZY) - v\,\mathrm{tr}\,(Z) + p^{-1}(v-u)\,\mathrm{tr}\,(Z)(Y'Y)\},
\end{aligned}
$$

$$b_2(Z, Y) = -2u \operatorname{tr}(Z^2) - 2p^{-1}(v-u)\operatorname{tr}^2(Z) + \{u + p^{-1}(v-u)\}(Y'Y)^2 - 2v(Y'Y) + pv,$$

and

$$(3.16)\quad \begin{aligned} c(U_1, Z, Y) &= u(U_1'ZU_1) + p^{-1}(v-u)\operatorname{tr}(Z)(U_1'U_1) - v\operatorname{tr}(Z) + U_1'Y, \\ d(U_1, Y) &= u(U_1'Y)^2 - v(U_1'U_1) + p^{-1}(v-u)(Y'Y)(U_1'U_1) \\ &\quad - v(Y'Y) + pv. \end{aligned}$$

The Jacobian of a transformation $(W, Y) \to (Z, Y)$ is $(1 - n^{-1})^{p(p+1)/2}$. Therefore, from lemma, the joint density function of Z and Y can be expanded as

$$(3.17)\quad f(Z, Y) = (2\pi)^{-q/2}|C_q|^{-1/2}\exp[-(1/2)r_0(Z, Y)] \cdot [1 + n^{-1/2}r_1(Z, Y) + n^{-1}r_2(Z, Y) + O(n^{-3/2})],$$

where

$$(3.18)\quad r_1(Z, Y) = E[(1/6)c(U_1, Z, Y)^3 - (1/2)c(U_1, Z, Y)a(U_1, U_1)] - (1/2)b_1(Z, Y),$$

$$\begin{aligned}(3.19)\quad r_2(Z, Y) = {} & (1/2)r_1(Z, Y)^2 - p(p+1)/2 - (1/2)b_2(Z, Y) \\ & - (1/8)\{q(q+2) - 2(q+2)r_0(Z, Y) + r_0(Z, Y)^2\} \\ & + E[(1/2)c(U_1, Z, Y)^2 d(U_1, Y) - (1/2)d(U_1, Y)a(U_1, U_1) \\ & - (1/8)\{c(U_1, Z, Y)^2 - a(U_1, U_1)\}a(U_1, U_2)\{c(U_2, Z, Y)^2 \\ & - a(U_2, U_2)\} - (1/12)a(U_1, U_2)^3 \\ & + (1/4)a(U_1, U_2)^2 c(U_1, Z, Y)c(U_2, Z, Y) + (1/24)c(U_1, Z, Y)^4 \\ & - (1/4)c(U_1, Z, Y)^2 a(U_1, U_1) + (1/8)a(U_1, U_1)^2]\end{aligned}$$

Taking these expectations, the joint density function of Z and Y can be expanded as the following theorem.

THEOREM. *Let Z and Y be given by* (1.1). *When the underlying distribution is elliptical with characteristic function* $\exp(it'\mu)\psi(t'\Gamma t)$ *and finite* 10-*th moments, the joint density function of Z and Y can be expanded as*

$$(3.20)\quad f(Z, Y) = (2\pi)^{-p(p+3)/4}2^{p(p-1)/4}u^{(p+2)(p-1)/4}v^{1/2}\exp\{-r_0(Z, Y)/2\} \cdot [1 + n^{-1/2}r_1(Z, Y) + n^{-1}r_2(Z, Y)] + O(n^{-3/2}),$$

where u and v are given by (3.9), *$r_0(Z, Y)$ is given by* (3.14) *and*

$$(3.21)\quad r_1(Z, Y) = \alpha_1 \operatorname{tr}(Z) + \alpha_2 \operatorname{tr}^3(Z) + \alpha_3 \operatorname{tr}(Z^3) + \alpha_4 \operatorname{tr}(Z)\operatorname{tr}(Z^2) + \alpha_5(Y'Y)\operatorname{tr}(Z) + \alpha_6 Y'ZY,$$

