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Pianigiani-Yorke measures for non-Hölder continuous potentials

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Abstract. We prove that each non-Hölder continuous potential has a Pianigiani-Yorke measure for a Markovian factor of a given topological Markov chain under some condition. We give a uniqueness condition of the Pianigiani-Yorke measure together with a concrete example which shows the condition is essential. Moreover we give absolutely continuous Pianigiani-Yorke measures for cookie-cutter Cantor sets generated by $\mathscr{C}^1$-maps on $[0, 1]$.

1. Introduction

Pianigiani and Yorke [12] introduced a conditionally invariant probability measure for a $\mathscr{C}^2$-map on a subset of a Euclidean space. The notion of conditionally invariant measure can be set in the context of sub-Markov chains with absorbing states. The probability measure is called a *Pianigiani-Yorke measure*. Lopes and Markarian [9] pointed out that the map is not necessarily in $\mathscr{C}^2$ but in $\mathscr{C}^{1+\gamma}$ for some $\gamma > 0$. More recently Collet, Martínez and Schmitt [6] proved that each Hölder continuous potential has a Pianigiani-Yorke measure for a Markovian factor of a topologically mixing Markov chain.

In this paper, we prove that each *non-Hölder* continuous potential has a Pianigiani-Yorke measure for a Markovian factor of a topologically mixing Markov chain under a weak condition (see Theorem 3.3 (i)). Proofs in this paper are more elementary and clearer than theirs. We refer to the tools in thermodynamic formalism introduced by Bowen [2], [4], Ruelle [13], [14], Keane [7] and Walters [16]. Especially we use g-measure to prove the convergence property (3.11) in Theorem 3.1. We show the uniqueness of the Pianigiani-Yorke measure under a certain condition (see Theorem 3.3 (iii)). We can see that the condition is essential by virtue of Example 2.

We can also construct a Pianigiani-Yorke measure for a Markovian factor which is not necessarily mixing (see Theorem 5.1). Applying Theorem 5.1 to a cookie-cutter map, we give its Pianigiani-Yorke measure, which is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$. Since potentials in

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our framework are not necessarily Hölder continuous, we can treat $\mathscr{C}^1$-map (see Theorem 5.2). As to cookie-cutter maps and Cantor sets, readers are referred to Bedford [1] and Nakata [11].

2. Preliminaries

Let S be a finite set whose cardinality is greater than two. Consider $X = S^{N \cup \{0\}}$ where the topology is given as the infinite product of discrete topology. Let $\sigma : X \to X$ be the shift transformation which is clearly continuous with respect to the topology.

For a *structure matrix* $L = (l_{ij} \in \{0,1\} : i, j \in S)$, put

$$X_L = \{\underline{x} = (x_0 x_1 \cdots) \in X : l_{x_n, x_{n+1}} = 1 \text{ for any } n \in \mathbf{N} \cup \{0\}\}$$

and $\sigma_L : X_L \to X_L$ the action of the left shift on X_L. We call (X_L, σ_L) a *topological Markov chain* with respect to L. Suppose that L is irreducible and aperiodic, namely there exists a positive number q such that all the entries of the matrix L^q are strictly positive. Then we have

$$\sigma_L^{-1} \underline{x} \neq \emptyset \quad \text{for any } \underline{x} \in X_L. \tag{2.1}$$

For a compact subset $Y \subset X$ we denote by $\mathscr{C}(Y)$ the space of continuous real functions, by $\|\cdot\|_Y$ the supremum norm and by $\mathscr{M}(Y)$ the space of probability measures defined on Y.

For a continuous map $\phi : Y \to \mathbf{R}$, we define

$$\begin{aligned} \mathrm{var}_k^Y(\phi) = \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : \underline{x} = (x_0 x_1 \cdots), \underline{y} = (y_0 y_1 \cdots) \in Y, \\ x_i = y_i \text{ for } i = 0, \ldots, k\}. \end{aligned}$$

Set

$$\phi_Y^* = \sum_{k=1}^{\infty} \mathrm{var}_k^Y(\phi). \tag{2.2}$$

A map ϕ is said to be *Hölder continuous* on Y if there exist $c_0 > 0$ and $\theta \in (0,1)$ such that

$$\mathrm{var}_k^Y(\phi) \le c_0 \theta^k \quad \text{for any } k \in \mathbf{N}. \tag{2.3}$$

Note that if ϕ is a Hölder continuous potential on Y then we always have $\phi_Y^* < +\infty$.

Let $\mathscr{L}_L$ be a *Ruelle-Perron-Frobenius operator* acting on $\mathscr{C}(X_L)$ for a continuous map $\phi \in \mathscr{C}(X_L)$, namely

$$\mathscr{L}_L f(\underline{x}) = \sum_{\underline{y} \in X_L, \sigma_L \underline{y} = \underline{x}} e^{\phi(\underline{y})} f(\underline{y}), \qquad \text{for } \underline{x} \in X_L, f \in \mathscr{C}(X_L). \tag{2.4}$$

It is clear that $\mathscr{L}_L$ is a bounded operator.

Bowen and Walters showed that each non-Hölder continuous potential $\phi : X_L \to \mathbf{R}$ which satisfies $\phi^*_{X_L} < +\infty$ has a unique equilibrium state. Especially Walters [15] showed the following theorem with the idea of g-measure.

THEOREM 2.1 (RUELLE'S OPERATOR THEOREM [4] [13] [15, Theorem 3.3]). *Assume that L is irreducible and aperiodic, and $\phi^*_{X_L} < +\infty$. Then there exist uniquely $\alpha_L > 0$, $h_L \in \mathscr{C}(X_L)$ and $\nu_L \in \mathscr{M}(X_L)$ such that*

$$\mathscr{L}_L h_L = \alpha_L h_L, \quad \mathscr{L}_L^* \nu_L = \alpha_L \nu_L, \quad \nu_L(h_L) = 1. \tag{2.5}$$

Moreover $h_L > 0$ on X_L and

$$\lim_{n\to\infty} \|\alpha_L^{-n} \mathscr{L}_L^n f - h_L \nu_L(f)\|_{X_L} = 0 \quad \textit{for any } f \in \mathscr{C}(X_L). \tag{2.6}$$

$\mathscr{L}_L^*$ denotes the adjoint of the operator $\mathscr{L}_L$ defined by $\mathscr{L}_L^*\mu(f) = \mu(\mathscr{L}_L f)$. Note that we identify $\mu(f)$ with $\int_{X_L} f\, d\mu$, especially, identify $\mu(1_D)$ with $\mu(D)$ for any Borel set $D \subset X_L$.

3. Pianigiani-Yorke measures for topological Markov chains

Pianigiani and Yorke [12] defined a conditionally invariant measure for a map T on A in Euclidean space such that T is an expanding $\mathscr{C}^2$-map and TA includes A strictly. If T satisfies some suitable conditions, then there exists a probability measure μ on A, which is called a *Pianigiani-Yorke measure*, satisfying

$$\mu \circ T^{-1} = \alpha\mu \quad \text{for a number } \alpha > 0.$$

The measure is conditionally invariant, i.e. $\alpha = \mu(T^{-1}A)$ and

$$\mu(T^{-1}B | T^{-1}A) = \mu(B) \quad \text{for any Borel set } B \subset A.$$

Collet, Martínez and Schmitt [6] showed each Hölder continuous potential has a Pianigiani-Yorke measure for topological Markov chain. We construct such a measure without the Hölder continuity of the potential under a weak condition.

