HIROSHIMA MATH. J.
29 (1999), 193–225

Contravariant forms on generalized Verma modules and *b*-functions

Akihito WACHI

(Received February 23, 1998)
(Revised May 14, 1998)

ABSTRACT. Two bilinear forms on a scalar generalized Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda$ are treated in this paper, where $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{p}$ is its parabolic subalgebra. They coincide on each $\mathfrak{l}$-irreducible component up to scalar multiple, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{p}$. These ratios have played important roles in the representation theory. We show intrinsically that these ratios are products of b-functions when the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is commutative. As an application we explain the reason why the b-functions control the irreducibility or $M(\lambda)$, the orbit decomposition of $\mathfrak{n}^+$ under the action of the Levi subgroup, and the unitarizability of $M(\lambda)$.

1. Introduction

Let G be a complex simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ its Cartan subalgebra. Let Δ and Δ^+ be the root system and the positive system, respectively. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{h}$ and all the positive root spaces. Then the pair $(\mathfrak{g}, \mathfrak{p})$ is said to be of commutative parabolic type if the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is commutative. In this paper, we exclusively consider $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type.

Let $M(\lambda)$ be the scalar generalized Verma module induced from $\lambda \in \operatorname{Hom}(\mathfrak{p}, \mathbf{C})$. Then $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$ as vector spaces. We therefore obtain the representation of $U(\mathfrak{g})$ on $\mathbf{C}[\mathfrak{n}^+]$, and denote it by $\Psi_\lambda : U(\mathfrak{g}) \to \operatorname{End} \mathbf{C}[\mathfrak{n}^+]$.

Let $\{X_\alpha, H_i\}$ be a Chevalley basis of $\mathfrak{g}$, where $X_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Delta$ and $H_i \in \mathfrak{h}$. To give the definition of contravariant forms, we define an involutive anti-automorphism ${}^t\cdot$ on $U(\mathfrak{g})$ by $X_\alpha \mapsto X_{-\alpha} (\alpha \in \Delta)$ and to be the identity on $\mathfrak{h}$. For a representation (π, V) of $\mathfrak{g}$, a bilinear form $(\,,\,)$ on V is called a contravariant form or a $\pi(U(\mathfrak{g}))$-contravariant form if it satisfies $(\pi(X)v, w) = (v, \pi({}^tX)w)$ for $X \in \mathfrak{g}$ and v, $w \in V$. We study a canonical $\Psi_\lambda(U(\mathfrak{g}))$-contravariant form $(\,,\,)_\lambda$ and a canonical $\operatorname{ad}(U(\mathfrak{l}))$-contravariant form $(\,,\,)$ on $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$, where $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$. Let $\mathbf{C}[\mathfrak{n}^+] = \bigoplus_\mu I_\mu$ be the irreducible decomposition as an $\operatorname{ad}(U(\mathfrak{l}))$-

1991 *Mathematical Subject Classification*: 17B10.
Key words and Phrases: generalized Verma modules, b-functions, contravariant forms, semisimple Lie algebras.

module. Then the above two contravariant forms coincide up to constant multiple on each I_μ. Let $q_\lambda(\mu)$ be the ratio of $(\ ,\ )_\lambda$ and $(\ ,\ )$ on I_μ.

On the other hand, there is a prehomogeneous vector space $(L, \mathrm{Ad}, \mathfrak{n}^+)$ associated with $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, where L is the connected subgroup of G corresponding to $\mathfrak{l}$. If $(L, \mathfrak{n}^+)$ is regular prehomogeneous vector space (see Definition 6.1 (3)) then there exists a relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ and the b-function $b(s)$ is defined by ${}^t f(\partial) f^{s+1} = b(s) f^s$. In Wallach [28], $q_\lambda(\mu)$ appears and is determined explicitly. Moreover the results of Kostant-Sahi [16], of Shimura [23], of Rubenthaler-Schiffmann [20] and of Faraut-Koranyi [4] are deduced from the explicit formula for $q_\lambda(\mu)$. Our main purpose is to show intrinsically that $q_\lambda(\mu)$ is a certain product of b-functions. As an application we explain the reason why the b-functions control the irreducibility of $M(\lambda)$, the orbit decomposition of $\mathfrak{n}^+$ under the action of the Levi subgroup, and the unitarizability of $M(\lambda)$.

The contents of this paper is as follows: In §2 to §5, we prepare basic definition such as scalar generalized Verma modules and contravariant forms. In §6 we recall the definition of b-functions and introduce another function, which is deeply related to b-functions. In §7 we state our main theorem (Theorem 7.1). In §8 we define subalgebras of $\mathfrak{g}$ and restate our main result at the end of the section. In §9 we derive an important conclusion Corollary 9.1 from our main theorem, which gives an expression of $q_\lambda(\mu)$ as a product of b-functions. In §10 we give another expression of $q_\lambda(\mu)$.

In §11 , we consider the irreducibility of $M(\lambda)$. It is known that $M(\lambda)$ is irreducible if and only if the contravariant form $(\ ,\ )_\lambda$ is nondegenerate or equivalently $q_\lambda(\mu) \neq 0$ for all μ. In Jantzen [12], the determinant of $(\ ,\ )_\lambda$ is calculated and the irreducibility criteria are described concretely. In Shapovalov [22], the determinant is calculated for the Verma module. It is observed that the values of λ which makes $M(\lambda)$ irreducible, are related to the zeros of the b-functions. The first published result which relates the irreducibility criteria and the b-functions, is Suga [24]. The necessary condition for the irreducibility is stated there in terms of b-functions in the case where $\mathfrak{g}$ is classical. Gyoja [7] and [8] conjectured an irreducibility criterion in terms of b-functions in a more general setting, and he proved this in some special cases including the commutative parabolic cases by a case study. In this paper we explain intrinsically why there exists such a relation between b-functions and the irreducibility.

In §12, we consider the one-to-one correspondence between $\mathrm{Ad}(L)$-orbit on $\mathfrak{n}^+$ and the zeros of a b-function. Tanisaki found this correspondence motivated by the study of hypergeometric systems (Tanisaki [25], [26]). His proof was a case study. We give an intrinsic proof of the correspondence.

In §13, we consider the unitarizability of the irreducible quotient of $M(\lambda)$, say $L(\lambda)$. This application is suggested by Professor Shuichi Suga. Only in §13, we work in 'real' situation, that is, we use a real form of the complex Lie algebra $\mathfrak{g}$. Most arguments, however, go well as in the 'complex' situation. There are many articles which treat the unitarizability (Wallach [27], Parthasarathy [19], Garland-Zuckerman [5], Enright-Howe-Wallach [3], Enright-Joseph [13] and many other articles). It is known that the values of λ such that $L(\lambda)$ is unitarizable, are related to the zeros of a b-function. We explain intrinsically the reason for this relation.

In §14 and §15, we prove the main theorem using Boe [1].

The author would like to express his gratitude to Professor Mutsumi Saito for his constant encouragement and valuable advice. He also would like to express his thanks to Professor Akihiko Gyoja, Professor Shuichi Suga and Professor Hirofumi Yamada for their valuable advice.

2. Commutative parabolic type

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote the root system and the set of positive roots by Δ and Δ^+, respectively. Let $\{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots and let $\{\varpi_1, \dots, \varpi_n\}$ be the set of fundamental weights corresponding to $\{\alpha_1, \dots, \alpha_n\}$. In other words, $\varpi_j \in \mathfrak{h}^*$ and $2(\varpi_i, \alpha_j) = \delta_{ij}(\alpha_j, \alpha_j)$. We take a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing all the positive root spaces and $\mathfrak{h}$. Let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$, and $\mathfrak{n}^+$ the nilpotent radical of $\mathfrak{p}$. *In this paper, we exclusively consider the case where* $\mathfrak{n}^+$ *is nonzero and commutative.* We say $(\mathfrak{g}, \mathfrak{p})$ in this case to be of *commutative parabolic type*. In this case, $\mathfrak{p}$ is a maximal parabolic subalgebra and there exists exactly one simple root α_{i_0} such that the root space $\mathfrak{g}^{-\alpha_{i_0}}$ is not contained in $\mathfrak{p}$. For all the possible pairs $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, corresponding pairs $(\mathfrak{g}, i_0)$ are listed in Figure 1, where the numbering of the simple roots follows Bourbaki [2], and white circles correspond to α_{i_0}. Let Δ_L be the root system of $\mathfrak{l}$ and $\Delta_N^+ = \Delta^+ \backslash \Delta_L$. Set $\mathfrak{n}^- = \sum_{\alpha \in \Delta_N^+} \mathfrak{g}^{-\alpha}$. Let G be the connected algebraic group corresponding to $\mathfrak{g}$, and L be the closed subgroup of G corresponding to $\mathfrak{l}$.

3. Generalized Verma modules

Definition 3.1. For $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$, we set $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda$, where $\mathbf{C}_\lambda$ is the representation space of λ. The $U(\mathfrak{g})$-module $M(\lambda)$ is called a *scalar generalized Verma module* with highest weight λ.

There is an identification $S(\mathfrak{n}^-) \simeq \mathbf{C}[\mathfrak{n}^+]$, since $\mathfrak{n}^-$ can be considered as the dual space of $\mathfrak{n}^+$ via the Killing form. Thus there is a vector space iso-

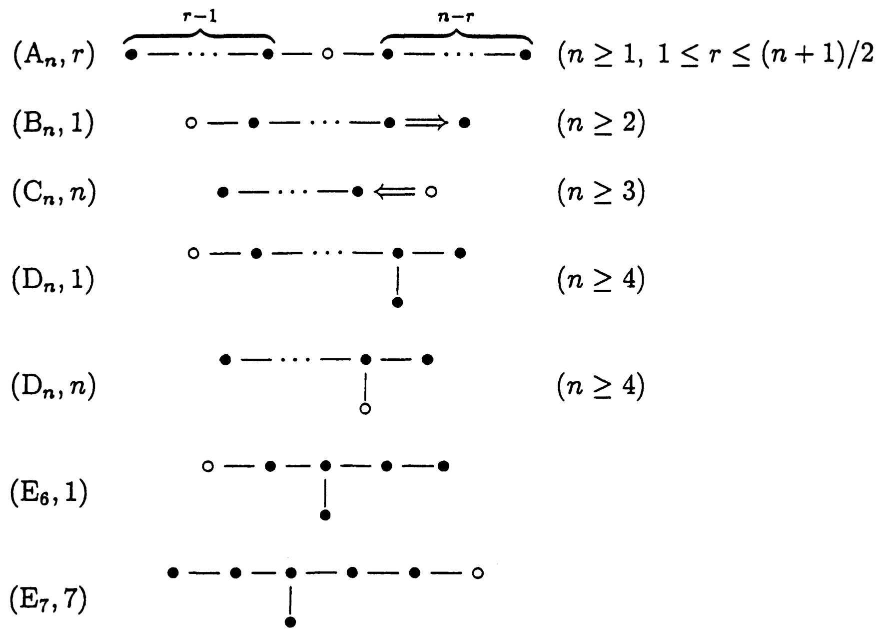

Fig. 1. Commutative parabolic type

morphism $M(\lambda) \simeq U(\mathfrak{n}^-) \otimes_{\mathbf{C}} \mathbf{C}_\lambda \simeq S(\mathfrak{n}^-) \simeq \mathbf{C}[\mathfrak{n}^+]$. We can therefore consider $\mathbf{C}[\mathfrak{n}^+]$ as a $U(\mathfrak{g})$-module. We denote this representation by $\Psi_\lambda : U(\mathfrak{g}) \to \operatorname{End} \mathbf{C}[\mathfrak{n}^+]$. We can find explicit form of $\Psi_\lambda(X)$ for $X \in \mathfrak{g}$ by a direct calculation.

LEMMA 3.2.
(1) $\Psi_\lambda(X) = X \qquad (X \in \mathfrak{n}^-)$
(2)
$$\Psi_\lambda(X) = \operatorname{ad}(X) + \lambda(X) = \sum_k [X, F_k] \frac{\partial}{\partial F_k} + \lambda(X) \qquad (X \in \mathfrak{l}),$$
(3)
$$\Psi_\lambda(X) = \frac{1}{2} \sum_{k,l} [[X, F_k], F_l] \frac{\partial}{\partial F_k} \frac{\partial}{\partial F_l} + \sum_k \lambda([X, F_k]) \frac{\partial}{\partial F_k} \qquad (X \in \mathfrak{n}^+),$$

where $\langle\ ,\ \rangle$ *is the Killing form on* $\mathfrak{g}$, $(\ ,\ )$ *is the inner product on* $\mathfrak{h}^*$ *induced from the Killing form,* $\{F_k\}$ *is a basis of* $\mathfrak{n}^-$ *and* λ^0 *is the complex number determined by* $\lambda = \lambda^0 \varpi_{i_0}$.

In particular, $\Psi_\lambda(U(\mathfrak{g}))$ is contained in $D_{\mathfrak{n}^+}$, the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. We identify $M(\lambda)$ with $\Psi_\lambda(U(\mathfrak{g}))$-module $\mathbf{C}[\mathfrak{n}^+]$ from now on.

4. Two contravariant forms

In this section, we give a definition of contravariant forms and then we introduce two contravariant forms on $M(\lambda)$.

DEFINITION 4.1. Define anti-automorphism ${}^{t}\cdot$ of $U(\mathfrak{g})$ by

$$X_\alpha \mapsto {}^{t}X_\alpha = X_{-\alpha} \qquad (\alpha \in \Delta),$$

$$H_i \mapsto {}^{t}H_i = H_i \qquad (i \in \{1, \dots, n\}),$$

where $H_i \in \mathfrak{h}$ is the coroot of α_i, that is, $H_i \in [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$ and $\alpha_i(H_i) = 2$, and $X_\alpha \in \mathfrak{g}^\alpha (\alpha \in \Delta)$ are the root vectors such that $\{H_i, X_\alpha\}$ forms a Chevalley basis of $\mathfrak{g}$.

Definition 4.1 depends on the choice of Chevalley bases. We fix a Chevalley basis $\{H_i, X_\alpha\}$ once and for all. Here we have an equality

$$\langle X_\alpha, X_{-\alpha} \rangle = \frac{2}{(\alpha, \alpha)} \qquad (\alpha \in \Delta), \tag{4.1}$$

Indeed, $2\langle X_\alpha, X_{-\alpha}\rangle = \langle [H_\alpha, X_\alpha], X_{-\alpha}\rangle = \langle H_\alpha, H_\alpha \rangle = (2\alpha/(\alpha,\alpha), 2\alpha/(\alpha,\alpha)) = 4/(\alpha,\alpha)$, where $H_\alpha \in \mathfrak{h}$ is the coroot of $\alpha \in \Delta^+$.