$$(3.22)\quad r_2(Z, Y) = (1/2)r_1(Z, Y)^2 + \beta_1 + \beta_2 \operatorname{tr}^2(Z) + \beta_3 \operatorname{tr}^4(Z) + \beta_4 \operatorname{tr}(Z^2) + \beta_5 \operatorname{tr}^2(Z)\operatorname{tr}(Z^2) + \beta_6 \operatorname{tr}^2(Z^2) + \beta_7 \operatorname{tr}(Z)\operatorname{tr}(Z^3) + \beta_8 \operatorname{tr}(Z^4) + \beta_9(Y'Y) + \beta_{10}(Y'Y)\operatorname{tr}^2(Z) + \beta_{11}(Y'Y)\operatorname{tr}(Z^2) + \beta_{12}(Y'Y)^2 + \beta_{13} Y'ZY\operatorname{tr}(Z) + \beta_{14} Y'Z^2 Y.$$

Coefficients α_j's and β_j's are as follows:

$$(3.23)\quad \alpha_1 = (v/4)(2 + 3p + p^2 - 4pv) + (\kappa_2/2)(2 + p)v(-1 + 3v) - (\kappa_3/2)v(2u + 5pu + p^2u + 2v + pv + 8uv),$$

$$\alpha_2 = (v^2/12)(3 - 6u - 2v) + (\kappa_3/6)uv^2(1 - 10u + 16u^2 - 2v + 8uv),$$

$$\alpha_3 = (4\kappa_3/3)u^3,$$

$$\alpha_4 = -uv/2 + \kappa_3 u^2(-1 + 4u)v,$$

$$\alpha_5 = (v/2)(1 - 2u),$$

$$\alpha_6 = (1 - 2u)/2,$$

$$\begin{aligned}\beta_1 = {} & \{p(-84 - 87p - 18p^2 - 3p^3 - 12v - 36pv - 39p^2v - 18p^3v \\ & - 3p^4v + 12pv^2 + 72p^2v^2 + 24p^3v^2 - 80p^2v^3)\}/96 \\ & + \{(2 + p)(2 - p + 2pv + 3p^2v + p^3v - 13p^2v^2 - 3p^3v^2 \\ & + 20p^2v^3)\kappa_2\}/8 \\ & + \{(2 + p)^2v(-2 - p + 6pv - 15pv^2)\kappa_2^2\}/8 \\ & + \{(2 + p)(4 + p)(-6u + 6pu + 12v - 3pv - 3p^2v - 6uv - 3puv \\ & + 6p^2uv + 3p^3uv - 24v^2 + 18pv^2 + 12p^2v^2 + 12puv^2 \\ & - 12p^2uv^2 - 20pv^3)\kappa_3\}/24 \\ & + \{(2 + p)^2(4 + p)v^2(-1 + 5v)\kappa_2\kappa_3\}/4 \\ & + \{(192u^3 + 128pu^3 - 48p^2u^3 - 16p^3u^3 - 32u^2v - 36pu^2v \\ & - 120p^2u^2v - 99p^3u^2v - 30p^4u^2v - 3p^5u^2v + 256u^3v \\ & + 160uv^2 - 264puv^2 - 240p^2uv^2 - 66p^3uv^2 - 6p^4uv^2 + 64u^2v^2 \\ & - 320v^3 - 260pv^3 - 60p^2v^3 - 5p^3v^3 - 320uv^3)\kappa_3^2\}/24\end{aligned}$$