Now we prepare some terminologies. For a given irreducible and aperiodic structure matrix $L' = (l'_{ij} \in \{0,1\} : i, j \in S)$, let $L = (l_{ij} \in \{0,1\} : i, j \in S)$ be an irreducible and aperiodic structure matrix such that $L \le L'$, i.e. $l_{ij} \le l'_{ij}$ for any $i, j \in S$. $L \le L'$ implies $X_L \subset X_{L'}$.

For $\phi \in \mathscr{C}(X_{L'})$ and the left shift $\sigma_{L'} : X_{L'} \to X_{L'}$, we also define the Ruelle-Perron-Frobenius operator $\mathscr{L}_{L'}$ acting on $\mathscr{C}(X_{L'})$ corresponding to (2.4). If $\phi^*_{X_{L'}} < +\infty$, then we have a unique $\alpha_{L'} > 0$, a unique $h_{L'} \in \mathscr{C}(X_{L'}), h_{L'} > 0$ on $X_{L'}$ and a unique $\nu_{L'} \in \mathscr{M}(X_{L'})$ satisfying (2.5) and (2.6) in Theorem 2.1. Put

$$\underline{X} = \{\underline{x} \in X_{L'} : l_{x_0 x_1} = 1\}.$$

Then $X_L \subset \underline{X} \subset X_{L'}$ and $\underline{X}$ is open and closed in $X_{L'}$. Let $\underline{\sigma} : \underline{X} \to X_{L'}$ be the restriction of $\sigma_{L'}$ to $\underline{X}$. By definition, it is clear that

$$\underline{\sigma} = \sigma_L \quad \text{on } X_L. \tag{3.1}$$

Since L is irreducible, any columns of L are non-zero vectors. Therefore $\underline{\sigma} : \underline{X} \to X_{L'}$ is onto, that is,

$$\underline{\sigma}^{-1}(\underline{x}) \neq \emptyset \quad \text{for any } \underline{x} \in X_{L'}. \tag{3.2}$$

For $\underline{x} \in X_{L'}$, we have

$$\begin{aligned} \underline{\sigma}^{-1}(\underline{x}) &= \{\underline{y} = (y_0 y_1 \cdots) \in X_{L'} : \sigma_{L'}\underline{y} = \underline{x}, l_{y_0,y_1} = 1\} \\ &= \{\underline{y} = y_0\underline{x} : l_{y_0x_0} = 1\} = \sigma_{L'}^{-1}(\underline{x}) \cap \underline{X}. \end{aligned} \tag{3.3}$$

Similarly for $\underline{x} \in X_{L'}$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} \underline{\sigma}^{-n}(\underline{x}) &= \{\underline{y} \in X_{L'} : \sigma_{L'}^n \underline{y} = \underline{x}, l_{y_k,y_{k+1}} = 1, k = 0, 1, 2, \dots, n-1\} \\ &= \{\underline{y} = y_0 \cdots y_{n-1}\underline{x} : l_{y_k y_{k+1}} = 1, k = 0, \dots, n-2, l_{y_{n-1}x_0} = 1\}. \end{aligned}$$

Therefore we obtain

$$X_L = \bigcap_{n=1}^{\infty} \underline{\sigma}^{-n} X_{L'}. \tag{3.4}$$

The operator $\underline{\mathscr{L}}$ on $\mathscr{C}(X_{L'})$ is defined by

$$\underline{\mathscr{L}}f(\underline{x}) = \sum_{\underline{y} \in \underline{\sigma}^{-1}\underline{x}} e^{\phi(\underline{y})}(\Pi_{\underline{X},X_{L'}} f)(\underline{y}) \qquad \text{for } \underline{x} \in X_{L'} \quad \text{and } f \in \mathscr{C}(X_{L'}), \tag{3.5}$$

where the projection $\Pi_{Y,Y'}f$ is the restriction of f from Y' to Y for $Y \subset Y'$. Clearly we have

$$\mathscr{L}_{L'}(f 1_{\underline{X}}) = \underline{\mathscr{L}}f \qquad \text{for } f \in \mathscr{C}(X_{L'}) \tag{3.6}$$

and

$$\mathscr{L}_{L'}\Pi_{L,L'} = \Pi_{L,L'}\underline{\mathscr{L}} \quad \text{on } \mathscr{C}(X_{L'}), \tag{3.7}$$

where $\Pi_{L,L'}$ denotes $\Pi_{X_L,X_{L'}}$. Generally for $f \in \mathscr{C}(X_{L'})$ and for any $n \in \mathbf{N}$, we have

$$\mathscr{L}_{L'}^n(f \cdot 1_{\underline{\sigma}^{-n}X_{L'}}) = \underline{\mathscr{L}}^n f \quad \text{and} \quad \mathscr{L}_L^n \Pi_{L,L'} = \Pi_{L,L'}\underline{\mathscr{L}}^n \quad \text{on } \mathscr{C}(X_{L'}) \tag{3.8}$$

Then we obtain the following theorem. It will be proved in the next section.

THEOREM 3.1. *Assume that a structure matrix L' is irreducible and aperiodic. Let L be a structure matrix with $L \leq L'$. For $\phi \in \mathscr{C}(X_{L'})$, suppose* *that* $\phi^*_{X_{L'}} < +\infty$. *Let* α_L, h_L *and* ν_L *be given by Theorem 2.1. Then there exist uniquely* $\underline{\alpha} > 0$, $\underline{h} \in \mathscr{C}(X_{L'})$ *and* $\underline{\nu} \in \mathscr{M}(X_{L'})$ *such that*

$$\underline{\mathscr{L}}\underline{h} = \underline{\alpha}\underline{h}, \quad \underline{\mathscr{L}}^*\underline{\nu} = \underline{\alpha}\underline{\nu}, \quad \underline{\nu}(\underline{h}) = 1. \tag{3.9}$$

Furthermore we have

$$\underline{h} > 0 \quad \text{on } X_{L'}, \quad \underline{\alpha} = \alpha_L, \quad \Pi_{L,L'}\underline{h} = h_L, \quad \underline{\nu}(X_L) = 1, \quad \underline{\nu}\Pi_{L,L'} = \nu_L \tag{3.10}$$

and

$$\lim_{n\to\infty} \|\alpha_L^{-n}\underline{\mathscr{L}}^n f - \underline{h}\underline{\nu}(f)\|_{X_{L'}} = 0 \quad \textit{for any } f \in \mathscr{C}(X_{L'}). \tag{3.11}$$

As to the assumption of Theorem 3.1, the irreducibility of L is not necessary. In § 5, to deal with cookie-cutter Cantor sets, we extend the last theorem so as to handle non-irreducible structure matrix L under some condition.

We deduce the following corollary similarly to [6].

COROLLARY 3.2. *Let* $\alpha_{L'}$, $h_{L'}$, $\nu_{L'}$ *be given by Theorem* 2.1 for L'. *Then we have* $\alpha_L \le \alpha_{L'}$ *and the following properties for any Borel set* $D \subset X_{L'}$:

$$\underline{h}\nu_{L'}(\underline{\sigma}^{-n}D) = (\alpha_L\alpha_{L'}^{-1})^n\underline{h}\nu_{L'}(D) \quad \textit{for any } n \in \mathbf{N}, \tag{3.12}$$

$$\lim_{n\to\infty} \nu_{L'}(\underline{\sigma}^{-n}D|\underline{\sigma}^{-n}X_{L'}) = \nu_{L'}(1_D\underline{h})/\nu_{L'}(\underline{h}),$$

$$\lim_{n\to\infty} \nu_{L'}(D|\underline{\sigma}^{-n}X_{L'}) = \underline{\nu}(D).$$

Now we have the following theorem, whose proof will be given in the next section.