DEFINITION 4.2. Let (π, V) be a $U(\mathfrak{g})$-module. A symmetric bilinear form $(\ ,\ )$ on V is called a *contravariant form* or a $\pi(U(\mathfrak{g}))$-*contravariant* form if $(\pi(u)v, v') = (v, \pi({}^{t}u)v')$ for all $u \in U(\mathfrak{g})$ and v, $v' \in V$.

The following propositions are fundamental on contravariant forms.

PROPOSITION 4.3. *Let V be a U $(\mathfrak{g})$-module and $\mathfrak{m}$ a reductive subalgebra of $\mathfrak{g}$. Assume that $(\ ,\ )$ is an $\mathfrak{m}$-contravariant form on V. If W_1 and W_2 are inequivalent irreducible $\mathfrak{m}$-submodule, then $(W_1, W_2) = 0$. In particular, different weight spaces of V are orthogonal with respect to $(\ ,\ )$.*

PROOF. See Garland-Zuckerman [5, Lemma 2.5]. □

PROPOSITION 4.4. *Let V be a $U(\mathfrak{g})$-module. Assume that V is a highest weight module. Then we have*
(1) *There exists a nonzero contravariant form on V, and it is unique up to constant multiple.*
(2) *The radical of a nonzero contravariant form on V coincides with the maximal proper submodule of V.*

PROOF. See Humphreys [11, §6] or Wallach [27]. □

We introduce two contravariant forms on $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$. One is a

$\Psi_\lambda(U(\mathfrak{g}))$-contravariant form and the other is an $\mathrm{ad}(U(\mathfrak{l}))$-contravariant form on $\mathbf{C}[\mathfrak{n}^+]$.

DEFINITION 4.5. Define $\mathbf{C}$-linear function $\varphi_\lambda : U(\mathfrak{g}) \to \mathbf{C}$ as a composite of the projection from $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{c}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{c}^+)$ to $U(\mathfrak{h})$ and $\lambda : U(\mathfrak{h}) \to \mathbf{C}$, where $\mathfrak{c}^\pm = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\pm\alpha}$. We define a $\Psi_\lambda(U(\mathfrak{g}))$-contravariant form $(\ ,\ )_\lambda$ by

$$(f, g)_\lambda = \varphi_\lambda({}^t g f) \quad \text{for } f, g \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-) \subset U(\mathfrak{g}).$$

See also Humphreys [11, §6].

We will define another symmetric $\mathbf{C}$-bilinear form on $\mathbf{C}[\mathfrak{n}^+]$. We shall identify $S(\mathfrak{n}^+)$ with the ring of constant coefficient differential operators on $\mathfrak{n}^+$ via the Killing form as follows: For $P \in S(\mathfrak{n}^+) \simeq \mathbf{C}[\mathfrak{n}^-]$, define a constant coefficient differential operator $P(\partial)$ on $\mathfrak{n}^+$ by

$$P(\partial) \exp\langle x, y\rangle = P(y) \exp\langle x, y\rangle \quad \text{for } x \in \mathfrak{n}^+ \text{ and } y \in \mathfrak{n}^-. \tag{4.2}$$

For $P \in S(\mathfrak{n}^+)$, we write it by $P(\partial)$ when it is regarded as a differential operator on $\mathfrak{n}^+$.

DEFINITION 4.6. Define symmetric $\mathbf{C}$-bilinear form $(\ ,\ )$ on $\mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$ by

$$(f, g) = ({}^t g(\partial) f)(0) \qquad \text{for } f, g \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-),$$

where ${}^t g(\partial)$ is the constant coefficient differential operator on $\mathfrak{n}^+$ identified with ${}^t g \in S(\mathfrak{n}^+)$, and $({}^t g(\partial) f)(0)$ means a differentiation followed by evaluation at $0 \in \mathfrak{n}^+$. This bilinear form is $\mathrm{ad}(U(\mathfrak{l}))$-contravariant, since the bilinear form defined by $\langle P, f\rangle = (P(\partial) f)(0)$ for $P \in S(\mathfrak{n}^+)$ and $f \in \mathbf{C}[\mathfrak{n}^+]$, is $\mathrm{Ad}(L)$-invariant. Moreover $(\ ,\ )$ is nondegenerate.

We summarize some properties of these forms.

LEMMA 4.7. (1) *$\Psi_\lambda(U(\mathfrak{l}))$-contravariance and $\mathrm{ad}(U(\mathfrak{l}))$-contravariance are the same notion.*

(2) *A $\Psi_\lambda(U(\mathfrak{g}))$-contravariant form is also $\Psi_\lambda(U(\mathfrak{l}))$-contravariant.*

(3) $(f, gh)_\lambda = (\Psi_\lambda({}^t g) f, h)_\lambda$ *for $f, g, h \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$.*

(4) $(f, gh) = ({}^t g(\partial) f, h)$ *for $f, g, h \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$.*

PROOF. (1) It follows from $\Psi_\lambda(X) = \mathrm{ad}(X) + \lambda(X)$ for $X \in \mathfrak{l}$. (2) It follows immediately from the definition of the contravariance. (3) Since $(\ ,\ )_\lambda$ is $\Psi_\lambda(U(\mathfrak{g}))$-contravariant, and since $\Psi_\lambda(u)$ is just a multiplying operator for $u \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$, we get the identity. (4) It follows immediately from Definition 4.6. □

5. The ratio of the two contravariant forms

We have defined two contravariant forms on $M(\lambda) \simeq S(\mathfrak{n}^-) \simeq \mathbf{C}[\mathfrak{n}^+]$. Since both $(\ ,\ )$ and $(\ ,\ )_\lambda$ are $\mathrm{ad}(U(\mathfrak{l}))$-contravariant by Lemma 4.7, it follows from Proposition 4.4 (1) that $(\ ,\ )_\lambda$ coincides with $(\ ,\ )$ up to constant multiple on each irreducible $\mathrm{ad}(U(\mathfrak{l}))$-submodule of $\mathbf{C}[\mathfrak{n}^+]$. In this section we define a function $q_\lambda(\mu)$ as the ratio of these two forms.

Definition 5.1. $\alpha, \beta \in \Delta$ are said to be *strongly orthogonal*, if α and β are non-proportional and both $\alpha + \beta \notin \Delta$ and $\alpha - \beta \notin \Delta$ hold.

If $\alpha, \beta \in \Delta$ and $(\alpha, \beta) < 0$, then $\alpha - \beta \in \Delta$. Thus if α is strongly orthogonal to β then α is orthogonal to β.

We take the family of mutually strongly orthogonal roots contained in Δ_N^+ as follows (Harish-Chandra [9]): Set $\gamma_1 = \alpha_{i_0}$. When we have taken $\gamma_1, \dots, \gamma_i$, let γ_{i+1} be the lowest root in

$$\{\alpha \in \Delta_N^+ \mid \alpha \text{ is strongly orthogonal to all } \gamma_1, \dots, \gamma_i\},$$

if this set is not empty. Let r be the index of γ_i which we could take last. Set $\lambda_i = -(\gamma_1 + \dots + \gamma_i)$ for $i \in \{1, \dots, r\}$.

Theorem 5.2. (Schmid [21]) *Let V_μ be the finite dimensional irreducible $\mathrm{ad}(U(\mathfrak{l}))$-module with highest weight μ. We denote by $\mathbf{C}^d[\mathfrak{n}^+]$ the homogeneous component of degree d of $\mathbf{C}[\mathfrak{n}^+]$. Then we have*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{l}}(V_\mu, \mathbf{C}^d[\mathfrak{n}^+]) = \begin{cases} 1 & (\mu = k_1\lambda_1 + \dots + k_r\lambda_r, \\ & \quad \textit{for some } k_j \in \mathbf{Z}_{\ge 0},\ d = \sum_j jk_j) \\ 0 & (\textit{otherwise}) \end{cases}$$

For $\mu = k_1\lambda_1 + \dots + k_r\lambda_r$ $(k_j \in \mathbf{Z}_{\ge 0})$, let I_μ be the unique $\mathrm{ad}(U(\mathfrak{l}))$-submodule in $\mathbf{C}[\mathfrak{n}^+]$ with the highest weight μ. Then we have an irreducible decomposition

$$\mathbf{C}[\mathfrak{n}^+] = \bigoplus_{\mu \in \sum_{j=1}^r \mathbf{Z}_{\ge 0}\lambda_j} I_\mu.$$

In particular $\mathbf{C}[\mathfrak{n}^+]$ is *multiplicity free*, that is, all the multiplicities of irreducible $\mathrm{ad}(U(\mathfrak{l}))$-submodules are equal to one. Let f_i be a highest weight vector of I_{λ_i} and $f_\mu = f_1^{k_1} \cdots f_r^{k_r}$ for $\mu = k_1\lambda_1 + \dots + k_r\lambda_r$. Then f_μ is a highest weight vector of I_μ.

As we stated before, two $\mathrm{ad}(U(\mathfrak{l}))$-contravariant forms $(\ ,\ )_\lambda$ and $(\ ,\ )$ on $\mathbf{C}[\mathfrak{n}^+]$, coincide on each irreducible submodule I_μ. For $\mu = k_1\lambda_1 + \dots + k_r\lambda_r$

($k_j \in \mathbf{Z}_{\geq 0}$) and $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathrm{C})$, we define $q_\lambda(\mu)$ by

$$(,)_\lambda = q_\lambda(\mu)(,) \quad \text{on } I_\mu \times I_\mu.$$

LEMMA 5.3. *For all* μ, (f_μ, f_μ) *is nonzero.*

PROOF. It is obvious from the definition that $(,)$ is nondegenerate on $\mathbf{C}[\mathfrak{n}^+]$. For $\mu \neq \nu$, it follows from Proposition 4.3 that $(I_\mu, I_\nu) = 0$ since $\mathbf{C}[\mathfrak{n}^+]$ is multiplicity free. Thus $(,)$ is nondegenerate on each I_μ. We have $(f_\mu, f_\mu) \neq 0$ since the highest weight space of I_μ is one-dimensional. $\square$

Thanks to Lemma 5.3, we have

$$q_\lambda(\mu) = (f_\mu f_\mu)_\lambda / (f_\mu, f_\mu). \tag{5.1}$$

6. *b*-Functions

In this section, we introduce prehomogeneous vector spaces and define *b*-functions of prehomogeneous vector spaces associated with $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type.

DEFINITION 6.1. (1) A finite dimensional G-module V is called a *prehomogeneous vector space* if there exists an open G-orbit on V.

(2) A nonzero function f on V is called a *relative invariant* of (G, V), if there exists a character χ of G such that $f(gv) = \chi(g)f(v)$ for all $g \in G$ and $v \in V$.

(3) A prehomogeneous vector space (G, V) is said to be *regular* if there exists a relative invariant f of (G, V) and the Hessian $\det(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a linear coordinate system of V.

REMARK 6.2. It is known that $(L, \mathfrak{n}^+)$ is a prehomogeneous vector space and the open L-orbit contains $X_{\gamma_1} + \cdots + X_{\gamma_r}$, where X_{γ_j} is an element of our fixed Chevalley basis (Muller-Rubenthaler-Schiffmann [18, Theorem 2.4]). The $(L, \mathfrak{n}^+)$ is regular if an only if Hermitian symmetric space G/L is of tube type (Koranyi-Wolf [15]).

All the pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type, where $(L, \mathfrak{n}^+)$ becomes regular prehomogeneous vector spaces, are listed in Figure 2. Notation is the same as before.

Let H_{γ_j} be the coroot of γ_j, that is, $H_{\gamma_j} \in [\mathfrak{g}^{\gamma_j}, \mathfrak{g}^{-\gamma_j}]$ and $\gamma_j(H_{\gamma_j}) = 2$. Set $\mathfrak{h}^- = \sum_{j=1}^r \mathbf{C}H_{\gamma_j}$.

THEOREM 6.3. (Moore [17, Theorem 2]) (1) *For* $\alpha \in \Delta_L \cap \Delta^+$, *the possible forms of* $\alpha|_{\mathfrak{h}^-}$ *are as follows:*

$$\frac{1}{2}(\gamma_j - \gamma_i)(1 \leq i < j \leq r), \quad -\frac{1}{2}\gamma_i(1 \leq i \leq r), \quad 0.$$

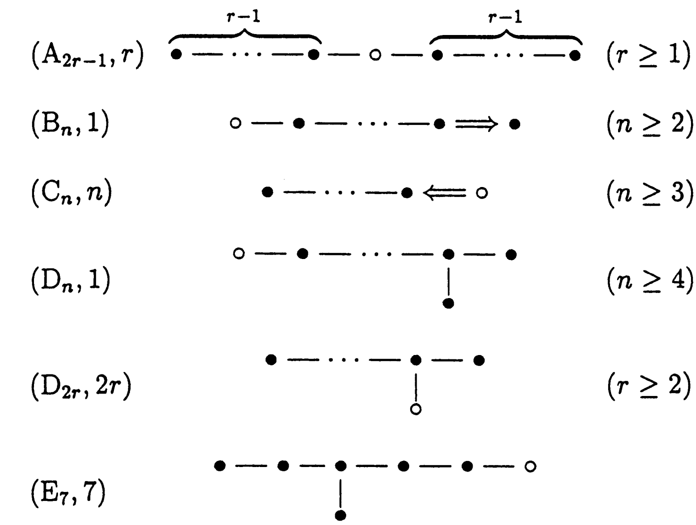

Fig. 2. Regular type

(2) *For $\alpha \in \Delta_N^+ \backslash \{\gamma_1, \dots, \gamma_r\}$, the possible forms of $\alpha|_{\mathfrak{h}^-}$ are as follows:*

$$\frac{1}{2}(\gamma_j + \gamma_i)(1 \leq i < j \leq r), \quad \frac{1}{2}\gamma_i(1 \leq i \leq r).$$

(3) *If $(L, \mathfrak{n}^+)$ is a regular prehomogeneous vector space, then $\pm(1/2)\gamma_i$ in the above possibilities do not occur.*

We exclusively deal with the case when $(L, \mathfrak{n}^+)$ is a regular prehomogeneous vector space for the rest of this section. We can find relative invariant of a regular prehomogeneous vector space $(L, \mathfrak{n}^+)$ using Theorem 6.3.