$$
\begin{aligned}
&+\{(-52u^2-28pu^2+21p^2u^2+10p^3u^2+p^4u^2+32uv+64puv\\
&\quad+22p^2uv+2p^3uv-48u^2v+20v^2+12pv^2+p^2v^2\\
&\quad+48uv^2)\kappa_4\}/8,\\
\beta_2=&\{-3u-pu+v+2pv+8uv-3puv-p^2uv-6v^2\\
&-3pv^2+4puv^2+2pv^3-6p^2v^3-2p^3v^3+16p^2v^4\}/8\\
&+\{v(-2+4v-12v^2+20pv^2+19p^2v^2\\
&\quad+3p^3v^2-96pv^3-48p^2v^3)\kappa_2\}/8\\
&+\{(2+p)v^2(1-6v-3pv+36v^2+18pv^2)\kappa_2^2\}/4\\
&+\{(2u^2-2uv-puv+12u^2v+12pu^2v+2p^2u^2v-6v^2-pv^2\\
&\quad+32uv^2+4puv^2-32u^2v^2-4pu^2v^2+44v^3-28pv^3-15p^2v^3\\
&\quad-2p^3v^3-64uv^3+4puv^3+10p^2uv^3+2p^3uv^3-32v^4\\
&\quad+48pv^4+8p^2v^4)\kappa_3\}/4\\
&+\{(2+p)(6+p)(1-12v)v^3\kappa_2\kappa_3\}/4\\
&+\{(-16u^4-68u^3v-18pu^3v-2p^2u^3v+32u^4v+8u^2v^2\\
&\quad+16pu^2v^2+2p^2u^2v^2-32u^3v^2+128u^4v^2+12uv^3+40puv^3\\
&\quad+11p^2uv^3+p^3uv^3-32u^2v^3+192u^3v^3+40v^4+24pv^4\\
&\quad+2p^2v^4+64uv^4+128u^2v^4)\kappa_3^2\}/4\\
&+\{(18u^3+2pu^3+16u^2v-28u^3v-20uv^2-11puv^2-p^2uv^2\\
&\quad-48u^3v^2-2v^3-pv^3+4uv^3-48u^2v^3)\kappa_4\}/4,\\
\beta_3=&\{v^2(-1+2u+14v-20uv-32v^2-16pv^3)\}/32\\
&+\{3v^4(3+2v+4pv)\kappa_2\}/8-\{9(2+p)v^5\kappa_2^2\}/8\\
&+\{v^2(3u^2-6u^3+7uv-70u^2v+88u^3v-9v^2\\
&\quad-26uv^2+80u^2v^2+6v^3-6pv^3+24uv^3)\kappa_3\}/12\\
&+\{3(2+p)v^5\kappa_2\kappa_3\}/4\\
&+\{v^2(-2u^3+4u^4+2u^2v-44u^3v+224u^4v-256u^5v+2uv^2\\
&\quad-4u^2v^2+96u^3v^2-192u^4v^2-2v^3-pv^3-4uv^3\\
&\quad-64u^3v^3)\kappa_3^2\}/8
\end{aligned}
$$

$$+ \{uv^2(3u - 12u^2 + 12u^3 - 2v + 32uv - 128u^2 v$$
$$+ 144u^3 v + 4v^2 - 28uv^2 + 48u^2 v^2)\kappa_4\}/24,$$

$$\beta_4 = \{2 + 12u + 3pu + p^2 u - 2pv - p^2 v + 4puv + 3p^2 uv + p^3 uv + 6pv^2$$
$$+ 3p^2 v^2 - 4p^2 uv^2\}/8$$
$$+ \{(4 + p)u(-2u + 2v + pv - 4uv - 4puv$$
$$- 2p^2 uv - 8v^2 - 4pv^2 + 4puv^2)\kappa_3\}/4$$
$$+ \{u^2(16u^2 + 8pu^2 + 26puv + 9p^2 uv + p^3 uv$$
$$+ 64u^2 v + 16v^2 + 10pv^2 + p^2 v^2 + 32uv^2)\kappa_3^2\}/2$$
$$+ \{u^2(-14u - 9pu - p^2 u + 2v - pv - 24uv)\kappa_4\}/2,$$

$$\beta_5 = \{v(u - 2uv - 6v^2 - 3pv^2 + 4puv^2)\}/8$$
$$+ uv(-u^2 - 5uv + 12u^2 v + 2v^2 + pv^2 + 4uv^2 - puv^2)\kappa_3$$
$$+ \{u^2 v(2u^2 - 4uv + 64u^2 v - 128u^3 v - 2v^2 - pv^2 - 32u^2 v^2)\kappa_3^2\}/2$$
$$+ \{u^2 v(-u + 2u^2 + 2v - 14uv + 24u^2 v)\kappa_4\}/2,$$

$$\beta_6 = -\{u^2(1 + pv)\}/8 + \{(4 + p)u^3 v\kappa_3\}/2$$
$$- \{u^4(p + 16u)v\kappa_3^2\}/2 + \{u^4\kappa_4\}/2,$$