THEOREM 3.3. *Assume the conditions in Theorem* 3.1.
(i) $\mu_{PY} = \underline{h}\nu_{L'}/\nu_{L'}(\underline{h}) \in \mathscr{M}(X_{L'})$ *is a Pianigiani-Yorke measure on* $X_{L'}$.
(ii) *Let* $m \in \mathscr{M}(X_{L'})$ *be*

$$dm = F\,d\nu_{L'} \qquad \text{for } F \in \mathscr{C}(X_{L'}). \tag{3.13}$$

Then m *is a Pianigiani-Yorke measure if and only if*

$$\underline{\mathscr{L}}F = \beta F \quad \textit{with some } \beta > 0. \tag{3.14}$$

(iii) *Suppose that* m *is a Pianigiani-Yorke measure given by* (3.13) *and*

$$\underline{\nu}(F) > 0. \tag{3.15}$$

Then $m = \mu_{PY}$, *that is*, $\underline{h} = F/\underline{\nu}(F)$.

We can give an example with distinct Pianigiani-Yorke measures which are absolutely continuous with respect to $\nu_{L'}$, if (3.15) in Theorem 3.3 is not required (see §5 Example 2). Therefore (3.15) is essential for the uniqueness. Now we get the following proposition.

PROPOSITION 3.4. *Assume the conditions in Theorem* 3.3. *Then we have*

$$\lim_{n\to\infty} \mu_{PY}(D|\underline{\sigma}^{-n}X_{L'}) = \underline{\nu}(\underline{h}1_D) \quad \textit{for any Borel set } D \subset X_{L'}. \tag{3.16}$$

PROOF. By (3.8) and (3.11), we get the following:

$$\mu_{PY}(D|\underline{\sigma}^{-n}X_{L'}) = \frac{\underline{h}\nu_{L'}(D\cap\underline{\sigma}^{-n}X_{L'})}{\underline{h}\nu_{L'}(\underline{\sigma}^{-n}X_{L'})} = \frac{\nu_{L'}((\alpha_L^{-n}\underline{\mathscr{L}}^n)(\underline{h}1_D))}{\nu_{L'}((\alpha_L^{-n}\underline{\mathscr{L}}^n)\underline{h})}$$

$$\to \frac{\nu_{L'}(\underline{h}\cdot\underline{\nu}(\underline{h}1_D))}{\nu_{L'}(\underline{h}\cdot\underline{\nu}(\underline{h}))} = \frac{\underline{\nu}(\underline{h}1_D)}{\underline{\nu}(\underline{h})} = \underline{\nu}(\underline{h}1_D) \quad \text{as } n\to\infty. \quad \blacksquare$$

We are very interested in the case where (3.15) is not satisfied. For any Pianigiani-Yorke measures m which satisfies (3.13), we do not know the validity of

$$\lim_{n\to\infty} m(D|\underline{\sigma}^{-n}X_{L'}) = \underline{\nu}(F1_D) \quad \text{for any Borel set } D\subset X_{L'}, \tag{3.17}$$

when $\underline{\nu}(F)=0$. We can not apply the above proof for m. Even if $\underline{\nu}(F)=0$, we can give an example for which (3.17) holds (see Example 2).

Using Theorem 3.1, Collet, Martínez and Schmitt gave a simple example of $\phi=0$ on $X_{L'}$. Now we give a simple example with a non-zero potential.

EXAMPLE 1. Let (p_0,p_1) be a positive stochastic vector, that is, $p_0p_1>0$ and $p_0+p_1=1$. Put $\phi(\underline{x})=\phi(x_0x_1\cdots)=\log p_{x_0}$,

$$L'=\begin{pmatrix}1&1\\1&1\end{pmatrix} \quad\text{and}\quad L=\begin{pmatrix}1&1\\1&0\end{pmatrix}.$$

Then $\alpha_{L'}=1, h_{L'}=1$ on $X_{L'}$ and $\nu_{L'}$ is the (p_0,p_1)-Bernoulli measure. We have

$$\alpha_L=\frac{p_0+\sqrt{p_0^2+4p_0p_1}}{2}, \quad h_L(\underline{x})=\begin{cases}(\alpha_L+2p_1)^{-1}(\alpha_L+p_1)(\alpha_L/p_0) & \text{if } x_0=0,\\ (\alpha_L+2p_1)^{-1}(\alpha_L+p_1) & \text{if } x_0=1,\end{cases}$$

for $\underline{x}\in X_L$, and

$$\nu_L([i_0\cdots i_n]_{X_L})=\begin{cases}(p_{i_0}\cdots p_{i_{n-1}}\alpha_L^{-n})(p_0\alpha_L^{-1}) & \text{if } i_n=0,\\ (p_{i_0}\cdots p_{i_{n-1}}\alpha_L^{-n})(p_0p_1\alpha_L^{-2}) & \text{if } i_n=1,\end{cases}$$

where $[i_0\cdots i_n]_Y=\{\underline{y}=(y_0y_1\cdots)\in Y: y_0=i_0,\dots,y_n=i_n\}$. Note that ν_L is the Markov measure whose initial distribution is $(p_0\alpha_L^{-1}, p_0p_1\alpha_L^{-2})$ and transition matrix is $\begin{pmatrix}p_0\alpha_L^{-1} & p_0p_1\alpha_L^{-2}\\ 1 & 0\end{pmatrix}$. The function $\underline{h}$ agrees with h_L on $X_{L'}$. Therefore a Pianigiani-Yorke measure $\mu_{PY} = \underline{h}\nu_{L'}/\nu_{L'}(\underline{h})$ is the following:

$$\mu_{PY}([j_0\cdots j_n]_{X_{L'}}) = \begin{cases} \alpha_L(\alpha_L+p_1)^{-1}p_{j_1}\cdots p_{j_n} & \text{if } j_0=0, \\ p_1(\alpha_L+p_1)^{-1}p_{j_1}\cdots p_{j_n} & \text{if } j_0=1. \end{cases}$$

Note that the measure is the (p_0, p_1)-Bernoulli measure whose initial distribution is $(\alpha_L(\alpha_L+p_1)^{-1}, p_1(\alpha_L+p_1)^{-1})$.

4. Proofs of theorems

To prove Theorem 3.1, we prepare Lemma 4.1, Lemma 4.2 and Lemma 4.4.