Lemma 6.4. *If $(L, \mathfrak{n}^+)$ is regular, then $\lambda_r = -2\varpi_{i_0}$. Thus f_r is a relative invariant of $(L, \mathfrak{n}^+)$.*

Proof. For $\alpha \in \Delta_L$, we have $\alpha|_{\mathfrak{h}^-} = (1/2)(\gamma_j - \gamma_i)(i, j \in \{1, \dots, r\})$ by Theorem 6.3, since $(L, \mathfrak{n}^+)$ is regular. We have that $(\lambda_r, \alpha) = -\sum_{k=1}^r (\gamma_k, \alpha) = -\sum_k \alpha(t_{\gamma_k}) = -\sum_k \alpha|_{\mathfrak{h}^-}(t_{\gamma_k}) = -\sum_k (1/2)(\gamma_j - \gamma_i)(t_{\gamma_k}) = -(1/2)\{(\gamma_j, \gamma_j) - (\gamma_i, \gamma_i)\} = 0$, where t_{γ_k} denotes the element in $\mathfrak{h}$ which is identified with γ_k via the Killing form. Thus λ_r is a constant multiple of ϖ_{i_0}. We can determine it by computing $(\lambda_r, \alpha_{i_0})/(\varpi_{i_0}, \alpha_{i_0})$. We have that $(\lambda_r, \alpha_{i_0})/(\varpi_{i_0}, \alpha_{i_0}) = -(\gamma_1 + \dots + \gamma_r, \gamma_1)/2^{-1}(\alpha_{i_0}, \alpha_{i_0}) = -(\gamma_1, \gamma_1)/2^{-1}(\gamma_1, \gamma_1) = -2$.

The highest weight of I_{λ_r} is equal to $-2\varpi_{i_0}$, and therefore I_{λ_r} is a trivial one-dimensional $\mathrm{ad}(U([\mathfrak{l}, \mathfrak{l}]))$-module, that is, $I_{\lambda_r} = \mathbf{C} f_r$. This means that f_r is a relative invariant. □

Here we define b-functions and b-function-like functions associated with the regular prehomogeneous vector spaces $(L, \mathfrak{n}^+) = (L, \mathrm{Ad}, \mathfrak{n}^+)$. Since f_r is the relative invariant, $g \in L$ acts on f_r by a certain scalar multiple, say $\chi(g)$. Dually for ${}^t f_r \in S(\mathfrak{n}^+)$, g acts by $\chi(g)^{-1}$. Thus ${}^t f_r f_r \in U(g)$ is $\mathrm{Ad}(L)$-invariant, and therefore the differential operator ${}^t f_r(\partial) f_r$ on $\mathfrak{n}^+$ is $\mathrm{Ad}(L)$-invariant. Then ${}^t f_r(\partial) f_r$ acts on $f_\mu \in \mathbf{C}[\mathfrak{n}^+]$ $(\mu = k_1\lambda_1 + \cdots + k_r\lambda_r)$ by a certain scalar multiple, since $\mathbf{C}[\mathfrak{n}^+]$ is multiplicity free.

As for $\mathrm{Ad}(L)$-invariance of $\Psi_\lambda({}^t f_r f_r)$, we need the following lemma.

LEMMA 6.5. *The representation* Ψ_λ *is* Ad(L)-*equivariant. Namely*, Ad $(g).\Psi_\lambda(u) := \mathrm{Ad}(g) \circ \Psi_\lambda(u) \circ \mathrm{Ad}(g^{-1}) = \Psi_\lambda(\mathrm{Ad}(g)u)$, *for* $u \in U(\mathfrak{g})$ *and* $g \in L$.

PROOF. We have a canonical linear isomorphism $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda \to \mathbf{C}[\mathfrak{n}^+]$. Thus we can define linear mapping $\alpha : U(\mathfrak{g}) \to \mathbf{C}[\mathfrak{n}^+]$ as a composite of the canonical surjection $U(\mathfrak{g}) \to M(\lambda)$ and the above canonical isomorphism $M(\lambda) \to \mathrm{C}[\mathfrak{n}^+]$.

First we show that α commutes with the $\mathrm{Ad}(L)$-action. Since $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{p})$ from PBW theorem, any $u \in U(\mathfrak{g})$ is a sum of elements such as np $(n \in U(\mathfrak{n}^-),\ p \in U(\mathfrak{p}))$. We may assume $u = np$ without loss of generality. For $g \in L$, we have $\alpha(\mathrm{Ad}(g)(np)) = \alpha(\mathrm{Ad}(g)n\,\mathrm{Ad}(g)p) = \mathrm{Ad}(g)n \cdot \lambda(\mathrm{Ad}(g)p) = \mathrm{Ad}(g)n \cdot \lambda(p) = \mathrm{Ad}(g)\alpha(np)$. Thus α commutes with the $\mathrm{Ad}(L)$-action.

It is easy to see that $\Psi_\lambda(u)f = \alpha(uf)$ for $u \in U(\mathfrak{g})$ and $f \in \mathbf{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$. Thus we have $\mathrm{Ad}(g) \circ \Psi_\lambda(u) \circ \mathrm{Ad}(g^{-1})f = \mathrm{Ad}(g)\alpha(u\mathrm{Ad}(g^{-1})f) = \alpha((\mathrm{Ad}(g)u)f) = \Psi_\lambda(\mathrm{Ad}(g)u)f$. The assertion is proved. □

By Lemma 6.5, $\Psi_\lambda({}^t f_r f_r)$ is $\mathrm{Ad}(L)$-invariant, and therefore $\Psi_\lambda({}^t f_r f_r)$ also acts on f_μ by a certain scalar multiple.

Then we can define functions $b_r(\mu)$ and $\beta_{\lambda,r}(\mu)$ by ${}^t f_r(\partial) f_r f_\mu = b_r(\mu) f_\mu$ and $\Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda,r}(\mu) f_\mu$, respectively. It is easily seen that b_r and $\beta_{\lambda,r}$ are polynomials.

Moreover we can define these functions for $\mu \in \sum \mathbf{C}\lambda_i$ as follows. Let A be a connected simply connected open subset of $\mathfrak{n}^+$ such that $f_1(a), \ldots, f_r(a) \neq 0$ for all $a \in A$. Set $\mathcal{O} = \mathrm{C}[\mathfrak{n}^+]$. For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathrm{C})$, $\mathcal{O}[f_1^{-1}, \ldots, f_r^{-1}]f_\mu$ on A becomes a D_A-module. Here a differential operator $\partial/\partial x \in D_A$ acts on $\mathcal{O}[f_1^{-1}, \ldots, f_r^{-1}]f_\mu$ by

$$\begin{aligned}\frac{\partial}{\partial x}(\varphi f_\mu) &= \frac{\partial \varphi}{\partial x} f_\mu + \varphi \frac{\partial f_\mu}{\partial x} \\ &= \frac{\partial \varphi}{\partial x} f_\mu + \varphi \frac{\partial \log f_\mu}{\partial x} f_\mu \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial \log f_\mu}{\partial x}\right)(\varphi) f_\mu,\end{aligned}$$

where $\varphi \in \mathcal{O}[f_1^{-1}, \ldots, f_r^{-1}]$ and $\partial \log f_\mu/\partial x = \sum_i k_i f_i^{-1} \partial f_i/\partial x$. Then there exists b-function $b'_r(\mu) \in \mathcal{O}[k_1, \ldots, k_r]$ such that ${}^t f_r(\partial) f_r f_\mu = b'_r(\mu) f_\mu$. Here $b_r(\mu)$ and $b'_r(\mu)$ coincide when $\mu \in \sum_i \mathbf{Z}_{\geq 0} \lambda_i$, and therefore they coincide for all $\mu \in \sum_i \mathbf{C} \lambda_i$. Similarly we can define $\beta_{\lambda, r}(\mu)$ for $\mu \in \sum_i \mathbf{C}\lambda_i$. In this way, we can define polynomials $b_r(\mu)$ and $\beta_{\lambda,r}(\mu)$ by the following definition.

Definition 6.6. Assume that $(L, \mathfrak{n}^+)$ is regular. Define polynomials b_r and $\beta_{\lambda,r}$ by

$$ {}^t f_r(\partial) f_r f_\mu = b_r(\mu) f_\mu, $$

$$ \Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda,r}(\mu) f_\mu, $$

for $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ and $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{C})$.

7. Main theorem

We continue to assume that $(L, \mathfrak{n}^+)$ is regular in this section. We state our main theorem. The theorem needs the normalization of f_r. By Muller-Rubenthaler-Schiffmann [18, Theorem 2.4], the open L-orbit on $(L, \mathfrak{n}^+)$ contains the vector $X_{\gamma_1} + \cdots + X_{\gamma_r}$. Thus $f_r(X_{\gamma_1} + \cdots + X_{\gamma_r}) \neq 0$. Then we normalize f_r by

$$ f_r(X_{\gamma_1} + \cdots + X_{\gamma_r}) = 1. \tag{7.1} $$

We fix this normalization for the rest of this paper.

We define the constants which will be used in our main theorem. Let $\rho_r \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ be the half sum of the roots of $\mathfrak{n}^+$, that is,

$$ \rho_r(X) = \left(\frac{1}{2} \sum_{\alpha \in \Delta_N^+} \alpha \right)(X) = \frac{1}{2} \mathrm{Tr}_{\mathfrak{n}^+} \mathrm{ad}(X) \quad (X \in \mathfrak{p}). $$

Since ρ_r is a constant multiple of ϖ_{i_0}, we define the complex number ρ_r^0 by

$$ \rho_r = \rho_r^0 \varpi_{i_0}. \tag{7.2} $$

The following theorem and its corollary (Corollary 9.1) are our main results. This theorem suggests that the structure of scalar generalized Verma modules has a certain connection with b-functions of prehomogeneous vector spaces associated with them.

Theorem 7.1. *Assume that $(L, \mathfrak{n}^+)$ is a regular prehomogeneous vector space. If f_r is normalized by* (7.1), *then for $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_i \in \mathbf{C})$,*

$$\beta_{\lambda,r}(\mu)=(-1)^r b_r(\mu)b_r(\mu-(\lambda^0+\rho_r^0)\lambda_r),$$

where λ^0 *is the complex number defined by* $\lambda=\lambda^0\varpi_{i_0}$.

8. Subalgebras of $\mathfrak{g}$

In this section, we will define certain subalgebras of $\mathfrak{g}$ and show some properties related to these subalgebras. Then we define b-functions and b-function-like functions associated with the subalgebras. We return to the situation where $(L,\mathfrak{n}^+)$ is not necessarily regular.

We define the subalgebras of $\mathfrak{g}$ following Wallach [27]. Set

$$\Delta_{N,i}^+=\{\alpha\in\Delta_N^+;\alpha|_{\mathfrak{h}^-}=(\gamma_k+\gamma_j)/2 \text{ for some } 1\le j<k\le i\}\cup\{\gamma_1,\dots,\gamma_i\},$$

$$\mathfrak{n}_i^{\pm}=\sum_{\alpha\in\Delta_{N,i}^+}\mathfrak{g}^{\pm\alpha}.$$

Let $\mathfrak{l}_i$ be $[\mathfrak{n}_i^+,\mathfrak{n}_i^-]$. It is easy to see that $\mathfrak{l}_i$ is a Lie algebra. Set

$$\mathfrak{p}_i=\mathfrak{l}_i+\mathfrak{n}_i^+,$$

$$\mathfrak{g}_i=\mathfrak{n}_i^-+\mathfrak{l}_i+\mathfrak{n}_i^+,$$

$$\mathfrak{h}_i=\mathfrak{h}\cap\mathfrak{g}_i,$$

and let L_i be the connected closed subgroup of G corresponding to $\mathfrak{l}_i$. Then $(\mathfrak{g}_i,\mathfrak{p}_i)$ is of commutative parabolic type, and $(L_i,\mathfrak{n}_i^+)$ is a regular prehomogeneous vector space. Obviously, the maximal family of mutually strongly orthogonal roots contained in $\Delta_{N,i}^+$, constructed in the same way, coincides with $\{\gamma_1|_{\mathfrak{h}_i},\dots,\gamma_i|_{\mathfrak{h}_i}\}$.

We can describe the decomposition of $\mathbf{C}[\mathfrak{n}_i^+]$ as an $\mathrm{ad}(U(\mathfrak{l}_i))$-module. For $\mu=k_1\gamma_1+\cdots+k_i\gamma_i$ $(k_j\in\mathbf{Z}_{\ge 0})$, f_μ is contained in $\mathbf{C}[\mathfrak{n}_i^+]$, although I_μ is not necessarily contained in $\mathbf{C}[\mathfrak{n}_i^+]$. We can show that f_μ is a maximal weight vector with respect to the action of $\mathrm{ad}(U(\mathfrak{l}_i))$, and f_μ runs over all the maximal weight vectors of $\mathrm{ad}(U(\mathfrak{l}_i))$-module $\mathbf{C}[\mathfrak{n}_i^+]$ by applying Theorem 5.2 to $\mathbf{C}[\mathfrak{n}_i^+]$. In other words, there is a decomposition into irreducible $\mathrm{ad}(U(\mathfrak{l}_i))$-modules

$$\mathbf{C}[\mathfrak{n}_i^+]=\bigoplus_{\mu\in\sum_{j=1}^{i}\mathbf{Z}_{\ge 0}\lambda_j}(I_\mu\cap\mathbf{C}[\mathfrak{n}_i^+]).$$

We consider $\mathbf{C}[\mathfrak{n}_i^+]$ as a $U(\mathfrak{g}_i)$-module in the following way. The restriction $\lambda|_{\mathfrak{p}_i}$ leads to the generalized Verma module $M(\lambda|_{\mathfrak{p}_i})=U(\mathfrak{g}_i)\otimes_{U(\mathfrak{p}_i)}\mathbf{C}_{\lambda|_{\mathfrak{p}_i}}$ which is isomorphic to $\mathbf{C}[\mathfrak{n}_i^+]$ as a vector space. We denote this representation of $U(\mathfrak{g}_i)$ on $\mathbf{C}[\mathfrak{n}_i^+]$ by $\Psi_{\lambda|_{\mathfrak{p}_i}}$. Note that this representation is not equivalent to the restriction of Ψ_λ to $U(\mathfrak{g}_i)$. By the same argument as in Lemma 6.4, we can show that $\lambda_i = -2\varpi_{i_0}$ on $\mathfrak{h}_i$, and therefore f_i is the relative invariant of $(L_i, \mathfrak{n}_i^+)$. The following definition is a generalization of Definition 6.6.

Definition 8.1. For $i \in \{1, \dots, r\}$, define polynomials b_i and $\beta_{\lambda,i}$ by

$$ {}^t f_i(\partial) f_i f_\mu = b_i(\mu) f_\mu, $$

$$ \Psi_{\lambda|_{\mathfrak{p}_i}}({}^t f_i f_i) f_\mu = \beta_{\lambda,i}(\mu) f_\mu, $$

for $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ and $\mu = k_1\lambda_1 + \cdots + k_i\lambda_i$ $(k_j \in \mathbf{C})$.

As in Theorem 7.1 we normalize f_i so that

$$ f_i(X_{\gamma_1} + \cdots + X_{\gamma_r}) = 1 \quad \text{for } i \in \{1, \dots, r\}, \tag{8.1} $$

and define a character ρ_i by

$$ \rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_{N,i}^+} \alpha \in \mathrm{Hom}(\mathfrak{p}_i, \mathbf{C}). $$

Since ρ_i is a constant multiple of ϖ_{i_0} on $\mathfrak{h}_i$, we define the complex number ρ_i^0 by

$$ \rho_i = \rho_i^0 \varpi_{i_0} \quad \text{on } \mathfrak{h}_i. \tag{8.2} $$

Since each $(L_i, \mathfrak{n}_i^+)$ is a regular prehomogeneous vector space, even if $(L, \mathfrak{n}^+)$ is not regular, Theorem 7.1 implies the following assertion.