$$\beta_7 = \{8u^3 v\kappa_3\}/3 + 8(1 - 4u)u^4 v\kappa_3^2 + \{8u^3(-1 + 3u)v\kappa_4\}/3,$$

$$\beta_8 = -8u^5\kappa_3^2 + 2u^4\kappa_4,$$

$$\beta_9 = \{4 + 3p + p^2 + 4v + 10pv + 5p^2 v + p^3 v - 8pv^2 - 4p^2 v^2\}/8$$
$$+ \{(2p - 12v - 20pv - 7p^2 v - p^3 v + 24v^2 + 32pv^2 + 10p^2 v^2)\kappa_2\}/8$$
$$+ \{(2 + p)v(4 + p - 6v - 3pv)\kappa_2^2\}/4$$
$$+ \{(-4u - 10pu - 2p^2 u - 8v + 10pv + 7p^2 v + p^3 v - 8uv - 24puv$$
$$- 14p^2 uv - 2p^3 uv - 8v^2 - 16pv^2 - 2p^2 v^2 - 32uv^2)\kappa_3\}/8$$
$$+ \{(2 + p)(6 + p)v^2\kappa_2\kappa_3\}/4,$$

$$\beta_{10} = \{v^2(5 - 6u - 2v + 2pv)\}/4 - \{v^2(3 + 4v + 5pv)\kappa_2\}/4$$
$$+ \{3(2 + p)v^3\kappa_2^2\}/4 + \{v^2(1 + 2u - 20u^2 + 32u^3 + pv - 4uv$$
$$+ 16u^2 v)\kappa_3\}/4$$
$$- \{(2 + p)v^3\kappa_2\kappa_3\}/4,$$

$$\beta_{11} = \{-2u + 2v + pv - 4uv - 2puv\}/8$$
$$+ \{u(2u - 2v - pv - 4uv + 2puv + 16u^2 v)\kappa_3\}/4,$$
$$\beta_{12} = (3 - 4u + 2v - pv - 4uv)/8 + \{(-1 + 2v + 2pv)\kappa_2\}/8$$
$$- \{(2 + p)v\kappa_2^2\}/8,$$
$$\beta_{13} = -(uv) + 2u^2(-1 + 4u)v\kappa_3,$$
$$\beta_{14} = -1/2 + 4u^3\kappa_3.$$

The coefficients α_j's are corresponding with a_j's in the theorem 2.1 of Wakaki [7]. If we substitute $\kappa = \kappa_2 - 1$ and $\psi_3 = \kappa_3 - 1$ and make some reduction using $(v - u)/p = uv - v/2$ and $u\kappa_2 = 1/2$, then we obtain $a_j = \alpha_j$, for $j = 2, 3, \ldots, 6$. But $a_1 \neq \alpha_1$. The coefficient a_1 should be corrected as

$$a_1 = -\psi_3\{uv(p^2 + 5p + 2 - 8p^{-1})/2 + v^2(p + 6 + 8p^{-1})/2\} \tag{3.24}$$
$$+ \kappa\{uv(p^2 + p - 2)/2 + v^2(3p + 6)/2 - v(p + 1)/2\}$$
$$- 2uv(p + 1 - 2p^{-1}) - 4v^2 p^{-1}.$$

If the underlying distribution is normal, $\kappa_2 = \kappa_3 = \kappa_4 = 1$ and $u = v = 1/2$. $r_1(Z, Y)$ and $r_2(Z, Y)$ are reduced to

$$r_1(Z, Y) = -(1 + p)/2 \operatorname{tr}(Z) + (1/6) \operatorname{tr}(Z^3), \tag{3.25}$$

$$r_2(Z, Y) = (1/2)\{r_1(Z, Y)\}^2 - p(5 + 9p + 2p^2)/24 \tag{3.26}$$
$$+ (2 + p)/4 \operatorname{tr}(Z^2) - (1/8) \operatorname{tr}(Z^4).$$

If we make a transformation Z to $\{(n - 1)/n\}^{1/2} Z$, we obtain the same result given by Fujikoshi [2] (see also Siotani, Hayakawa and Fujikoshi [5]).

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