LEMMA 4.1. *There exists* $\underline{\nu} \in \mathcal{M}(X_{L'})$ *such that* $\underline{\mathcal{L}}^*\underline{\nu} = \underline{\alpha}\underline{\nu}$, *where* $\underline{\alpha} = \underline{\mathcal{L}}^*\underline{\nu}(1) > 0$. *Moreover we have* $\underline{\nu}(X_L) = 1$. *Let* $\underline{\nu}' \in \mathcal{M}(X_L)$ *be a measure which satisfies*

$$\underline{\nu}'\Pi_{L,L'} = \underline{\nu}. \tag{4.1}$$

Then

$$\mathcal{L}_L^*\underline{\nu}' = \underline{\alpha}\underline{\nu}'. \tag{4.2}$$

PROOF. For $\mu \in \mathcal{M}(X_{L'})$, we have $\underline{\mathcal{L}}^*\mu(1) > 0$ by (3.2). It is well-known that $\mathcal{M}(X_{L'})$ is compact and convex in the weak-$*$-topology. Put $F(\mu) = \frac{\underline{\mathcal{L}}^*\mu}{\underline{\mathcal{L}}^*\mu(1)}$. Then $F : \mathcal{M}(X_{L'}) \to \mathcal{M}(X_{L'})$ is continuous in the topology. Using the Schauder-Tychonoff fixed point theorem, there exists a fixed point $\underline{\nu} \in \mathcal{M}(X_{L'})$ of F. Set $\underline{\alpha} = \underline{\mathcal{L}}^*\underline{\nu}(1) > 0$. Then we obtain

$$\underline{\mathcal{L}}^*\underline{\nu} = \underline{\alpha}\underline{\nu}. \tag{4.3}$$

Since $\underline{\sigma}^{-1}\underline{x} \subset \underline{X}$ and $\underline{\sigma}^{-1}\underline{x} \subset \underline{\sigma}^{-n}X_{L'}$ for any $\underline{x} \in X_{L'}$ and $n \in \mathbf{N}$, we get

$$\underline{\mathcal{L}}(1_{X_{L'}\setminus\underline{\sigma}^{-n}X_{L'}}) = \sum_{\underline{y}\in\underline{\sigma}^{-1}\underline{x}} e^{\phi(\underline{y})}(\Pi_{\underline{X},X_{L'}}1_{X_{L'}\setminus\underline{\sigma}^{-n}X_{L'}})(\underline{y}) = 0 \quad \text{for any } \underline{x} \in X_{L'}.$$

Therefore we have

$$\underline{\alpha}\underline{\nu}(1_{X_{L'}\setminus\underline{\sigma}^{-n}X_{L'}}) = \underline{\mathcal{L}}^*\underline{\nu}(1_{X_{L'}\setminus\underline{\sigma}^{-n}X_{L'}}) = \underline{\nu}(\underline{\mathcal{L}}(1_{X_{L'}\setminus\underline{\sigma}^{-n}X_{L'}})) = 0 \qquad \text{for } n \in \mathbf{N}.$$

Hence we obtain $\underline{\nu}(\underline{\sigma}^{-n}X_{L'}) = 1$ for any $n \in \mathbf{N}$. That is, by (3.4),

$$\underline{\nu}(X_L) = \underline{\nu}\left(\bigcap_{n=1}^{\infty} \underline{\sigma}^{-n}X_{L'}\right) = \lim_{n\to\infty} \underline{\nu}(\underline{\sigma}^{-n}X_{L'}) = 1.$$

By (3.7), (4.1) and (4.3), we have

$$\begin{aligned}\mathscr{L}_L^* \underline{\nu}' \Pi_{L,L'} &= \underline{\nu}' \mathscr{L}_L \Pi_{L,L'} = \underline{\nu}' \Pi_{L,L'} \underline{\mathscr{L}} \\ &= \underline{\nu \mathscr{L}} = \underline{\mathscr{L}}^* \underline{\nu} = \underline{\alpha \nu} = \underline{\alpha \nu}' \Pi_{L,L'} \quad \text{on } \mathscr{C}(X_{L'}).\end{aligned}$$

Hence we deduce (4.2). ■

LEMMA 4.2. *Let $\underline{\alpha}$, $\underline{\nu}$ and $\underline{\nu}'$ be given by Lemma 4.1. Then there exists $\underline{h} \in \mathscr{C}(X_{L'})$ such that*

$$\underline{\mathscr{L} h} = \underline{\alpha h}, \quad \underline{\nu}(\underline{h}) = 1. \tag{4.4}$$

Moreover if $\underline{h} \in \mathscr{C}(X_{L'})$ satisfies (4.4), *then*

$$\underline{h} > 0 \quad on\ X_{L'}, \quad \Pi_{L,L'} \underline{h} = h_L, \quad \underline{\nu}' = \nu_L \quad and\ \underline{\alpha} = \alpha_L. \tag{4.5}$$

PROOF. This proof is an adaptation of that of Bowen [2, Theorem 1.7].

We prepare some terminologies. Put $B_m = \exp\left[\sum_{k=m+1}^{\infty} \operatorname{var}_k^{X_{L'}}(\phi)\right]$ and for $\underline{x} = (x_0 x_1 x_2 \cdots)$, $\underline{x}' = (x_0' x_1' x_2' \cdots) \in X_{L'}$,

$$\Lambda = \{f \in \mathscr{C}(X_{L'}) : f \geq 0, \underline{\nu}(f) = 1, f(\underline{x}) \leq B_m f(\underline{x}') \text{ if } x_i = x_i' \text{ for } i = 0, \dots, m\}.$$

Obviously we have $1 \in \Lambda$, so that $\Lambda \neq \emptyset$. Now we prove that there exists $\underline{h} \in \Lambda$ which satisfies (3.9). By (3.2), we can use the Bowen's method with respect to B_m and Λ, so that we get

$$\underline{\alpha}^{-1} \underline{\mathscr{L}} : \Lambda \to \Lambda \tag{4.6}$$

and that Λ is uniformly bounded and equicontinuous. Hence by the Ascoli-Arzela theorem, Λ is compact. By definition, Λ is convex. Since the operator $\underline{\alpha}^{-1} \underline{\mathscr{L}}$ in (4.6) is clearly continuous on Λ, there exists a fixed point $\underline{h} \in \Lambda$ thanks to the Schauder-Tychonoff fixed point theorem. Therefore

$$\underline{\mathscr{L} h} = \underline{\alpha h}. \tag{4.7}$$

By a similar argument to Bowen's proof, we deduce $\inf\{\underline{h}(\underline{x}) : \underline{x} \in X_{L'}\} > 0$. By (3.7) and (4.7), we have

$$\mathscr{L}_L(\Pi_{L,L'} \underline{h}) = \Pi_{L,L'}(\underline{\mathscr{L} h}) = \underline{\alpha}(\Pi_{L,L'} \underline{h}). \tag{4.8}$$

By (4.1), we obtain

$$\underline{\nu}'(\Pi_{L,L'} \underline{h}) = \underline{\nu}(\underline{h}) = 1. \tag{4.9}$$

Hence by (4.2), (4.8) and (4.9), the uniqueness of h_L, ν_L and α_L in Theorem 2.1 implies $\Pi_{L,L'} \underline{h} = h_L$, $\underline{\nu}' = \nu_L$ and $\underline{\alpha} = \alpha_L$. ■

To prove (3.11), we prepare the theory of g-measure studied by Keane [7]. Set

$$G = \left\{ g \in \mathscr{C}(X_{L'}) : g(\underline{x}) > 0, \sum_{\underline{y} \in \underline{\sigma}^{-1}\underline{x}} g(\underline{y}) = 1 \quad \text{for any } \underline{x} \in X_{L'} \right\}.$$

For $g \in G$, let $\underline{\mathscr{L}}_{\log g} : \mathscr{C}(X_{L'}) \to \mathscr{C}(X_{L'})$ be an operator such that

$$\underline{\mathscr{L}}_{\log g} f(\underline{x}) = \sum_{\underline{y} \in \underline{\sigma}^{-1}\underline{x}} g(\underline{y}) f(\underline{y}) \quad \text{for } f \in \mathscr{C}(X_{L'}).$$

Then a probability measure $\mu \in \mathscr{M}(X_{L'})$ which satisfies $\underline{\mathscr{L}}^*_{\log g}\mu = \mu$ is called *g-measure*. Using g-measure, we claim the following lemma to prove Lemma 4.4.