Theorem 8.2. *Assume that f_j $(j \in \{1, \dots, r\})$ is normalized as* (8.1) *and that* $(\mathfrak{g}, \mathfrak{p})$ *is of commutative parabolic type, where the prehomogeneous vector space* $(L, \mathfrak{n}^+)$ *is not necessarily regular. We fix $i \in \{1, \dots, r\}$. Then for $\mu = k_1\lambda_1 + \cdots + k_i\lambda_i (k_j \in \mathbf{C})$,*

$$ \beta_{\lambda,i}(\mu) = (-1)^i b_i(\mu) b_i(\mu - (\lambda^0 + \rho_i^0)\lambda_i), $$

where λ^0 is the complex number defined by $\lambda|_{\mathfrak{p}_i} = \lambda^0 \varpi_{i_0}$.

9. An expression of $q_\lambda(\mu)$ in terms of b-function

In this section we give a corollary to the main theorem. The corollary is a part of our main results. It indicates that a contravariant form on a scalar generalized Verma module is deeply related to b-functions.

COROLLARY 9.1. *If f_i's are normalized by* (8.1), *then for* $\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r$ $(k_i \in \mathbf{Z}_{\geq 0})$,

$$q_\lambda(\mu) = (-1)^{\deg f_\mu} \prod_{i=1}^{r} \prod_{j=0}^{k_i-1} b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i - (\lambda^0 + \rho_i^0)\lambda_i).$$

PROOF. We can calculate $q_\lambda(\mu)$ using (5.1): $q_\lambda(\mu) = (f_\mu, f_\mu)_\lambda / (f_\mu, f_\mu)$. First we compute $(f_\mu, f_\mu)_\lambda$.

By Definition 4.5, the construction of $(\ ,\ )_\lambda$, we can compute $(f_\mu, f_\mu)_\lambda$ within the subalgebra $U(\mathfrak{g}_r)$ of $U(\mathfrak{g})$, since $f_\mu \in \mathbf{C}[\mathfrak{n}_r^+]$. In other words, $(f_\mu, f_\mu)_\lambda = (f_\mu, f_\mu)_{\lambda|_{\mathfrak{p}_r}}$, where $(\ ,\ )_{\lambda|_{\mathfrak{p}_r}}$ is the $\Psi_{\lambda|_{\mathfrak{p}_r}}(U(\mathfrak{g}_r))$-contravariant form on $\mathbf{C}[\mathfrak{n}_r^+]$ constructed in the same way as in Definition 4.5. Then we have

$$\begin{aligned}
(9.1) \qquad (f_\mu, f_\mu)_\lambda &= (f_\mu, f_\mu)_{\lambda|_{\mathfrak{p}_r}} \\
&= (f_r f_{\mu-\lambda_r}, f_r f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\
&= (\Psi_{\lambda|_{\mathfrak{p}_r}}(f_r) f_{\mu-\lambda_r}, \Psi_{\lambda|_{\mathfrak{p}_r}}(f_r) f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\
&= (\Psi_{\lambda|_{\mathfrak{p}_r}}({}^t f_r f_r) f_{\mu-\lambda_r}, f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\
&= \beta_{\lambda,r}(\mu - \lambda_r)(f_{\mu-\lambda_r}, f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\
&= \cdots \\
&= \beta_{\lambda,r}(\mu - \lambda_r) \cdots \beta_{\lambda,r}(\mu - k_r\lambda_r)(f_{\mu-k_r\lambda_r}, f_{\mu-k_r\lambda_r})_{\lambda|_{\mathfrak{p}_r}},
\end{aligned}$$

by Lemma 4.7 (3). Here $f_{\mu-k_r\lambda_r} = f_1^{k_1} \cdots f_{r-1}^{k_{r-1}} \in \mathbf{C}[\mathfrak{n}_{r-1}^+]$. Thus $(f_{\mu-k_r\lambda_r}, f_{\mu-k_r\lambda_r})_{\lambda|_{\mathfrak{p}_r}} = (f_{\mu-k_r\lambda_r}, f_{\mu-k_r\lambda_r})_{\lambda|_{\mathfrak{p}_{r-1}}}$ as before. Then we can apply Lemma 4.7 (3) again to (9.1), and at last we have

$$(f_\mu, f_\mu)_\lambda = \prod_{i=1}^{r} \prod_{j=0}^{k_i-1} \beta_{\lambda,i}(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i).$$

Similarly, it follows from Lemma 4.7 (4) that

$$(f_\mu, f_\mu) = \prod_{i=1}^{r} \prod_{j=0}^{k_i-1} b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i).$$

Then we have

$$q_\lambda(\mu) = (f_\mu f_\mu)_\lambda/(f_\mu, f_\mu)$$

$$= \prod_{i=1}^{r}\prod_{j=0}^{k_i-1} \beta_{\lambda,i}(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i)/b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i)$$

$$= \prod_{i=1}^{r}\prod_{j=0}^{k_i-1} (-1)^i b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i - (\lambda^0 + \rho_i^0)\lambda_i).$$

Here $\sum_{i=1}^{r}\sum_{j=0}^{k_i-1} i = \sum_{i=1}^{r} ik_i = \deg f_\mu$. Thus the corollary is proved. □

10. Another expression of $q_\lambda(\mu)$ in terms of b-function

We give some applications in sections from 11 to 13. In this section we show some formulas for $b_i(\mu)$ and $q_\lambda(\mu)$ which will be used in the later sections.

Set

(10.1) $$b_i(s) = b_i(s\lambda_i) \quad (s \in \mathbf{C}).$$

First we give a formula which expresses $b_r(\mu)$ in terms of $b_i(s)$ $(i \in \{1, \dots, r\})$. Note that this formula does not depend on the main theorems (Theorem 7.1, Theorem 8.2), and we will use it to prove the main theorems.

PROPOSITION 10.1. *For* $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{C})$,

$$b_r(\mu) = \frac{b_r(k_1 + \cdots + k_r)}{b_{r-1}(k_1 + \cdots + k_r)} \cdots \frac{b_2(k_{r-1} + k_r)}{b_1(k_{r-1} + k_r)} b_1(k_r).$$

In particular, $b_i(s)$ divides $b_{i+1}(s)$ and therefore $b_r(\mu)$ is a polynomial of degree i in k_i. Moreover the total degree of $b_r(\mu)$ in $k_1, \dots, k_r$ is equal to r.

PROOF. If we know the above equality, then it is easy to see that $b_i(s)$ divides $b_{i+1}(s)$, since $b_r(\mu)$ and $b_i(s)$ are polynomials. Then, in addition, it is obvious that $b_r(\mu)$ is a polynomial of degree i in k_i $(i \in \{1, \dots, r\})$, since the degree of $b_i(s)$ is equal to i. Moreover it follows that the total degree of $b_r(\mu)$ in $k_1, \dots, k_r$ is equal to r.

We therefore have only to prove the equality. For $i \in \{0, \dots, r-1\}$, let $\tilde{f}_i$ be a lowest weight vector of $I_{\lambda_{r-i}}$. The longest element of the Weyl group of $(\mathfrak{l}, \mathfrak{h})$ maps γ_i to γ_{r-i+1}, which can be proved by using Theorem 6.3. This fact implies that $\tilde{f}_i \in \mathbf{C}[\mathfrak{g}^{\Delta_{Q,i}^+}]$, where $\Delta_{Q,i}^+ = \{\alpha \in \Delta_N^+; \alpha|_{\mathfrak{h}^-} = (\gamma_k + \gamma_j)/2$ for some $i < j < k \le r\} \cup \{\gamma_{i+1}, \dots, \gamma_r\}$ and $\mathfrak{g}^A = \sum_{\alpha \in A} \mathfrak{g}^\alpha$ for a subset A in Δ.

Here we also normalize $\tilde{f}_i$'s so that

(10.2) $$\tilde{f}_i(X_{\gamma_1} + \cdots + X_{\gamma_r}) = 1 \qquad \text{for } i \in \{1, \dots, r\}.$$

Then we have

$$f_r|_{\mathfrak{n}_i^+ \oplus \mathfrak{g}^{\Delta_{Q,i}^+}}(x+y) = f_i(x)\tilde{f}_i(y) \quad \text{for } x \in \mathfrak{n}_i^+ \text{ and } y \in \mathfrak{g}^{\Delta_{Q,i}^+}.$$

Indeed, $f_r(x+y)$ is a relative invariant with respect to $\mathrm{Ad}(L_i)$-action as a function of $x \in \mathfrak{n}_i^+$, since $\mathrm{Ad}(L_i)$ acts trivially on $\mathfrak{g}^{\Delta_{Q,i}^+}$. It has a weight $\lambda_r|_{\mathfrak{h}_i}$, which is equal to $\lambda_i|_{\mathfrak{h}_i}$ by Theorem 6.3. Thus $f_r|_{\mathfrak{n}_i^+ \oplus \{y\}}$ is a scalar multiple of $f_i|_{\mathfrak{n}_i^+}$. Similarly, $f_r|_{\{x\} \oplus \mathfrak{g}^{\Delta_{Q,i}^+}}$ is a scalar multiple of $\tilde{f}_i|_{\mathfrak{g}^{\Delta_{Q,i}^+}}$. Thus we have the above equality thanks to the normalizations.

Here we show that

$$(10.3) \qquad {}^t\tilde{f}_i(\partial) f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1} = b_{r-i}(m) f_1^{k_1} \cdots f_{i-1}^{k_{i-1}} f_i^{k_i+1} f_r^m,$$

for $i \in \{1, \ldots, r-1\}$, $k_1, \ldots, k_i$, $m \in \mathbf{Z}_{\geq 0}$ (Rubenthaler-Schiffmann [20]). Let N_L be the nilpotent subgroup of L corresponding to the nilpotent subalgebra $\mathfrak{g}^{\Delta_{L_i} \cap \Delta^+}$ of $\mathfrak{l}$. Then both sides are $\mathrm{Ad}(N_L)$-stable, and they have the same weight. Thus they coincide up to a constant multiple. Let A be an affine space $X_{\gamma_1} + \cdots + X_{\gamma_i} + \mathfrak{g}^{\Delta_{Q,i}^+} \subset \mathfrak{n}^+$. Then we have

$$\begin{aligned}
{}^t\tilde{f}_i(\partial) f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1}|_A &= {}^t\tilde{f}_i(\partial)(f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1}|_A) \\
&= {}^t\tilde{f}_i(\partial)\tilde{f}_i^{m+1}|_A \\
&= b_{r-i}(m)\tilde{f}_i^m|_A \\
&= b_{r-i}(m) f_1^{k_1} \cdots f_{i-1}^{k_{i-1}} f_i^{k_i+1} f_r^m|_A.
\end{aligned}$$

We proved (10.3).

Applying the equality (10.3) repeatedly, we have

$${}^t\tilde{f}_{r-1}^{k_{r-1}}(\partial) \cdots {}^t\tilde{f}_1^{k_1}(\partial) f_r^{k_1+\cdots+k_r+1} = \prod_{i=1}^{r-1}\prod_{j=1}^{k_i} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_1^{k_1} \cdots f_r^{k_r} f_r.$$

Applying ${}^tf_r(\partial)$ to this equality, we get

$$\begin{aligned}
&b_r(k_1 + \cdots + k_r) \prod_{i=1}^{r-1}\prod_{j=0}^{k_i-1} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_\mu \\
&= b_r(\mu) \prod_{i=1}^{r-1}\prod_{j=1}^{k_i} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_\mu
\end{aligned}$$

Then the proposition is proved by comparing both sides. □

Second we show the formula for the b-function $b_i(s)$, although

Rubenthaler-Schiffmann [20] gives an intrinsic proof of the formula. We need two known lemmas to show the formula.

Lemma 10.2. *For $i \in \{1, \dots, r-1\}$, $I_{\lambda_1} I_{\lambda_i} \supset I_{\lambda_{i+1}}$.*

Proof. Let $\mathbf{C}^d[\mathfrak{n}^+]$ denote the homogeneous component of degree d of $\mathbf{C}[\mathfrak{n}^+]$. Then $I_{\lambda_{i+1}} \subset \mathbf{C}^{i+1}[\mathfrak{n}^+] = \mathbf{C}^i[\mathfrak{n}^+]\mathbf{C}^1[\mathfrak{n}^+] = \mathbf{C}^i[\mathfrak{n}^+] I_{\lambda_1}$. Thus there exists $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ such that $I_\mu \subset \mathbf{C}^i[\mathfrak{n}^+]$ and $I_{\lambda_{i+1}} \subset I_\mu I_{\lambda_1}$. Assume that $\mu \neq \lambda_i$. Then $\mu = k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} = -(m_1\gamma_1 + \cdots + m_{i-1}\gamma_{i-1})$, where $m_j = k_j + \cdots + k_r$. Any $\mathrm{ad}(U(\mathfrak{l}))$-maximal weight occurring in $I_\mu I_{\lambda_1}$ is a sum of μ and a weight of I_{λ_1}. Thus $\lambda_{i+1} = \mu + \alpha$ for some $\alpha \in \Delta_N^-$. Here $(\lambda_{i+1} - \mu)|_{\mathfrak{h}^-} = \{(m_1 - 1)\gamma_1 + \cdots + (m_{i-1} - 1)\gamma_{i-1} - \gamma_i - \gamma_{i+1}\}|_{\mathfrak{h}^-}$. This can not be equal to $\alpha|_{\mathfrak{h}^-}$ by Theorem 6.3. Thus μ must be λ_i and we prove the lemma. □

Lemma 10.3. *Let Y be the maximal submodule of $M(\lambda)$. For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{Z}_{\geq 0})$, I_μ occurs as a component of the irreducible decomposition of Y regarded as an $\mathrm{ad}(U(\mathfrak{l}))$-module, if and only of $q_\lambda(\mu) = 0$.*

Proof. It follows from Proposition 4.3 and Proposition 4.4 (2) that $I_\mu \subset Y$ if and only if $(I_\mu, I_\mu)_\lambda = 0$. Since I_μ is an irreducible $\mathrm{ad}(U(\mathfrak{l}))$-module, the nonzero contravariant form (,) defined in Definition 4.6 is nondegenerate on I_μ. Thus it follows from the definition of $q_\lambda(\mu)$ that $I_\mu \subset Y$ if and only if $q_\lambda(\mu) = 0$. □

Definition 10.4. For $1 \leq i < j \leq r$, define $c = \# \ \{\alpha \in \Delta_L \cap \Delta^+; \alpha|_{\mathfrak{h}^-} = (\gamma_j - \gamma_i)/2\}$. It is known that c is independent of i or j.

It is easily seen that $c = \#\{\alpha \in \Delta_N^+; \alpha|_{\mathfrak{h}^-} = (\gamma_j + \gamma_i)/2\}$ for $1 \leq i < j \leq r$. Then we can determine the constant ρ_i^0 defined in (8.2) in the same way as in Lemma 6.4:

$$\rho_i^0 = \frac{i-1}{2}c + 1. \tag{10.4}$$

Proposition 10.5. (Rubenthaler-Schiffmann [20], Wallach [28]) *For $i \in \{1, \dots, r\}$*

$$b_i(s) = d_i \prod_{j=0}^{i-1} \left(s + 1 + \frac{j}{2}c\right),$$

where $d_i \in \mathbf{C}^\times$ is constant.

Proof. Let Y be the maximal submodule of $M(\lambda)$. By Lemma 10.2 if $I_{\lambda_i} \subset Y$ then $I_{\lambda_{i+1}} \subset Y$ since $\Psi_\lambda(X)$ is a multiplying operator for

$X \in \mathfrak{n}^-$. Then it follows from Lemma 10.3 that $q_\lambda(\lambda_i) = 0$ implies $q_\lambda(\lambda_{i+1}) = 0$ for $i \in \{1, \dots, r-1\}$. By Corollary 9.1, if $b_i(-\lambda^0 - \rho_i^0) = 0$ then $b_{i+1}(-\lambda^0 - \rho_{i+1}^0) = 0$. Obviously $b_j(s)$ has $s+1$ in its factors for any j and it is known that $b_j(s)$ is of degree j in s. Since $b_1(s) = s+1$ up to constant, if $b_1(-\lambda^0 - \rho_1^0) = 0$ then $-\lambda^0 - \rho_1^0 + 1 = 0$. In this case, we have $b_2(-\lambda^0 - \rho_2^0) = 0$. Since $-\lambda^0 - \rho_2^0 + 1 \neq 0$, $b_2(-\lambda^0 - \rho_2^0)$ has a factor $-\lambda^0 - \rho_1^0 + 1$. Thus we have $b_1(-\lambda^0 - \rho_2^0) = (-\lambda^0 - \rho_1^0 + 1)(-\lambda^0 - \rho_2^0 + 1)$ up to constant. Inductively we have $b_i(-\lambda^0 - \rho_i^0) = (-\lambda^0 - \rho_1^0 + 1) \cdots (-\lambda^0 - \rho_i^0 + 1)$ for $i \in \{1, \dots, r\}$ up to constant multiple. The proposition is proved by replacing $-\lambda^0 - \rho_i^0$ by s and using (10.4) for ρ_j^0. □

We conclude this section with some consequences of Corollary 9.1, Proposition 10.1 and Proposition 10.5.

PROPOSITION 10.6. *For $i \in \{1, \dots, r\}$ and $\mu = k_1\lambda_1 + \cdots + k_i\lambda_i$ ($k_j \in \mathbf{C}$) we have*

$$b_i(\mu) = d_i \prod_{j=0}^{i-1} \left(k_{i-j} + \cdots + k_i + 1 + \frac{j}{2} c \right), \tag{10.5}$$

where d_i's are defined in Proposition 10.5.

PROPOSITION 10.7. *For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$) we have*

$$q_\lambda(\mu) = \left(\prod_{i=1}^{r} d_i^{k_i} \right) \prod_{i=0}^{r-1} \prod_{m=0}^{k_{i+1}+\cdots+k_r-1} \left(\frac{i}{2} c + \lambda^0 - m \right) \tag{10.6}$$

$$q_\lambda(\mu) = \prod_{i=1}^{r} \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \cdots + k_r)), \tag{10.7}$$

where d_i's are defined in Proposition 10.5.

11. Irreducibility criteria for scalar generalized Verma modules

In this section we consider two irreducibility criteria for scalar generalized Verma modules. One is in terms of b-functions and the other is in terms of contravariant forms. We see how these two criteria relate to each other through Corollary 9.1 or Proposition 10.7.

PROPOSITION 11.1. *For $\lambda = \lambda^0 \varpi_{i_0} \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$, the following are equivalent:*

(1) *The scalar generalized Verma module $M(\lambda)$ with a highest weight λ is irreducible.*

(2) $\prod_{i=1}^{r}\prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \cdots + k_r)) \neq 0$ *for all* $k_1, \dots, k_r \in \mathbf{Z}_{\geq 0}$.

(3) $b_r(\lambda^0 - m) \neq 0$ *for all* $m \in \mathbf{Z}_{>0}$.

Proof. By Lemma 10.3 and (10.7), is follows that (1) and (2) are equivalent. Next we assume (2). Then $b_i(\lambda^0 - m) \neq 0$ for all $i \in \{1, \dots, r\}$ and $m \in \mathbf{Z}_{>0}$ and therefore (3) follows. Conversely if we assume (3), then $b_i(\lambda^0 - m) \neq 0$ for all $i \in \{1, \dots, r\}$ and $m \in \mathbf{Z}_{>0}$, since b_i divides b_r. Then (2) follows. □

12. *b*-Functions and *L*-orbits on the nilpotent radical

In this section we consider $\mathrm{Ad}(L)$-orbits on $\mathfrak{n}^+$. The set of $\mathrm{Ad}(L)$-orbits on $\mathfrak{n}^+$ and the set of zeros of the b-function of the relative invariant are in one-to-one correspondence (Tanisaki [26]). We give another proof of the correspondence and it explains why there exists a correspondence between the orbits and the zeros of the b-function. We give another proof of the correspondence, which is intrinsic.

First we investigate $\mathrm{Ad}(L)$-orbits on $\mathfrak{n}^+$. For $i \in \{0, \dots, r\}$, set $C_i = V(I_{\lambda_{i+1}}) \setminus V(I_{\lambda_i})$, where $V(I_{\lambda_i}) = \{X \in \mathfrak{n}^+ \mid f(X) = 0 \text{ for all } f \in I_{\lambda_i}\}$, $V(I_{\lambda_0}) = \varnothing$ and $V(I_{\lambda_{r+1}}) = \mathfrak{n}^+$. Then we have $\overline{C_i} = V(I_{\lambda_{i+1}})$ and

$$\{0\} = \overline{C_0} \subset \cdots \subset \overline{C_r} = \mathfrak{n}^+, \tag{12.1}$$

where the overlines denote the Zariski closures. In fact, the disjoint union $\mathfrak{n}^+ = C_0 \cup \cdots \cup C_r$ is the $\mathrm{Ad}(L)$-orbit decomposition of $\mathfrak{n}^+$ (Goncharov [6, Proposition 3.5]).

Lemma 12.1. (1) $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i} = \bigoplus_\mu I_\mu$, *where* $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{Z}_{\geq 0})$ *runs over such* μ *satisfying* $k_i + \cdots + k_r > 0$.

(2) $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ *is a radical ideal of* $\mathbf{C}[\mathfrak{n}^+]$.

(3) *The defining ideal* $I(\overline{C_i})$ of $\overline{C_i}$ *is equal to* $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ for $i \in \{0, \dots, r-1\}$. *(See also Tanisaki [26, Proposition 1.5].)*

Proof. Set $R_i = \bigoplus_\mu I_\mu$ $(\mu = k_1\lambda_1 + \cdots + k_r\lambda_r, k_j \in \mathbf{Z}_{\geq 0}, k_i + \cdots + k_r > 0)$.

(1) $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ contains $f_i, \dots, f_r$ from Lemma 10.2. Thus $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ contains all the maximal weight vectors which appear in R_i. Since both sides are $\mathrm{ad}(U(\mathfrak{l}))$-stable, we have $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i} \supset R_i$.

On the other hand, assume that μ is an $\mathrm{ad}(U(\mathfrak{l}))$-maximal weight occurring in $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$. Then we can write

$$\begin{aligned} \mu &= k_1\lambda_1 + \cdots + k_r\lambda_r \quad (k_j \in \mathbf{Z}_{\geq 0}) \\ &= -m_1\gamma_1 - \cdots - m_r\gamma_r \quad (m_j = k_j + \cdots + k_r), \end{aligned}$$

by Theorem 5.2. Also μ is a sum of λ_i and a weight occurring in $\mathbf{C}[\mathfrak{n}^+]$, say ν. Here ν is a sum of roots in Δ_N^-, and therefore $\nu|_{\mathfrak{h}^-}$ is a certain sum of $-(1/2)\gamma_j$'s $(1 \le j \le r)$ by Theorem 6.3. Then we have that $m_i > 0$, or equivalently $k_i + \cdots + k_r > 0$, since μ has $\lambda_i = -(\gamma_1 + \cdots + \gamma_i)$ as its summand. Thus the maximal weight μ occurs in R_i.

(2) We show R_i is a radical ideal. Since R_i is $\mathrm{ad}(U(\mathfrak{l}))$-stable and this action is derived from $\mathrm{Ad}(L)$-action, its radical also $\mathrm{ad}(U(\mathfrak{l}))$-stable. If R_i is not a radical ideal then there exists $\nu = k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1}$ such that I_ν is contained in the radical of R_i. Then there exists $m \in \mathbf{Z}_{>0}$ such that $(f_\nu)^m \in R_i$, where f_ν is the highest weight vector in I_ν defined before. However $(f_\nu)^m \in I_{m\nu}$, indeed $(f_\nu)^m$ is the highest weight vector in $I_{m\nu}$. Since $I_{m\nu} \cap R_i = 0$, this is a contradiction, and we proved (2).

(3) $I(\overline{C_i}) = I(V(I_{\lambda_{i+1}})) \supset \mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$. Here $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ is a radical ideal by (2). Thus $I(\overline{C_i}) = \mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$. □

PROPOSITION 12.2. (Tanisaki [26]) *There exists a one-to-one correspondence between non-open* $\mathrm{Ad}(L)$*-orbits on* $\mathfrak{n}^+$ *and the zeros of the b-function* $b_r(s)$,

$$\begin{array}{ccc} (\textit{non-open } \mathrm{Ad}(L)\textit{-orbits on } \mathfrak{n}^+) & \to & (\textit{zeros of } b_r(s)) \\ C & \mapsto & \varphi(C) - 1, \end{array}$$

where $\varphi(C)$ *is the unique complex number* λ^0 *such that* $I(\bar{C}) \subset \mathbf{C}[\mathfrak{n}^+]$ *becomes the maximal submodule of* $M(\lambda^0\varpi_{i_0}) \simeq \mathbf{C}[\mathfrak{n}^+]$. *More concretely for* $i \in \{0, \ldots, r-1\}$, $\varphi(C_i) = a_{i+1} + 1$, *where* C_i *is the non-open* $\mathrm{Ad}(L)$*-orbit as in* (12.1) *and* a_{i+1} *is the unique zero of* $b_{i+1}(s)$ *which is not a zero of* $b_i(s)$.

PROOF. Let C_i and a_{i+1} be as above for $i \in \{0, \ldots, r-1\}$. Then the defining ideal $I(\overline{C_i})$ is equal to $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ by Lemma 12.1 (3). We assume that $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ is the maximal submodule of $M(\lambda)$ for some λ. Then it follows from Lemma 10.3 and Lemma 12.1 (1) that $q_\lambda(\mu) = 0$ if and only if $k_{i+1} + \cdots + k_r > 0$ for $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$. In particular, we have $q_\lambda(\lambda_i) \ne 0$ and $q_\lambda(\lambda_{i+1}) = 0$. Then by (10.7) we have $b_i(\lambda^0 - 1) \ne 0$ and $b_{i+1}(\lambda^0 - 1) = 0$, and therefore it follows that there exists at most one λ such that $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ becomes the maximal submodule of $M(\lambda)$. Namely, the unique possibility for such λ is given by $\lambda^0 = a_{i+1} + 1$.

Conversely we assume that $\lambda = (a_{j+1} + 1)\varpi_{i_0}$ for $j \in \{0, \ldots, r-1\}$. We find $a_{j+1} = -jc/2 - 1$ from Proposition 10.5. For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$, if $k_{j+1} + \cdots + k_r > 0$ then $q_\lambda(\mu)$ in (10.6) has a factor in which $i = j$ and $m = 0$, that is, $(jc/2 - jc/2 - 0)$. Thus $q_\lambda(\mu) = 0$. On the other hand, if $k_{j+1} = \cdots = k_r = 0$ then a factor $(ic/2 - jc/2 - m)$ in (10.6) occurs only when $i < j$. Thus all these factors are negative and therefore $q_\lambda(\mu) \ne 0$.

Then we have that $q_\lambda(\mu) = 0$ if and only if $k_{j+1} + \cdots + k_r > 0$ for

$\lambda^0 = a_{j+1} + 1$. In other words, the maximal submodule of $M(\lambda^0 \varpi_{i_0})$ is equal to $\mathbf{C}[\mathfrak{n}^+] I_{\lambda_{i+1}}$ for $\lambda^0 = a_{i+1} + 1$ by Lemma 12.1 (1). In this way we obtain the desired correspondence. □

13. The unitarizability of the irreducible quotient of $M(\lambda)$

Let $L(\lambda)$ be the irreducible quotient of $M(\lambda)$. In this section, we consider the unitarizability of $L(\lambda)$. This application is suggested by Professor Shuichi Suga. This problem is considered in Wallach [27], Parthasarathy [19], Garland-Zuckerman [5], Enright-Howe-Wallach [3], Enright-Joseph [13] and many other articles. It is known from these articles that the values of λ such that $L(\lambda)$ is unitarizable, are related to the zeros of b-functions. We will explain in terms of our main theorem, the reason why there exists such a relation.

We must consider real Lie algebras. We take the real form $\mathfrak{g}_0$ such that

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{l}_0 \oplus \mathfrak{n}_0 \quad \text{(Cartan decomposition)}, \\ (\mathfrak{g}_0)_{\mathbf{C}} &= \mathfrak{g}, \\ (\mathfrak{l}_0)_{\mathbf{C}} &= \mathfrak{l}, \\ (\mathfrak{n}_0)_{\mathbf{C}} &= \mathfrak{n}^+ \oplus \mathfrak{n}^-, \end{aligned}$$

where subscript $\mathbf{C}$ means the complexification.

When we work in the 'real' situation, the definitions in previous sections must be slightly modified.

DEFINITION 13.1. Define a conjugate linear anti-automorphism $\cdot^*$ of $U(\mathfrak{g})$ by

$$\begin{aligned} H_i^* &= H_i \quad (i \in \{1, \dots, n\}), \\ X_\alpha^* &= X_{-\alpha} \quad (\alpha \in \Delta_L), \\ X_\alpha^* &= -X_{-\alpha} \quad (\alpha \in \Delta_N). \end{aligned}$$

We extend $\cdot^*$ conjugate linearly to $\mathfrak{g}$ and anti-automorphically to $U(\mathfrak{g})$. Note that $\cdot^*$ is a composite of ${}^t\cdot$ and the Cartan involution. Note also that $\cdot^*$ is a complex conjugation by regarding $\mathfrak{g}_0$ as a purely imaginary part and $\sqrt{-1}\mathfrak{g}_0$ as a real part. See also Wallach [28], Garland-Zuckerman [5], Enright-Joseph [13] or Shapovalov [22].