LEMMA 4.3. *For $g \in G$, suppose that the sum of the variation of* $\log g$ *is finite, that is,* $(\log g)^*_{X_{L'}} < +\infty$. *Then $\underline{\mathscr{L}}^n_{\log g} f$ converges uniformly to a constant $\mu(f)$ for each $f \in \mathscr{C}(X_{L'})$. Moreover μ is a g-measure, i.e.,*

$$\underline{\mathscr{L}}^*_{\log g}\mu = \mu. \tag{4.10}$$

PROOF. Firstly, we mention that for any $f \in \mathscr{C}(X_{L'})$, we have $m_{L'}(f) \le m_{L'}(\underline{\mathscr{L}}_{\log g} f)$ by (3.2), where $m_{L'}(f) = \min\{f(\underline{x}) : \underline{x} \in X_{L'}\}$. Moreover if $\{\underline{\mathscr{L}}^n_{\log g} f\}_{n=0}^{\infty}$ has a limit point f_*, then

$$m_{L'}(f) \le m_{L'}(\underline{\mathscr{L}}^n_{\log g} f) \le m_{L'}(f_*) \quad \text{for any } n \in \mathbf{N}. \tag{4.11}$$

Similarly to the argument of [15, Theorem 3.1], we can prove that $\{\underline{\mathscr{L}}^n_{\log g} f\}_{n=0}^{\infty}$ is uniformly bounded and is an equicontinuous subset of $\mathscr{C}(X_{L'})$ for a fixed $f \in \mathscr{C}(X_{L'})$. By the Ascoli-Arzela theorem, there exists $f_* \in \mathscr{C}(X_{L'})$ and subsequence $\{n_i\}_{i=1,2,\dots}$ such that $n_i \to +\infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} \|\underline{\mathscr{L}}^{n_i}_{\log g} f - f_*\|_{X_{L'}} = 0 \qquad \text{for } f \in \mathscr{C}(X_{L'}). \tag{4.12}$$

Now we show that f_* is a constant. We may assume that the sequence of (4.12) satisfies $n_i > 2n_{i-1}$ for any $i \in \mathbf{N}$. Then

$$\begin{aligned}
\|\underline{\mathscr{L}}^{n_i-n_{i-1}}_{\log g} f_* - f_*\|_{X_{L'}} &= \|(\underline{\mathscr{L}}^{n_i-n_{i-1}}_{\log g} f_* - \underline{\mathscr{L}}^{n_i}_{\log g} f) + (\underline{\mathscr{L}}^{n_i}_{\log g} f - f_*)\|_{X_{L'}} \\
&\le \|\underline{\mathscr{L}}^{n_i-n_{i-1}}_{\log g}(\underline{\mathscr{L}}^{n_{i-1}}_{\log g} f - f_*)\|_{X_{L'}} + \|\underline{\mathscr{L}}^{n_i}_{\log g} f - f_*\|_{X_{L'}} \\
&\le \|\underline{\mathscr{L}}^{n_{i-1}}_{\log g} f - f_*\|_{X_{L'}} + \|\underline{\mathscr{L}}^{n_i}_{\log g} f - f_*\|_{X_{L'}} \to 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Since $\lim_{i\to\infty} n_i - n_{i-1} = +\infty$, we deduce that f_* is a limit point of $\{\underline{\mathscr{L}}^n_{\log g} f_*\}_{n=0}^{\infty}$. Hence by (4.11), we have $m_{L'}(f_*) = m_{L'}(\underline{\mathscr{L}}^n_{\log g} f_*)$ for any

$n \in \mathbf{N}$. Now put $\underline{x}^n \in X_{L'}$ which satisfies

$$\mathscr{L}^n_{\log g} f_*(\underline{x}^n) = m_{L'}(f_*) \quad \text{for any } n \in \mathbf{N}. \tag{4.13}$$

Then by the definition of $\mathscr{L}_{\log g}$, we have for any $n \in \mathbf{N}$,

$$f_*(\underline{y}) = m_{L'}(f_*) \qquad \text{for } \underline{y} \in \underline{\sigma}^{-n}\underline{x}^n. \tag{4.14}$$

For $f \in \mathscr{C}(X_{L'})$, we put $m_L(f) = \min\{f(\underline{x}) : \underline{x} \in X_L\}$. Using the analogous argument of $m_{L'}(f)$, there exists $\underline{x}^m_L \in X_L$ such that $f_*(\underline{z}) = m_L(f_*)$ for $\underline{z} \in \underline{\sigma}^{-m}\underline{x}^m_L$ for any $m \in \mathbf{N}$. By (3.1), $\underline{\sigma}$ is topologically mixing on X_L, so that any cylinder set contains a point where f_* attains its minimum on X_L. Therefore

$$f_*(\underline{x}) = m_L(f_*) \quad \text{for any } \underline{x} \in X_L. \tag{4.15}$$

Since $\mathscr{L}^{n_i}_{\log g} f$ uniformly converges on $X_{L'}$, f_* is continuous in $X_{L'}$. By (4.15) and $f_* \in \mathscr{C}(X_{L'})$, for $\underline{x} \in X_L$ and for any $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that if $\underline{x}' \in X_{L'}$ and $x_i = x'_i$ for $0 \le i \le N$ then

$$|f_*(\underline{x}) - f_*(\underline{x}')| = |m_L(f_*) - f_*(\underline{x}')| < \varepsilon. \tag{4.16}$$

Clearly we have $m_L(f_*) \ge m_{L'}(f_*)$. Now we assume

$$m_L(f_*) > m_{L'}(f_*). \tag{4.17}$$

By (4.14), there exists $\underline{x}^{N+q} \in X_{L'} \backslash X_L$ such that

$$f_*(\underline{y}) = m_{L'}(f_*) \qquad \text{for } \underline{y} \in \underline{\sigma}^{-N-q}\underline{x}^{N+q} \not\subset X_L. \tag{4.18}$$

In fact, if $\underline{x}^{N+q} \in X_L$ then $\underline{\sigma}^{-N-q}\underline{x}^{N+q} \subset X_L$, so that we have $m_{L'}(f_*) = m_L(f_*)$ by (4.15). It contradicts (4.17). Therefore $\underline{x}^{N+q} \in X_{L'} \backslash X_L$. For $\underline{x} = (x_0 x_1 \cdots) \in X_L$, if we choose $\underline{y} = (y_0 y_1 \cdots) \in \underline{\sigma}^{-N-q}\underline{x}^{N+q} \not\subset X_L$ such that $y_i = x_i$ for $0 \le i \le N$, then $|f_*(\underline{x}) - f_*(\underline{y})| = |m_L(f_*) - f_*(\underline{y})| = |m_L(f_*) - m_{L'}(f_*)| < \varepsilon$ by (4.16). It contradicts (4.17). Hence $m_L(f_*) = m_{L'}(f_*)$. Using the same argument, we deduce $\max\{f_*(\underline{x}) : \underline{x} \in X_L\} = \max\{f_*(\underline{x}) : \underline{x} \in X_{L'}\}$. Therefore by (4.15), we claim that f_* is constant on $X_{L'}$. Since we can get $\mathscr{L}_{\log g} f_*(\underline{x}) = f_*$, we have $\lim_{n\to\infty} \|\mathscr{L}^n_{\log g} f - f_*\|_{X_{L'}} = 0$ for $f \in \mathscr{C}(X_{L'})$.

Set $\mu(f) = f_*$. Then by the Riez representation theorem μ is a probability measure on $X_{L'}$. Clearly we claim $\mathscr{L}^*_{\log g}\mu = \mu$. So the measure is only one. In fact, if a probability measure $\mu' \in \mathscr{M}(X_{L'})$ satisfies $\mathscr{L}^*_{\log g}\mu' = \mu'$, then

$$\mu'(f) = \lim_{n\to\infty} \mu'(\mathscr{L}^n_{\log g} f) = \mu'\Big(\lim_{n\to\infty} \mathscr{L}^n_{\log g} f\Big) = f_* = \mu(f) \quad \text{for any } f \in \mathscr{C}(X_{L'}),$$

by the Lebesgue convergence theorem. ■

LEMMA 4.4. *Suppose that* $\underline{\alpha} > 0$, $\underline{h} \in \mathscr{C}(X_{L'})$ *and* $\underline{\nu} \in \mathscr{M}(X_{L'})$ *satisfy* (3.9). *Then*

$$\lim_{n\to\infty} \|\underline{\alpha}^{-n}\underline{\mathscr{L}}^n f - \underline{h}\underline{\nu}(f)\|_{X_{L'}} = 0 \qquad \textit{for any } f \in \mathscr{C}(X_{L'}). \tag{4.19}$$

PROOF. Here we give $g \in G$ as follows:

$$g(\underline{x}) = \frac{e^{\phi(\underline{x})}\underline{h}(\underline{x})}{\underline{\alpha}\underline{h}(\underline{\sigma}(\underline{x}))} \in \mathscr{C}(X_{L'}).$$

Then we claim $g(\underline{x}) > 0$ and $\sum_{\underline{y}\in\underline{\sigma}^{-1}\underline{x}} g(\underline{y}) = 1$ for $\underline{x} \in X_{L'}$. We also have $\underline{h}(\underline{x})\underline{\mathscr{L}}^n_{\log g}(f/\underline{h})(\underline{x}) = \underline{\alpha}^{-n}\underline{\mathscr{L}}^n f(\underline{x})$ for any $n \in \mathbf{N}$. By a similar argument to [15, P. 384], the function g satisfies the condition $(\log g)^*_{X_{L'}} < +\infty$. By Lemma 4.3, we have

$$\lim_{n\to\infty} \|\underline{\alpha}^{-n}\underline{\mathscr{L}}^n f - \underline{h}\mu(f/\underline{h})\|_{X_{L'}} = 0, \tag{4.20}$$

where μ is a unique g-measure for $f \in \mathscr{C}(X_{L'})$. However the measure $\underline{h}\underline{\nu}$, say $\underline{\mu}$, is a g-measure, because it is a probability measure and

$$\underline{\mu}(\underline{\mathscr{L}}_{\log g} f(\underline{x})) = \underline{\nu}\left(\underline{h}(\underline{x}) \sum_{\underline{y}\in\underline{\sigma}^{-1}\underline{x}} \frac{e^{\phi(\underline{y})}\underline{h}(\underline{y})}{\underline{\alpha}\underline{h}(\underline{\sigma}(\underline{y}))} f(\underline{y})\right) = \underline{\alpha}^{-1}\underline{\nu}\left(\sum_{\underline{y}\in\sigma^{-1}\underline{x}} e^{\phi(\underline{y})}\underline{h}(\underline{y})f(\underline{y})\right)$$

$$= \underline{\alpha}^{-1}\underline{\mathscr{L}}^*\underline{\nu}(\underline{h}f) = \underline{\nu}(\underline{h}f) = \underline{\mu}(f) \quad \text{for any } f \in \mathscr{C}(X_{L'}).$$

Therefore by the uniqueness of g-measure, we get $\mu(f/\underline{h}) = \underline{\mu}(f/\underline{h}) = \underline{\nu}(\underline{h}f/\underline{h}) = \underline{\nu}(f)$ for $f \in \mathscr{C}(X_{L'})$. Hence by (4.20), we obtain (4.19). ■

PROOF OF THEOREM 3.1. By Lemma 4.2 and Lemma 4.4, we have (3.9) and (3.11) respectively. Since $\underline{\mathscr{L}}^*\underline{\nu} = \underline{\alpha}\underline{\nu}$, we have $\underline{\alpha} = \alpha_L$ by Lemma 4.2. Suppose that $\hat{\underline{\alpha}} \in \mathbf{R}$, $\hat{\underline{h}} \in \mathscr{C}(X_{L'})$ and $\hat{\underline{\nu}} \in \mathscr{M}(X_{L'})$ satisfy (3.9). Then we have $\hat{\underline{\alpha}} = \alpha_L$, too. Applying (3.11) to both $(\hat{\underline{\alpha}}, \hat{\underline{h}}, \hat{\underline{\nu}})$ and $(\underline{\alpha}, \underline{h}, \underline{\nu})$, we have $\lim_{n\to\infty} \underline{\mathscr{L}}^n 1 = \underline{h} = \hat{\underline{h}}$. Again by (3.11), $\lim_{n\to\infty} \underline{\mathscr{L}} f = \underline{h}\underline{\nu}(f) = \hat{\underline{h}}\hat{\underline{\nu}}(f) = \underline{h}\hat{\underline{\nu}}(f)$. Therefore we have $\underline{\nu} = \hat{\underline{\nu}}$. The rest of Theorem 3.1 follows from Lemma 4.2. ■

PROOF OF THEOREM 3.3. (i) Put $\mu_{PY}(D) = \frac{\underline{h}\nu_{L'}(1_D)}{\underline{h}\nu_{L'}(1)} = \frac{\nu_{L'}(\underline{h}1_D)}{\nu_{L'}(\underline{h})}$ and $\alpha = \alpha_L\alpha_{L'}^{-1} > 0$. Then μ_{PY} is a probability measure on $X_{L'}$ and by (3.12) of $n = 1$, we have

$$\mu_{PY}(\underline{\sigma}^{-1}D) = \frac{\nu_{L'}(\underline{h}1_{\underline{\sigma}^{-1}D})}{\nu_{L'}(\underline{h})} = \frac{\alpha_L}{\alpha_{L'}}\frac{\nu_{L'}(\underline{h}1_D)}{\nu_{L'}(\underline{h})} = \alpha\mu_{PY}(D)$$

for any Borel set $D \subset X_{L'}$.

PROOF OF THEOREM 3.3. (ii) It is clear that

$$\nu_{L'}(f\cdot(g\circ\sigma_{L'}))=\nu_{L'}((\alpha_{L'}^{-1}\mathscr{L}_{L'}f)\cdot g)\quad\text{for any } f,g\in\mathscr{C}(X_{L'}). \tag{4.21}$$

If m is a Pianigiani-Yorke measure, then

$$m(\underline{\sigma}^{-1}D)=\beta' m(D)\quad\text{for some } \beta'>0\quad\text{and any Borel set } D\subset X_{L'} \tag{4.22}$$

and $\nu_{L'}(F)=1$. By (3.13), and (4.22), we have $\nu_{L'}(F1_{\underline{\sigma}^{-1}D})=\beta'\nu_{L'}(F1_D)$ for any Borel set $D\subset X_{L'}$. By (3.3), (3.6) and (4.21), we deduce

$$\begin{aligned}\nu_{L'}((\beta' F)1_D)&=\nu_{L'}(F1_{\underline{\sigma}^{-1}D})=\nu_{L'}(F1_{\underline{X}}1_D\circ\sigma_{L'})=\nu_{L'}(\alpha_{L'}^{-1}\mathscr{L}_{L'}(F1_{\underline{X}})1_D)\\&=\nu_{L'}((\alpha_{L'}^{-1}\underline{\mathscr{L}}F)1_D).\end{aligned} \tag{4.23}$$

Set $\beta=\alpha_{L'}\beta'$. Then we have (3.14). It is clear that if (3.14) holds then m is a Pianigiani-Yorke measure.

PROOF OF THEOREM 3.3. (iii) Since m is a Pianigiani-Yorke measure, we have $\alpha_{L'}^{-1}\underline{\mathscr{L}}F=\beta' F$ by (4.23). Recalling (3.15), we set $\tilde{F}=F/\underline{\nu}(F)$. Then we get $\underline{\nu}(\tilde{F})=1$ and $\underline{\mathscr{L}}\tilde{F}=(\alpha_{L'}\beta')\tilde{F}$. By the uniqueness of $\underline{h}$ and $\underline{\nu}$ in (3.9), we deduce $\beta'=\alpha_L\alpha_{L'}^{-1}$ and $\tilde{F}=\underline{h}$. This completes the proof of Theorem 3.3. ■

5. Pianigiani-Yorke measures for cookie-cutter Cantor sets

We wish to investigate Pianigiani-Yorke measures for cookie-cutter Cantor sets. Especially we are interested in the absolutely continuous Pianigiani-Yorke measures with respect to the Lebesgue measure for the set generating by $\mathscr{C}^1$-maps on $[0,1]$. Since we can not directly apply Theorem 3.1 to cookie-cutter sets, we prepare a useful theorem.

We deal with a special type of a non-irreducible matrix. For $1\le k\le |S|-2$, put

$$L=\left(\begin{array}{c|c} O_{k,k} & O_{k,|S|-k}\\ \hline Q_{|S|-k,k} & \tilde{L}\end{array}\right), \tag{5.1}$$

where $O_{p,q}$ is the $p\times q$ zero matrix, $\tilde{L}$ is an $(|S|-k)\times(|S|-k)$ structure matrix and $Q_{|S|-k,k}$ is an $(|S|-k)\times k$ matrix whose components are 0 or 1 and whose columns are non-zero vectors. It is clear that L is not irreducible. Now we suppose that $\tilde{L}$ is irreducible and aperiodic. Obviously $(X_{\tilde{L}},\sigma_{\tilde{L}},\mathscr{L}_{\tilde{L}})$ is identified with $(X_L,\sigma_L,\mathscr{L}_L)$. For a given potential, we apply Theorem 2.1 to $(X_{\tilde{L}},\sigma_{\tilde{L}})$. Therefore there exist $\alpha_L,h_L>0$ and ν_L which satisfy (2.5). Under the situation, we claim the following theorem.

THEOREM 5.1. *Let L' be an irreducible and aperiodic structure matrix. Suppose that L of* (5.1) *with an irreducible and aperiodic matrix $\tilde{L}$ satisfies*

$L \leq L'$. *For* $\phi \in \mathscr{C}(X_{L'})$, *assume* $\phi^*_{X_{L'}} < +\infty$. *Then we obtain the same results as Theorem* 3.1.

The proof of this theorem is similar to the proof of Theorem 3.1. However in Lemma 4.2, we must deal with $\mathscr{L}\Lambda = \{\mathscr{L}f : f \in \Lambda\}$ instead of Λ. Note that the irreducibility of L is not necessary. To prove Theorem 5.1, we have to prepare α_L, h_L and ν_L which satisfy (2.5). We also need (3.2). Since any columns of L are non-zero vectors, we have (3.2). Theorem 5.1 implies the same claims as Corollary 3.2 and Corollary 3.3. Using Theorem 5.1, we prepare an effective example for constructing Pianigiani-Yorke measures for cookie-cutter sets.

EXAMPLE 2. Set $N = |S| - 1$. Let $(p_0, \dots, p_N)$ be a positive stochastic vector, that is $\sum_{i=0}^N p_i = 1$ and $p_i > 0$ for $i = 0, \dots, N$. Put $\phi(\underline{x}) = \phi(x_0 x_1 \cdots) = \log p_{x_0}$ for $\underline{x} \in X_{L'}$ and

$$L' = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad L = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{array}\right). \tag{5.2}$$

Then

$$\alpha_{L'} = 1, \quad h_{L'} = 1 \text{ on } X_{L'}, \quad \nu_{L'} \text{ is the } (p_0, \dots, p_N)\text{-Bernoulli measure},$$

$$\alpha_L = 1 - p_0, \quad h_L = 1 \text{ on } X_L, \quad \nu_L \text{ is the } \left(0, \tfrac{p_1}{1-p_0}, \dots, \tfrac{p_N}{1-p_0}\right)\text{-Bernoulli measure},$$

and $\underline{h} = 1$ on $X_{L'}$. Therefore $\mu_{PY} = \underline{h}\nu_{L'}/\nu_{L'}(\underline{h}) = \nu_{L'}$ is a Pianigiani-Yorke measure.

We claim that Pianigiani-Yorke measure is not unique in the class of continuous densities with respect to $\nu_{L'}$ (see Theorem 3.3). For any Pianigiani-Yorke measure $m \in \mathscr{M}(X_{L'})$ with continuous density $F \in \mathscr{C}(X_{L'})$, i.e., $m(\underline{\sigma}^{-1}D) = \beta m(D)$ for any Borel set $D \subset X_{L'}$, we can give another Pianigiani-Yorke measure. For $0 < \gamma < 1$, define

$$\rho_\gamma(\underline{x}) = \rho_\gamma((x_0 x_1 \cdots)) = \begin{cases} \gamma^{\min\{i \geq 0 : x_i = 0\}} & \text{if } \underline{x} \in X_{L'} \backslash X_L, \\ 0 & \text{if } \underline{x} \in X_L. \end{cases}$$

Set $F_\gamma = \rho_\gamma F / \nu_{L'}(\rho_\gamma F)$ and $dm_\gamma = F_\gamma d\nu_{L'}$. Obviously we have $\rho_\gamma, F_\gamma \in \mathscr{C}(X_{L'})$. Note that $\mathscr{L}F_\gamma = (\gamma\beta)F_\gamma$. By Theorem 3.3 (ii), m_γ is another Pianigiani-Yorke measure.

Since $\nu_{L'}$ is a Pianigiani-Yorke measure in this case, $m_\gamma = \rho_\gamma \nu_{L'}/\nu_{L'}(\rho_\gamma)$ is also a Pianigiani-Yorke measure. By elementary calculus, we have

$$\lim_{n\to\infty} m_\gamma(D|\underline{\sigma}^{-n}X_{L'}) = \underline{\nu}(D) \quad \text{for any Borel set } D \subset X_{L'}, \tag{5.3}$$

nevertheless we can not use the proof of Proposition 3.4 because of $\underline{\nu}(F) = 0$.

Using the argument of Example 2, we construct Pianigiani-Yorke measures of cookie-cutter sets on $I = [0,1]$ for $\mathscr{C}^1$-maps.

Divide I into $0 = x_0 < x_1 < \cdots < x_m = 1$ for $m \geq 3$. Put $I_i = [x_i, x_{i+1})$ for $i = 0, \dots, m-2$ and $I_{m-1} = [x_{m-1}, x_m] = [x_{m-1}, 1]$. We treat $T : I \to I$ which satisfies the following properties: For $i = 0, \dots, m-1$,

(i) $T|_{\operatorname{int} I_i} : \operatorname{int} I_i \to (0,1)$ is one-to-one and onto,

(ii) $T|_{I_i} \in \mathscr{C}^1(I_i)$,

(iii) $1 < \lambda < \inf\{|T'(x)| : x \in I_i\}$,

(iv) $|T'(x) - T'(y)| \leq Const(\log|x-y|)^{-2}$ for $Const > 0$, $x, y \in I_i$, $x \neq y$.

REMARK 1. For the left endpoint x of I_i, $T'(x)$ denotes the right derivative at x for $i = 0, \dots, m-2$ and $T'(1)$ denotes the left derivative of T at 1.

Set $\tilde{\phi}(x) = -\log|T'(x)|$,

$$\operatorname{Var}_n(\tilde{\phi}) = \sup\{|\tilde{\phi}(x) - \tilde{\phi}(y)| : x, y \in I_{x_0 \cdots x_n}\}$$

and

$$I_{x_0 \cdots x_n} = \{x \in I : x \in I_0, Tx \in I_{x_1}, \dots, T^n x \in I_{x_n}\}.$$

We wish to treat maps which are not in $\mathscr{C}^{1+\gamma}$ for any $\gamma > 0$; that is, in the class:

(NH) $\tilde{\phi}(x)$ is non-Holder continuous, that is, for any $c_1 > 0$ and $\eta \in (0,1)$, there exists $n_0 \in \mathbf{N}$ such that $\operatorname{Var}_{n_0}(\tilde{\phi}) > c_1 \eta^{n_0}$.

REMARK 2. If T satisfies (i)–(iv) and (NH) then $T \notin \mathscr{C}^{1+\gamma}$ for any $\gamma > 0$.

For a strict subset U of S whose cardinality is greater than one, we define $\check{T} : \check{I} \to I$ as $T|_{\check{I}} = \check{T}$, where $\check{I} = I \setminus \bigcup_{i \in U} I_i$. Let $C(\check{T})$ be a generalized cookie-cutter set of $\check{T}$, that is, $C(\check{T}) = \overline{\{x \in I : \check{T}^n(x) \in I \text{ for any } n \in \mathbf{N}\}}$.

Let $\varphi_i : \bar{I}_i \to I$ be a continuous extension of each $T^{-1}|_{I_i}$ for $i = 0, \dots, m-1$. Then by (iii), each φ_i satisfies

$$|\varphi_i'| \leq \lambda^{-1} < 1 \qquad \text{for } i = 0, \dots, m-1, \tag{5.4}$$

so that $\bigcap_{n=0}^{\infty} \varphi_{x_0} \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(I)$ is a singleton in I. We call it $\pi(\underline{x})$. It is clear that $\pi : X_{L'} \to I$ is continuous and onto. If $l'_{ij} = 1$ for any $i, j \in S$, then we get the following diagram:

$$\begin{array}{ccc} X_{L'} & \xrightarrow{\sigma_{L'}} & X_{L'} \\ \pi \big\downarrow & & \big\downarrow \pi \\ I & \xrightarrow{T} & I. \end{array}$$

By (iii) and (iv), we have $\sum_{n=1}^{\infty} \mathrm{Var}_n(\tilde{\phi}) < +\infty$. Put $\phi(\underline{x}) = \tilde{\phi} \circ \pi(\underline{x})$. Then $\mathrm{var}_n^{X_{L'}}(\phi) = \mathrm{Var}_n(\tilde{\phi})$ for any $n \in \mathbf{N}$. Therefore we have $\phi^*_{X_{L'}} < +\infty$. Using the above preparation, we get the following theorem.

THEOREM 5.2. *Suppose that T satisfies* (i)–(iv) *and $\check{T}$ is defined as above. Then there exists a Pianigiani-Yorke measure for $\check{T}$ on I, which is absolutely continuous with respect to the Lebesgue measure.*

PROOF. Let L be the same type of matrix as in (5.1), which satisfies $l_{ij} = 0$ for any $i \in U$ and $j \in S$. Then the following diagram is commuting:

$$\begin{array}{ccc} X_{L} & \xrightarrow{\sigma_{L}} & X_{L} \\ \pi \big\downarrow & & \big\downarrow \pi \\ C(\check{T}) & \xrightarrow{\check{T}} & c(\check{T}). \end{array}$$

Clearly we have

$$\check{T} \circ \pi = \pi \circ \underline{\sigma} \quad \text{on } \underline{X}. \tag{5.5}$$

Using Theorem 5.1, there exists a Pianigiani-Yorke measure μ_{PY} on $X_{L'}$, that is, μ_{PY} is a probability measure and there exists $\alpha > 0$ such that

$$\mu_{PY} \circ \underline{\sigma}^{-1} = \alpha \mu_{PY} \quad \text{on } X_{L'}. \tag{5.6}$$

In this case, $\tilde{\mu}_{PY} = \mu_{PY}\pi^{-1}$ is a Pianigiani-Yorke measure on I. In fact, by (5.5) and (5.6), we have

$$\begin{aligned} \tilde{\mu}_{PY}(\check{T}^{-1}B) &= \mu_{PY}(\pi^{-1}\check{T}^{-1}B) = \mu_{PY}(\underline{\sigma}^{-1}\pi^{-1}B) \\ &= \alpha\mu_{PY}(\pi^{-1}B) = \alpha\tilde{\mu}_{PY}(B) \quad \text{for any Borel set } B \subset I. \end{aligned}$$

On the other hand, the equilibrium state μ_ϕ is $h_{L'}\nu_{L'}$ (see Bowen [2, P. 21], Walters [16]). It is clear that $\mu_{PY} = \underline{h}\nu_{L'}/\nu_{L'}(\underline{h})$ is mutually absolutely continuous with respect to the equilibrium state μ_ϕ. Moreover we have $\mu_\phi \circ \pi^{-1}$ is absolutely continuous with respect to 1-dimensional Hausdorff measure, i.e. the Lebesgue measure on I by Nakamura [10]. Bedford [1, Theorem 3.1] showed the case that T is in $\mathscr{C}^{1+\gamma}$ for $\gamma > 0$. Nakamura [10] showed the case that $\sum_{n=1}^{\infty} \mathrm{Var}_n(\tilde{\phi}) < +\infty$. Therefore we have $\tilde{\mu}_{PY} = \mu_{PY} \circ \pi^{-1}$ is absolutely continuous with respect to the Lebesgue measure. ■

REMARK 3. We can also prove Theorem 5.2 under the condition $\sum_{n=1}^{\infty} \mathrm{Var}_n(\tilde{\phi}) < +\infty$ instead of (iv).

EXAMPLE 3. If we give $T : I \to I$ as follows,

$$T(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{x}{(\log x)^2} + \lambda x & \text{if } 0 < x < \alpha, \\ \frac{k}{1-\alpha}(x - \alpha)(\mathrm{mod}\, 1) & \text{if } \alpha \le x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

where $\lambda > 1, \alpha = \min\left\{x > 0 : \frac{x}{(\log x)^2} + \lambda x = 1\right\}$, $\beta = (\log \alpha)^{-2} - 2(\log \alpha)^{-3} + \lambda$ and $k = [(1 - \alpha)\beta] + 1$. Then T satisfies (i)–(iv) and (NH). Because of

$$\frac{C_1}{n^2} \le \mathrm{Var}_n(\tilde{\phi}) \le \frac{C_2}{n^2} \quad \text{for some } C_1, C_2 > 0 \quad \text{and } n \in \mathbf{N},$$

T satisfies the condition (NH). Since $\sum_{n=1}^{\infty} \mathrm{Var}_n(\tilde{\phi}) < +\infty$, we have (iv). It is clear that all of other conditions are satisfied. The research was supported in part by Research Aid of Inoue Foundation for Science.

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