LEMMA 13.2. *Under the normalization* (8.1), $f_\mu^* = (-1)^{\deg f_\mu}\, {}^t f_\mu$ *for* $\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r$ $(k_j \in \mathbf{Z}_{\geq 0})$.

PROOF. We have only to prove the lemma for $\mu = \lambda_i$ $(i \in \{1, \dots, r\})$, since $f_\mu = f_1^{k_1} \cdots f_r^{k_r}$. For $\alpha \in \Delta_L \cap \Delta^+$ we have $[X_{-\alpha}, f_i^*] = -[X_{-\alpha}^*, f_i]^* = -[X_\alpha, f_i]^* = 0$. Clearly the weight of f_i^* is equal to $-\lambda_i$ and therefore $f_i^* \in \mathbf{C}\,{}^t\!f_i$.

Set $X_+ = X_{\gamma_1} + \cdots + X_{\gamma_r}$ and $X_- = {}^tX_+ = X_{-\gamma_1} + \cdots + X_{-\gamma_r}$. We compute $f_i^*(X_-)$, where $f_i^* \in S(\mathfrak{n}^+)$ is regarded as a function on $\mathfrak{n}^-$. We expand f_i as a polynomial in $X_{-\alpha}(\alpha \in \Delta_N^+)$.

$$f_i = \sum_{B=(\beta_1,\dots,\beta_r)} a_B X_{-\beta_1} \cdots X_{-\beta_i} \quad (\beta_j \in \Delta_N^+),$$

where B runs over the set $\mathscr{B}_i$ which satisfies that $\{X_{-\beta_1} \cdots X_{-\beta_i} \mid B \in \mathscr{B}_i\}$ forms a basis for $\mathbf{C}^i[\mathfrak{n}^+]$, the homogeneous component of degree i. If $a_B X_{-\beta_1} \cdots X_{-\beta_i}(X_+) \neq 0$ then $(\beta_1, \dots, \beta_i)$ must be equal to $(\gamma_1, \dots, \gamma_i)$ up to order. Since $X_{-\gamma_j}(X_+) = 2(\gamma_1, \gamma_1)^{-1}$ and $f_i(X_+) = 1$, we have $a_{B_i} \in \mathbf{R}$, where $B_i = (\gamma_1, \dots, \gamma_i)$. Thus we have $f_i^*(X_-) = \sum_B (a_B X_{\beta_1} \cdots X_{\beta_i})^*(X_-) = \overline{a_{B_i}}(-X_{\beta_1}) \cdots (-X_{\beta_i})(X_-) = (-1)^i a_{B_i} X_{\beta_1} \cdots X_{\beta_i}\ (X_-) = (-1)^i\, {}^t\!f_i(X_-)$, and we proved the lemma. $\square$

We define 'real' analogues of b-functions $b_i(\mu)$ and $\beta_{\lambda,i}(\mu)$. See section 9 for the notation.

DEFINITION 13.3. For $i \in \{1, \dots, r\}$, define polynomials b_i^* and $\beta_{\lambda,i}^*$ by

$$f_i^*(\partial) f_i f_\mu = b_i^*(\mu) f_\mu,$$

$$\Psi_{\lambda|_{\mathfrak{p}_i}}(f_i^* f_i) f_\mu = \beta_{\lambda,i}^*(\mu) f_\mu,$$

for $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ and $\mu = k_1\lambda_1 + \cdots + k_i\lambda_i$ $(k_j \in \mathbf{C})$.

LEMMA 13.4. *For* $i \in \{1, \dots, r\}$ *and* $\mu = k_1\lambda_1 + \cdots + k_i\lambda_i$ $(k_j \in \mathbf{C})$,

$$b_i^*(\mu) = (-1)^i b_i(\mu),$$

$$\beta_{\lambda,i}^*(\mu) = (-1)^i \beta_{\lambda,i}(\mu).$$

PROOF. The lemma immediately follows from Lemma 13.2. $\square$

Next we consider a 'real' analogue of $q_\lambda(\mu)$.

DEFINITION 13.5. Let (π, V) be a representation of $\mathfrak{g}$.

(1) A Hermitian form $(\ ,\ )$ on V is called a *contravariant sesquilinear form* or a $\pi(U(\mathfrak{g}))$*-contravariant sesquilinear form* if $(\ ,\ )$ satisfies

$$(\pi(u)v, w) = (v, \pi(u^*)w) \tag{13.1}$$

for $u \in U(\mathfrak{g})$ and $v,\ w \in V$.

(2) We say V to be $\mathfrak{g}_0$-*infinitesimally unitary* if there exists a positive definite Hermitian form $(\ ,\ )$ on V such that

$$(\pi(X)v, w) = (v, -\pi(X)w) \tag{13.2}$$

for $X \in \mathfrak{g}_0$ and $v,\ w \in V$.

Note that if $(\ ,\ )$ satisfies (13.1) then it satisfies (13.2).

LEMMA 13.6. *Let V be a highest weight $U(\mathfrak{g})$-module with a highest weight $\nu \in \mathfrak{h}^*$.*

(1) *Contravariant sesquilinear forms on V are unique up to constant multiples.*

(2) *If there exists a nonzero contravariant sesquilinear form $(\ ,\ )$ on V, then $\nu \in \mathbf{R}\varpi_1 + \cdots + \mathbf{R}\varpi_n$, where ϖ_j's are fundamental weights.*

PROOF. (1) See Humphreys [11, §6], Wallach [27].

(2) Let $v_+ \in V$ be a highest weight vector. Then $\nu(H_i)(v_+, v_+) = (H_i.v_+, v_+) = (v_+, H_i^* v_+) = (v_+, H_i.v_+) = \overline{\nu(H_i)}(v_+, v_+)$, where H_i is the coroot of a simple root α_i, that is, an element of our fixed Chevalley basis. If $(v_+, v_+) \neq 0$ then $\nu(H_i) = \overline{\nu(H_i)}$ and we have $\nu \in \mathbf{R}\varpi_1 + \cdots + \mathbf{R}\varpi_n$. □

We assume that $\lambda \in \mathbf{R}\varpi_1 + \cdots + \mathbf{R}\varpi_n$, that is, $\lambda \in \mathbf{R}\varpi_{i_0}$.

DEFINITION 13.7. Define two bilinear form $(\ ,\ )_\lambda^*$ and $(\ ,\ )^*$ on $\mathbf{C}[\mathfrak{n}^+]$ by

$$(f, g)_\lambda^* = \varphi_\lambda(g^* f)$$

$$(f, g)^* = g^*(\partial) f(0)$$

for $f, g \in \mathbf{C}[\mathfrak{n}^+]$, where φ_λ is the same as in Definition 4.5. See also Enright-Joseph [13] as for $(\ ,\ )_\lambda^*$.

The following lemma gives important properties of these forms.

LEMMA 13.8. (1) *The bilinear form $(\ ,\ )^*$ is an $\mathrm{ad}(U(\mathfrak{l}))$-contravariant sesquilinear form.*

(2) *For $\lambda \in \mathbf{R}\varpi_{i_0}$, $(\ ,\ )_\lambda^*$ is a $\Psi_\lambda(U(\mathfrak{g}))$-contravariant sesquilinear form.*

(3) *The bilinear form $(\ ,\ )^*$ is positive definite on $\mathbf{C}^d[\mathfrak{n}^+]$ if d is even, and is negative definite if d is odd, where $\mathbf{C}^d[\mathfrak{n}^+]$ is a homogeneous component of degree d. Therefore $(-1)^{\deg f_\mu}(\ ,\ )^*$ is positive definite on I_μ.*

(4) *The radical of $(\ ,\ )_\lambda^*$ is equal to the maximal submodule of $M(\lambda)$.*

(5) $(fg, h)^* = (g, f^*(\partial)h)^*$ *for $f, g, h \in \mathbf{C}[\mathfrak{n}^+]$.*

PROOF. (1) It is easy to show that $(\ ,\ )^*$ is a Hermitian form. We have to show that $(\mathrm{ad}(u)f, g)^* = (f, \mathrm{ad}(u^*)g)^*$. This holds for $u \in \sum_i \mathbf{R}H_i + \sum_{\alpha \in \Delta_L}$

$\mathbf{R}X_\alpha$ and $f, g \in \sum_{d\in\mathbf{Z}_{\geq 0},\beta_j\in\Delta_N^+}(\sqrt{-1})^d\mathbf{R}X_{-\beta_1}\cdots X_{-\beta_d}$, since in this case ${}^t u = u^*$, ${}^t f = f^*$ and ${}^t g = g^*$. Thus it holds for all u, f, and g.

(2) It is easy to see that $(\ ,\ )_\lambda^*$ is a Hermitian form when $\lambda \in \mathbf{R}\varpi_{i_0}$. Then the assertion follows by the same argument as (1).

(3) The first assertion is clear from the definition of $\cdot^*$. Since $(\ ,\ )^*$ is nondegenerate on I_μ, the second one follows.

(4) We have the assertion by a similar argument to (1).

(5) is a direct consequence of the definition. □

Here we shall define the analogue of $q_\lambda(\mu)$. Define $q_\lambda^*(\mu)$ by

$$q_\lambda^*(\mu) = (f_\mu, f_\mu)_\lambda^* / (f_\mu, f_\mu)^*,$$

where $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{Z}_{\geq 0})$.

LEMMA 13.9. $q_\lambda^*(\mu) = q_\lambda(\mu)$.

PROOF.

$$\begin{aligned} q_\lambda^*(\mu) &= (f_\mu, f_\mu)_\lambda^* / (f_\mu, f_\mu)^* \\ &= \varphi_\lambda(f_\mu^* f_\mu) / f_\mu^*(\partial) f_\mu(0) \\ &= (-1)^{\deg f_\mu}\varphi_\lambda({}^t f_\mu f_\mu) / (-1)^{\deg f_\mu}\, {}^t f_\mu(\partial) f_\mu(0) \\ &= (f_\mu, f_\mu)_\lambda / (f_\mu, f_\mu) \\ &= q_\lambda(\mu). \end{aligned}$$

□

PROPOSITION 13.10. *The following conditions are equivalent*:

(1) *The irreducible quotient $L(\lambda)$ of $M(\lambda)$ is infinitesimally unitary.*

(2) $(\ ,\ )_\lambda^*$ *is nonnegative definite.*

(3) *For all $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ $(k_j \in \mathbf{Z}_{\geq 0})$, $(-1)^{\deg f_\mu} q_\lambda^*(\mu) \geq 0$.*

(4) $\lambda^0 = a_i + 1$ $(i \in \{1, \ldots, r\})$ *or* $\lambda^0 < a_r + 1$, *where λ^0 is the complex number determined by $\lambda = \lambda^0\varpi_{i_0}$ and a_i is the unique zero of $b_i(s)$ which is not a zero of $b_{i-1}(s)$.*

PROOF. The contravariant sesquilinear form $(\ ,\ )_\lambda^*$ on $M(\lambda)$ induces a nonzero contravariant sesquilinear form on $L(\lambda)$. This induced form on $L(\lambda)$ can not be negative definite, since $(1,1)_\lambda^* = 1$. By Lemma 13.6 (1), any contravariant sesquilinear form on $L(\lambda)$ is its constant multiple. Thus $L(\lambda)$ is infinitesimally unitary if and only if the induced form on $L(\lambda)$ is positive definite. Then by Lemma 13.8 (4), (1) and (2) are equivalent.

By Lemma 13.8 (3) and the definition of $q_\lambda^*(\mu)$, (2) and (3) are equivalent.

Next we show that (3) implies (4). By Lemma 13.8 (3) we have $(-1)^i(f_i, f_i)^* > 0$. Then $0 < (-1)^i(f_i^*(\partial)f_i, 1)^* = (-1)^i b_i^*(0) = b_i(0) = d_i \prod_{j=0}^{i-1} (1 + jc/2)$, by Proposition 10.5. Here $c \in \mathbf{Z}_{\geq 0}$ and therefore we have $d_i > 0$.

Thus (3) says

$$(-1)^{\deg f_\mu} \prod_{i=1}^{r} \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \cdots + k_r)) \geq 0$$

for all $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$). In Particular, if $\mu = \lambda_t$ then $(-1)^t b_t (\lambda^0 - 1) \geq 0$ for $t \in \{1, \ldots, r\}$. Thus if $\lambda^0 \neq a_i + 1$ for all $i \in \{1, \ldots, r\}$ then

$$0 < (-1)^t d_t \prod_{j=0}^{t-1} \left(\lambda^0 + \frac{j}{2}c\right) = d_t \prod_{j=0}^{t-1} \left(-\lambda^0 - \frac{j}{2}c\right)$$

for all t. Here $d_i > 0$ and $c \in \mathbf{Z}_{\geq 0}$, and therefore we have $\lambda^0 < -(r-1) c/2 = a_r + 1$. We proved that (3) implies (4).

Lastly we show that (4) implies (3). By (10.6)

$$(13.3) \qquad (-1)^{\deg f_\mu} q_\lambda(\mu) = \prod_{i=0}^{r-1} \prod_{m=0}^{k_{i+1}+\cdots+k_{r-1}} \left(-\lambda^0 - \frac{i}{2}c + m\right)$$

up to positive constant multiple. If $\lambda^0 < a_r + 1 = -(r-1)c/2$ then $-\lambda^0 > (r-1)c/2$ and therefore all the factors appearing on the right hand side of (13.3) is nonnegative. Namely (3) is satisfied. Next we assume that $\lambda^0 = a_t + 1$ for some $t \in \{1, \ldots, r\}$. If a factor $(-\lambda^0 - ic/2 + m)$ occurs in (13.3) for $i \geqq t-1$ then a factor $(-\lambda^0 - (t-1)c/2 + 0)$ also occurs. In this case (13.3) is equal to zero and (3) is satisfied. If all the factors $(-\lambda^0 - ic/2 + m)$ in (13.3) satisfy that $i < t-1$, then every factor is positive, and then (3) is satisfied. Thus we have shown that (3) implies (4). We proved the proposition. □

14. Factors contained in $\beta_{\lambda,i}(\mu)$

The sections 14 and 15 are devoted to proving the main theorems (Theorem 7.1 and Theorem 8.2). We have only to prove Theorem 7.1 since Theorem 8.2 is a direct consequence of Theorem 7.1. Thus we may assume that the prehomogeneous vector space $(L, \mathfrak{n}^+)$ is regular.

The proof of the theorem requires several steps. First, we show that $b_r(\mu)$ divides $\beta_{\lambda,r}(\mu)$ in the ring $\mathbf{C}[k_1, \ldots, k_r, \lambda^0]$. Second, we show $\beta_{\lambda,r}(\mu) = \beta_{-\lambda-2\rho_r,r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ by using Boe's theorem. Then we know $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ divides $\beta_{\lambda,r}(\mu)$ in the ring $\mathbf{C}[k_1, \ldots, k_r, \lambda^0]$. Thus $b_r(\mu)b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ divides $\beta_{\lambda,r}(\mu)$. Third, we show that $\beta_{\lambda,r}(\mu) =$

$b_r(\mu)b_r(\mu-(\lambda^0+\rho_r^0)\lambda_r)$ up to constant multiple. Lastly, we calculate the principal symbol of $\Psi_\lambda({}^tf_rf_r)$ to determine the constant.

LEMMA 14.1. *Let $D_{\mathfrak{n}^+}$ be the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. Then $\Psi_\lambda({}^tf_r)=P_\lambda{}^tf_r(\partial)$ for some $P_\lambda\in D_{\mathfrak{n}^+}^L$, where $D_{\mathfrak{n}^+}^L$ is the subspace of L-invariant elements in $D_{\mathfrak{n}^+}$.*

PROOF. In general, let $g^*\in S(\mathfrak{n}^+)$ and $h\in D_{\mathfrak{n}^+}\simeq\mathbf{C}[\mathfrak{n}^+]\otimes S(\mathfrak{n}^+)$ be relative invariants with respect to the $\mathrm{Ad}(L)$-action, and we assume that they correspond to the same character $\chi\in\mathrm{Hom}(L,\mathbf{C}^\times)$. Then h is a sum of several relative invariants $h_i\in I_{\mu_i}\otimes{}^tI_{\nu_i}$ which correspond to the same character χ, where ${}^tI_\nu=\{{}^tf\mid f\in\mathbf{C}[\mathfrak{n}^+]\}\subset S(\mathfrak{n}^+)$.

Set $g={}^t(g^*)\in\mathbf{C}[\mathfrak{n}^+]$. Then $\mathbf{C}gh_i\subset gI_{\mu_i}\otimes{}^tI_{\nu_i}$ is a trivial $\mathrm{Ad}(L)$-submodule. Namely, $gI_{\mu_i}\otimes{}^tI_{\nu_i}$ contains an $\mathrm{Ad}(L)$-invariant nonzero element. Thus it follows from Schur's lemma that gI_{μ_i} is the dual module of ${}^tI_{\nu_i}$, since both gI_{μ_i} and ${}^tI_{\nu_i}$ are irreducible. Therefore we have ${}^tI_{\nu_i}={}^t(gI_{\mu_i})={}^tI_{\mu_i}g^*$, since $\mathbf{C}[\mathfrak{n}^+]$ is multiplicity free.

Thus $h_i\in I_{\mu_i}\otimes{}^tI_{\nu_i}=I_{\mu_i}\otimes{}^tI_{\mu_i}g^*$ and there exists $P_i\in D_{\mathfrak{n}^+}$ such that $h_i=P_ig^*$. Here h_i and g^* have the same character, and therefore P_i is $\mathrm{Ad}(L)$-invariant. Then $h=(\sum P_i)g^*$, where $\sum P_i$ is an $\mathrm{Ad}(L)$-invariant.

Finally, we take ${}^tf_r(\partial)$ and $\Psi_\lambda({}^tf_r)$ as g^* and h, respectively, to prove the lemma. □

We have the following proposition from Lemma 14.1 and the definition of $\beta_{\lambda,r}$ and b_r. This is the goal of the first step.

PROPOSITION 14.2. *In $\mathbf{C}[k_1,\dots,k_r]$, $b_r(\mu)$ divides $\beta_{\lambda,r}(\mu)$ $(\mu=k_1\lambda_1+\cdots+k_r\lambda_r)$.*

Next, we show that $\beta_{\lambda,r}$ have another factor related to b_r. We use the theorem of Boe [1].

THEOREM 14.3. (Boe [1, Theorem 4.4]) *If $(\mathfrak{g},\mathfrak{p})$ is of commutative parabolic type and $(L,\mathfrak{n}^+)$ is a regular prehomogeneous vector space, then for λ, $\lambda'\in\mathrm{Hom}(\mathfrak{p},\mathbf{C})$ the necessary and sufficient condition for $\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda'),M(\lambda))$ to be nonzero is that $\lambda=\lambda'$ or $\lambda=l\varpi_{i_0}-\rho_r$ and $\lambda'=-l\varpi_{i_0}-\rho_r$ for some $l\in\mathbf{Z}_{>0}$.*

If $\lambda^0+\rho_r^0\in\mathbf{Z}_{\geq 0}$ then $\mathrm{Hom}(M(-\lambda-2\rho_r),M(\lambda))$ contains the mapping

$$\mathbf{C}[\mathfrak{n}^+]\to\mathbf{C}[\mathfrak{n}^+]$$

$$f\mapsto ff_r^{\lambda^0+\rho_r^0}$$

by the proof of the theorem. Thus it follows that

$$(14.1) \qquad f_r^{\lambda^0+\rho_r^0}\Psi_{-\lambda-2\rho_r}(u) = \Psi_\lambda(u)f_r^{\lambda^0+\rho_r^0} \quad (u \in U(\mathfrak{g})),$$

if $\lambda^0 + \rho_r^0 \in \mathbf{Z}_{\geq 0}$. In fact, this equality holds even if $\lambda^0 + \rho_r^0$ is a complex number.

Now we can prove that $\beta_{\lambda,r}$ have another factor related to b_r.

PROPOSITION 14.4. *The equality* $\beta_{\lambda,r}(\mu) = \beta_{-\lambda-2\rho_r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ *holds, and therefore* $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ *divides* $\beta_{\lambda,r}(\mu)$ *in* $\mathbf{C}[k_1, \dots, k_r, \lambda^0]$.

PROOF. We have $\Psi_\lambda({}^t f_r f_r)f_\mu = \beta_{\lambda,r}(\mu)f_\mu$ by the definition of $\beta_{\lambda,r}$. On the other hand, by (14.1), $\Psi_\lambda({}^t f_r f_r)f_\mu = f_r^{\lambda^0+\rho_r^0}\Psi_{-\lambda-2\rho_r}({}^t f_r f_r)f_r^{-\lambda^0-\rho_r^0}f_\mu = \beta_{-\lambda-2\rho_r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r)f_\mu$. This proves the first statement of the proposition. The second statement follows from Proposition 14.2. □

Next we get a proposition which proves the third step.

LEMMA 14.5. $\beta_{\lambda,r}(\mu) = b_r(\mu)b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$ *(up to constant multiple).*

PROOF. We have that $\beta_{\lambda,r}(\mu)$ has factors $b_r(\mu)$ and $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ from Proposition 14.2 and Proposition 14.4. Here $b_r(\mu)$ and $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ are prime each other, since all irreducible factors in $b_r(\mu)$ are different from that in $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$. Thus $\beta_{\lambda,r}(\mu)$ is divisible by $b_r(\mu)b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$. Then by Proposition 10.1 the total degree of $\beta_{\lambda,r}(\mu)$ in $k_1, \dots, k_r$ is at least $2r$, and the degree in λ^0 is at least r.

On the other hand, it follows from Lemma 3.2 that the operator $\Psi_\lambda({}^t f_r)$ is of order at most $2r$ and of degree at most r in λ^0, since ${}^t f_r$ is of degree r. Thus we have that the total degree of $\beta_{\lambda,r}(\mu)$ in $k_1, \dots, k_r$ is at most $2r$, and the degree of λ^0 is at most r. Then $\beta_{\lambda,r}(\mu)$ must be a constant multiple of $b_r(\mu)b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$, and we get the proposition. □

15. The principal symbol of $\Psi_\lambda({}^t f_r)$

In this section, we determine the constant which appeared in Lemma 14.5 and prove our main theorem. To determine it, we show that the principal symbol of $\Psi_\lambda({}^t f_r)$ and that of $f_r\,{}^t f_r(\partial)\,{}^t f_r(\partial)$ coincide up to a certain constant multiple.

First, we write f_i and ${}^t f_i$ in polynomials in root basis. We give an arbitrary total order to Δ_N^+ satisfying $\gamma_1 < \cdots < \gamma_r$. Set $\mathscr{B}_i = \{(\beta_1, \dots, \beta_i) \mid \beta_j \in \Delta_{N,i}^+, \beta_j \leq \beta_{j+1}\}$. We use this order only to define $\mathscr{B}_i$.

For $i \in \{1, \dots, r\}$, define $a_B \in \mathbf{C}(B \in \mathscr{B}_i)$ by

$$f_i = \sum_{B \in \mathscr{B}_i} a_B X_{-\beta_1} \cdots X_{-\beta_i} \quad (B = (\beta_1, \dots, \beta_i)).$$

Here a_B is uniquely determined thanks to the order. Then obviously we have

$$ {}^t f_i = \sum_{B \in \mathscr{B}_i} a_B X_{\beta_1} \cdots X_{\beta_i} \quad (B = (\beta_1, \ldots, \beta_i)). $$

We can determine special a_B's thanks to the normalization (8.1). Set $B_i = (\gamma_1, \ldots, \gamma_i)$. Here B_i is the unique element in $\mathscr{B}_i$ which consists of only γ_j's and makes a_B nonzero. We denote the Killing form on $\mathfrak{g}$ by $\langle\, ,\, \rangle$. Then we have

$$ \begin{aligned} (15.1) \qquad 1 &= f_i(X_{\gamma_1} + \cdots + X_{\gamma_i}) \\ &= \sum_{B \in \mathscr{B}_i} a_B \langle X_{-\beta_1}, X_{\gamma_1} + \cdots + X_{\gamma_i} \rangle \cdots \langle X_{-\beta_i}, X_{\gamma_1} + \cdots + X_{\gamma_i} \rangle. \end{aligned} $$

Here in the nonzero summands of (15.1), each β_j must be equal to some γ_k. By the property of B_i, (15.1) is equal to

$$ \begin{aligned} a_{B_i} \langle X_{-\gamma_1}, X_{\gamma_1} \rangle \cdots \langle X_{-\gamma_i}, X_{\gamma_i} \rangle &= a_{B_i} \frac{2}{(\gamma_1, \gamma_1)} \cdots \frac{2}{(\gamma_i, \gamma_i)} \\ &= 2^i a_{B_i} (\gamma_1, \gamma_1)^{-i}, \end{aligned} $$

where we used (4.1). Thus we have

$$ a_{B_i} = 2^{-i} (\gamma_1, \gamma_1)^i. $$

Next, we introduce the principal symbol of a differential operator. Let $D_{\mathfrak{n}^+}$ be the ring of polynomial coefficient differential operators, and $D^d_{\mathfrak{n}+}$ the subspace of $D_{\mathfrak{n}^+}$ consisting of operators of order at most d. We define a linear mapping $\sigma_d : D^d_{\mathfrak{n}^+} \to \mathbf{C}[\mathfrak{n}^+ \oplus \mathfrak{n}^-] \simeq \mathbf{C}[\mathfrak{n}^+] \otimes \mathbf{C}[\mathfrak{n}^-]$ as follows: If the dth order part of $P \in D^d_{\mathfrak{n}^+}$ is a certain sum of $g \partial / \partial X_{-\beta_1} \cdots \partial / \partial X_{-\beta_d}$ $(g \in \mathbf{C}[\mathfrak{n}^+], \beta_j \in \Delta^+_N)$, then $\sigma_d(P)$ is the sum of $g \xi_{\beta_1} \cdots \xi_{\beta_d}$. Where $\xi_\beta \in \mathbf{C}[\mathfrak{n}^-]$ is the linear mapping defined by

$$ \xi_\beta(X_{-\delta}) = \frac{\partial}{\partial X_{-\beta}} (X_{-\delta}) = \begin{cases} 1 & (\delta = \beta) \\ 0 & (\delta \neq \beta) \end{cases} \quad \text{for } \delta \in \Delta^+_N. $$

In particular, for $P \in D_{\mathfrak{n}^+}$ of order d, $\sigma_d(P)$ is the principal symbol of P.

PROPOSITION 15.1. *$\sigma(\Psi_\lambda({}^t f_r)) = (-1)^r \sigma(f_r {}^t f_r(\partial) {}^t f_r(\partial))$, under the normalization* (8.1).

PROOF. [Step 1] First we show that $\Psi_\lambda({}^t f_r)$ is a differential operator of order $2r$, namely that, $\sigma_{2r}(\Psi_\lambda({}^t f_r)) \neq 0$.

Set $X_+ = X_{\gamma_1} + \cdots + X_{\gamma_r}$ and $X_- = {}^tX_+ = X_{-\gamma_r} + \cdots + X_{-\gamma_r}$. Symbols can be considered as polynomial functions on the cotangent bundle of $\mathfrak{n}^+$, which can be identified with $\mathfrak{n}^+ \times \mathfrak{n}^-$. Then we evaluate $\sigma_{2r}(\Psi_\lambda({}^tf_r))$ at (X_+, X_-) and we have

$$
\begin{aligned}
(15.2) \qquad & \sigma_{2r}(\Psi_\lambda({}^tf_r))(X_+, X_-) \\
&= \sum_{B=(\beta_1,\dots,\beta_r)\in\mathscr{B}_r} a_B\sigma_{2r}(\Psi_\lambda(X_{\beta_1}\cdots X_{\beta_r}))(X_+, X_-) \\
&= \sum_{B\in\mathscr{B}_r} a_B\left\{\frac{1}{2}\sum_{\delta,\eta\in\Delta_N^+}[[X_{\beta_1}, X_{-\delta}], X_{-\eta}]\xi_\delta\xi_\eta\right\} \\
&\qquad \cdots\left\{\frac{1}{2}\sum_{\delta,\eta\in\Delta_N^+}[[X_{\beta_r}, X_{-\delta}], X_{-\eta}]\xi_\delta\xi_\eta\right\}(X_+, X_-),
\end{aligned}
$$

where we adopted the basis $\{X_\delta \mid \delta \in \Delta_N^+\}$ as $\{F_k\}$ in Lemma 3.2, since $\{F_k\}$ is any basis of $\mathfrak{n}^-$ there.

Here we compute the jth factor of (15.2).

$$
\begin{aligned}
&\left\{\frac{1}{2}\sum_{\delta,\eta\in\Delta_N^+}[[X_{\beta_j}, X_{-\delta}], X_{-\eta}]\xi_\delta\xi_\eta\right\}(X_+, X_-) \\
&\quad = \frac{1}{2}\sum_{\delta,\eta}\langle[[X_{\beta_j}, X_{-\delta}], X_{-\eta}], X_+\rangle\xi_\delta(X_-)\xi_\eta(X_-) \\
&\quad = \frac{1}{2}\sum_{k,l=1}^{r}\langle[[X_{\beta_j}, X_{-\gamma_k}], X_{-\gamma_l}], X_+\rangle \\
&\quad = \frac{1}{2}\sum_{k,l=1}^{r}\langle[X_{\beta_j}, X_{-\gamma_k}], -H_{\gamma_l}\rangle \\
&\quad = -\sum_{k=1}^{r}\langle X_{\beta_j}, X_{-\gamma_k}\rangle,
\end{aligned}
$$

where H_{γ_l} is the coroot of γ_l, that is, $H_{\gamma_l} = [X_{\gamma_l}, X_{-\gamma_l}]$. This is equal to zero if $\beta_j \notin \{\gamma_1, \dots, \gamma_r\}$. If $\beta_j = \gamma_m$ for some m, then $\langle X_{\beta_j}, X_{-\gamma_m}\rangle = 2/(\gamma_m, \gamma_m) = 2/(\gamma_1, \gamma_1)$.

Therefore the summand of (15.2) which does not vanish, is given only by $B = B_r = (\gamma_1, \dots, \gamma_r)$, and we have

$$(15.3)\quad \sigma_{2r}(\Psi_\lambda({}^t f_r))(X_+, X_-) = a_{B_r}\left\{-\sum_{k=1}^{r}\langle X_{\beta_1}, X_{-\gamma_k}\rangle\right\}\cdots\left\{-\sum_{k=1}^{r}\langle X_{\beta_r}, X_{-\gamma_k}\rangle\right\}$$
$$= 2^{-r}(\gamma_1,\gamma_1)^r\cdot\{-2/(\gamma_1,\gamma_1)\}^r$$
$$= (-1)^r.$$

Moreover we find that $\delta=\eta=\gamma_j$ in the jth factor of the summand of (15.2) which does not vanish.

[Step 2] By Lemma 14.1, there exists $\mathrm{Ad}(L)$-invariant operator $P_\lambda\in D_{\mathfrak{n}^+}^L$ such that $\Psi_\lambda({}^t f_r)=P_\lambda{}^t f_r(\partial)$. Here P_λ is of order r by Step 1. In this step, we show that $\sigma_r(P_\lambda)$ is a certain constant multiple of $\sigma_r(f_r{}^t f_r(\partial))$.

Let $(I_\mu\otimes{}^t I_\mu)^L$ denote the subspace of $I_\mu\otimes{}^t I_\mu$ consisting of $\mathrm{Ad}(L)$-invariant elements. By Schur's lemma, it is one-dimensional and spanned by $g_1\otimes g_1^*+g_2\otimes g_2^*+\cdots$, where $\{g_j\}$ is a basis of I_μ and $\{g_j^*\}$ is the dual basis with respect to $\langle\,,\,\rangle$ which appeared in Definition 4.6. Here we may assume that all g_j's are weight vectors, and that g_1 is the highest weight vector f_μ, and we put $f_\mu^*=g_1^*$. We embed $I_\mu\otimes{}^t I_\mu$ in $D_{\mathfrak{n}^+}$ by $g\otimes P\mapsto gP(\partial)$ as before. Then $D_{\mathfrak{n}^+}^L=\bigoplus_\mu(I_\mu\otimes{}^t I_\mu)^L$ and we can write $P_\lambda=\sum_\mu z_\mu$ $(z_\mu\in(I_\mu\otimes{}^t I_\mu)^L)$. Since the order of P_λ is equal to r, we have

$$P_\lambda=\sum_{\deg f_\mu\le r} z_\mu.$$

Moreover

$$\sigma_r(P_\lambda)=\sum_{\deg f_\mu=r}\sigma_r(z_\mu).$$

We have only to show that $\sigma_r(z_\mu)$ is equal to zero if $\deg f_\mu=r$ and $\mu\ne\lambda_r$, since $\dim I_{\lambda_r}=1$ implies that $z_{\lambda_r}\in\mathbf{C}f_r{}^t f_r(\partial)$.

A summand of $\sigma_r(P_\lambda)$ has a form $gQ(\partial)$, where $g\in\mathbf{C}[\mathfrak{n}^+]$ is a polynomial of degree r and $Q\in\mathbf{C}[\mathfrak{n}^-]$ is a monomial of degree r. We may assume that g and Q are weight vectors. We call such g a coefficient polynomial. Let ν_0 be a maximal element among the weights of coefficient polynomials in $\sigma_r(P_\lambda)$. Then ν_0 is an $\mathrm{Ad}(L)$-maximal weight occurring in $\mathbf{C}[\mathfrak{n}^+]$ by the maximality, since $\sigma_r(P_\lambda)$ is a sum of $\sigma_r(z_\mu)=f_\mu f_\mu^*(\partial)+g_2g_2^*(\partial)+\cdots$. In particular, all the terms of $\sigma_r(P_\lambda)$ in which the weight of the coefficient polynomial is ν_0, come from $\mathbf{C}f_{\nu_0}f_{\nu_0}^*=\mathbf{C}f_{\nu_0}{}^t f_{\nu_0}$. Let $P_0\in D_{\mathfrak{n}^+}$ be the sum of terms of P_λ in which the weight of the coefficient polynomial is equal to ν_0. Here $P_0\ne 0$ by the property of ν_0. Then we have $P_0=sf_{\nu_0}{}^t f_{\nu_0}(\partial)$ $(s\in\mathbf{C}^\times)$. Here we compute $\sigma(P_0{}^t f_r(\partial))\cdot(X_+,X_-)$ in two different ways.

First, we define h_j by $\nu_0 = h_1\lambda_1 + \cdots + h_r\lambda_r$. Then $f_{\nu_0} = f_1^{h_1} \cdots f_r^{h_1}$ and it follows from the normalization (8.1) that

$$\begin{aligned}(15.4)\qquad \sigma_{2r}(P_0{}^t f_r(\partial))(X_+, X_-) &= \sigma_{2r}(s f_{\nu_0}{}^t f_{\nu_0}(\partial){}^t f_r(\partial))(X_+, X_-)\\ &= s f_{\nu_0}(X_+){}^t f_{\nu_0}(X_-){}^t f_r(X_-)\\ &= s \neq 0,\end{aligned}$$

where ${}^t f_{\nu_0}$ and ${}^t f_r \in S(\mathfrak{n}^+) \simeq \mathbf{C}[\mathfrak{n}^-]$ are regarded as functions on $\mathfrak{n}^-$.

Second, we compute $\sigma(P_0{}^t f_r(\partial))(X_+, X_-)$ by using the explicit formula for $\Psi_\lambda({}^t f_r)$. In the formula

$$\begin{aligned}(15.5)\qquad \sigma_{2r}(\Psi_\lambda({}^t f_r)) = \sum_{B=(\beta_1,\ldots,\beta_r) \in \mathscr{B}_r} a_B &\left\{\frac{1}{2} \sum_{\delta_1,\eta_1 \in \Delta_N^+} [[X_{\beta_1}, X_{-\delta_1}], X_{-\eta_1}]\xi_{\delta_1}\xi_{\eta_1}\right\}\\ \cdots &\left\{\frac{1}{2} \sum_{\delta_r,\eta_r \in \Delta_N^+} [[X_{\beta_r}, X_{-\delta_r}], X_{-\eta_r}]\xi_{\delta_r}\xi_{\eta_r}\right\},\end{aligned}$$

the terms of $\sigma_{2r}(P_0{}^t f_r(\partial))$ precisely correspond to the sum of the terms in (15.5) in which the coefficient polynomials have the weight ν_0, that is, in which $\sum_j(\beta_j - \delta_j - \eta_j) = \nu_0$ or equivalently $\sum_j(\delta_j + \eta_j) = -\nu_0 - \lambda_r$. Thus we have

$$\begin{aligned}\sigma_{2r}(P_0{}^t f_r(\partial)) = \sum_B a_B 2^{-r} &\sum_{\substack{\delta_j,\eta_j \in \Delta_N^+\\ \sum(\delta_j+\eta_j)=-\nu_0-\lambda_r}} [[X_{\beta_1}, X_{-\delta_1}], X_{-\eta_1}] \cdots\\ &\cdots [[X_{\beta_r}, X_{-\delta_r}], X_{-\eta_r}]\xi_{\delta_1}\xi_{\eta_1}\cdots\xi_{\delta_r}\xi_{\eta_r}.\end{aligned}$$

As is stated in the last paragraph of Step 1, the nonzero summand of $\sigma_{2r}(\Psi_\lambda({}^t f_r))(X_+, X_-)$ is given only by $B = B_r$ and $\delta_j = \eta_j = \gamma_j$ for all j. Thus the nonzero summand of $\sigma_{2r}(P_0{}^t f_r(\partial))(X_+, X_-)$ also satisfies $B = B_r$, and $\delta_j = \eta_j = \gamma_j$. Such a summand occurs only if $\nu_0 = \lambda_r$, since $\sum_j(\delta_j + \eta_j) = -\nu_0 - \lambda_r$. By (15.4) $\sigma_{2r}(P_0{}^t f_r(\partial))(X_+, X_-)$ is nonzero and ν_0 must be equal to λ_r. Then we proved this step, since all the $\mathrm{Ad}(L)$-maximal weight satisfying $\deg f_\mu = r$ are equal to or higher than λ_r.

[Step 3] At last, we can prove the proposition. We have

$$\sigma_{2r}(f_r{}^t f_r(\partial){}^t f_r(\partial))(X_+, X_-) = f_r(X_+){}^t f_r(X_-){}^t f_r(X_-) = 1.$$

We combine this with (15.3), and obtain the proposition. □

Now we can prove Theorem 7.1. For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$, there exists a complex number a by Lemma 14.5 such that

$$\Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda,r}(\mu) f_\mu = a\, b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r) f_\mu, \tag{15.6}$$

and obviously we have

$$({}^t f_r(\partial) f_r\, {}^t f_r(\partial) f_r) f_\mu = b_r(\mu)^2 f_\mu. \tag{15.7}$$

Here $a \neq 0$. Indeed, if $a = 0$, then $\Psi_\lambda({}^t f_r f_r) f_\mu = 0$ for all μ, and $\Psi_\lambda({}^t f_r f_r) = 0$ as an operator, since $\Psi_\lambda({}^t f_r f_r)$ commutes with the $\mathrm{Ad}(L)$-action and f_μ's generate $\mathbf{C}[\mathfrak{n}^+]$ as an $\mathrm{Ad}(L)$-module. This contradicts the fact that the order of $\Psi_\lambda({}^t f_r)$ is equal to $2r$. Thus $a \neq 0$.

When we consider the top degree parts of $a\, b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ and of $b_r(\mu)^2$ in (15.6) and (15.7), they come from the top order parts of $\Psi_\lambda({}^t f_r f_r)$ and of ${}^t f_r(\partial) f_r\, {}^t f_r(\partial) f_r$, respectively. The relation between these top order parts is described in Proposition 15.1. Thus we have $a = (-1)^r$, since the top degree parts of $b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ and $b_r(\mu)^2$ coincide. We have proved Theorem 7.1.

References

[1] Boe, B. D., Homomorphisms between generalized Verma modules, Trans. Amer. Math. Soc. **288** (1985) vol. 2, 791–799.

[2] Bourbaki, N., Groupes et algèbres de Lie, Chapitres 4, 5, et 6, Hermann, Paris (1968).

[3] Enright, T., Howe, R. and Wallach, R., A classification of unitary highest weight modules, Representation theory of reductive groups, Ed. P. C. Trombi, Progress in Mathematics, Birkhäuser (1983).

[4] Faraut, J. and Koranyi, A., Function spaces and reproducing kernels on bounded symmetric domains, J. Func. Anal. **88** (1990), 64–89.

[5] Garland, H. and Zuckerman, G. J., On unitarizable highest weight modules of Hermitian pairs, J. Fac. Sci. Tokyo **28** (1982), 877–889.

[6] Goncharov, A. B., Generalized conformal structures on manifolds, Selecta Math. Soviet. **6** (1987) No. 4, 307–340.

[7] Gyoja, A., Further generalization of generalized Verma modules, Publ. RIMS, Kyoto Univ. **29** (1993), 349–395.

[8] Gyoja, A., Highest weight modules and b-functions of semi-invariants, Publ. RIMS, Kyoto Univ. **30** (1994), 353–400.

[9] Harish-Chandra, Representation of semisimple Lie groups. VI, Amer. J. Math. **78** (1956), 564–628.

[10] Howe, R. and Umeda, T., The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann. **290** (1991), 565–619.

[11] Humphreys, J. E., Finite and infinite dimensional modules for semisimple Lie algebra, Lie theories and their applications, Queen's papers in Pure and Appl. Math. No. 48, Queen's Univ., Kingston, Ont., (1978), 1–64.

[12] Jantzen, J. C., Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren, Math. Ann. **226** (1977), 53–65.

[13] Enright, T. J. and Joseph, A., An intrinsic analysis of unitarizable highest weight modules, Math. Ann. **288** (1990), 571–594.

[14] Kashiwara, M., The universal Verma module and the b-function, Advanced studies in pure Math. **6** (1985), 67–81.

[15] Koranyi, A. and Wolf, J. A., Realization of hermitian symmetric spaces as generalized half-planes, Ann. of Math. (2) **81** (1965), 265–288.

[16] Kostant, B. and Sahi, S., The Capelli identity, tube domains and the generalized Laplace transform, Advances in Math. **87** (1991), 71–92.

[17] Moore, C. C., Compactifications of symmetric spaces. II: the Cartan domains, Amer. J. Math. **86** (1964), 358–378.

[18] Muller, I. Rubenthaler, H. and Schiffmann, G., Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées, Math. Ann. **274** (1986), 95–123.

[19] Parthasarathy, R., Criteria for the unitarizability of some highest weight modules, Proc. Indian Acad. Sci. **89** No. 1 (1980), 1–24.

[20] Rubenthaler, H. and Schiffmann, G., Opérateurs différentiels de Shimura et espaces préhomogènes, Invent. Math. **90** (1987), 409–442.

[21] Schmid, W., Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent. Math. **9** (1969), 61–80.

[22] Shapovalov, N. N., On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra, Func. Anal. Appl. **6** (1972), 307–312.

[23] Shimura, G., On differential operators attached to certain representations of classical groups, Invent. Math. **77** (1984), 463–488.

[24] Suga, S., Highest weight modules associated with classical irreducible regular prehomogeneous vector spaces of commutative parabolic type, Osaka J. Math. **28** (1991), 323–346.

[25] Tanisaki, T., Hypergeometric systems and Radon transforms for Hermitian symmetric spaces, preprint, (1997).

[26] Tanisaki, T., Highest weight modules associated to parabolic subgroups with commutative unipotent radicals, preprint, (1997).

[27] Wallach, N. R., The analytic continuation of the discrete series. II, Trans. Amer. Math. Soc. **251** (1979), 19–37.

[28] Wallach, N. R., Polynomial differential operators associated with Hermitian symmetric spaces, Proceedings of Fuji-Kawaguchiko conference on representation theory of Lie groups and Lie algebras, World Scientific (1992).

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan