Operator approach for orthogonality in linear spaces

Mahdi Iranmanesh[1] and Maryam Saeedi Khojasteh[2,*]

Department of Mathematics, Shahrood University of Technology, Shahrood, IRAN
E-mail: m.iranmanesh2012@gmail.com[1], m.saeedi64@gmail.com[2]

Abstract

In this paper, we introduce the operator approach for orthogonality in linear spaces. In particular, we represent the concept of orthogonal vectors using an operator associated with them, in normed linear spaces.

2010 Mathematics Subject Classification. **46B20**. 46E99
Keywords. Linear space, normed space, orthogonality, Birkhoff orthogonality, angle, Minkowski plane, projection.

1 Introduction

Orthogonality, is one of the important concepts in mathematical and numerical analysis. Perhaps, it is the main property in linear spaces, normed spaces and inner product spaces. There are some various kinds of orthogonality. In fact, it has been defined different kinds in mathematical spaces.

In inner product spaces, it is easily said that two vectors x, y are orthogonal if

$$\langle x, y\rangle = 0.$$

But, in normed spaces, there is no simple tool for define orthogonal vectors. However, there are some good suggestions. One of them, is the Birkhoff James orthogonality [1].

Let X be a real normed space, and x, y be in X. We say that x is Birkhoff orthogonal to y if for every constant a,

$$\|x\| \leqslant \|x + ay\|. \tag{1.1}$$

It is not difficult to show that this definition is the same in inner product spaces [6].

In 1993, Milicic [9] introduced g-orthogonality in normed spaces via Gateaux derivatives. In fact, one has the notion of g-angle related to g-orthogonality.

Definition 1.1. The functional $g : X \times X \to \mathbb{R}$ is defined by

$$g(x, y) = \frac{1}{2}\|(\tau_+(x, y) + \tau_-(x, y))\|, \tag{1.2}$$

where

$$\tau_\pm(x, y) = \lim_{t\to\pm 0} \frac{\|x + ty\| - \|x\|}{t}. \tag{1.3}$$

The *g-angle* between two vectors x and y, denoted by $A_g(x, y)$, is given by

$$A_g(x, y) = \arccos \frac{g(x, y)}{\|x\|\|y\|}. \tag{1.4}$$

*The authors wish to thank the referees for reading the article so carefully and for the comments made on this article.

Tbilisi Mathematical Journal 11(4) (2018), pp. 103–112.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 11 May 2018 .
Accepted for publication: 17 September 2018.

Furthermore, x is said to be *g-orthogonal* to y, denoted by $x \perp_g y$, if

$$g(x, y) = 0,$$

i.e.,

$$A_g(x, y) = \frac{\pi}{2}.$$

In an inner product space $(X, \langle ., . \rangle)$, the angle $A(x, y)$ between two nonzero vectors x and y in X is usually given by

$$A(x, y) = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}, \tag{1.5}$$

where $\|x\| = \langle x, x \rangle^{1/2}$ denotes the induced norm in X.

One may observe that the angle $A(x, y)$ in X satisfies the following basic properties [4]:

(1) Parallelism: $A(x, y) = 0$ if and only if x and y are of the same direction; $A(x, y) = \pi$ if and only if x and y are of opposite direction.

(2) Symmetry: $A(x, y) = A(y, x)$ for every $x, y \in X$.

(3) Homogeneity:

$$A(ax, by) = \begin{cases} A(x, y) & ab > 0; \\ \pi - A(x, y) & ab < 0. \end{cases}$$

(4) Continuity: If $x_n \to x$ and $y_n \to y$ (in norm), then $A(x_n, y_n) \to A(x, y)$.

The g-angle is identical with the usual angle in an inner product space and has the following properties:

(1) Part of parallelism property: If x and y are of the same direction, then

$$A_g(x, y) = 0;$$

if x and y are of opposite direction, then

$$A_g(x, y) = \pi.$$

(2) Part of homogeneity property:

$$A_g(ax, by) = A_g(x, y), \qquad x, y \in X, a, b \in \mathbb{R};$$

(3) Homogeneity property:

$$A_g(ax, by) = \begin{cases} A_g(x, y) & ab > 0; \\ \pi - A_g(x, y) & ab < 0. \end{cases}$$

(4) Part of continuity property: If $y_n \to y$ (in norm), then $A_g(x, y_n) \to A_g(x, y)$.

However, g-orthogonality is not equivalent to Birkhoff orthogonality.

In [2], projections are used to give a definition of the p-angle $A_p(x, y)$ between two vectors x and y such that x is Birkhoff orthogonal to y if and only if

$$A_p(x, y) = \frac{\pi}{2}.$$

Since the angle between two vectors in a normed space is also the angle between these two vectors in the subspace spanned by them, it suffices to consider the Minkowski plane, i.e., real two dimensional normed linear space. More about the geometry of Minkowski plane could be found in [7] and [8].

Let X be the Minkowski plane. Denote by $\|.\|$ the norm of X. Fix a basis $\{e_1, e_2\}$ of X. Then we can write each $x \in X$ as $x = (x_1, x_2)$ under this basis, where $x_1, x_2 \in \mathbb{R}$. Moreover, $\{\delta_{e_1}, \delta_{e_2}\}$ is a basis of the dual space X^*, where δ_{e_i} for $i = 1, 2$ is a bounded linear function on X with

$$\delta_{e_i}(e_j) = \begin{cases} 0 & i \neq j; \\ 1 & i = j. \end{cases}$$

Denote by $L(X)$ the set of all bounded linear operators from X to X. For $T \in L(X)$, the operator $T^* \in L(X^*)$ is said to be the Banach conjugate operator of T if for any $z \in X$ and any $z^* \in X^*$, there must be $(T^* z^*)(z) = z^*(Tz)$. Note that if we use the following notation

$$f(x) = \langle x, f \rangle$$

then the property of conjugate can be rewritten as the following way

$$\langle x, T^* f \rangle = \langle Tx, f \rangle$$

as usual in inner product spaces.

Suppose that $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ are two linearly independent vectors in X under the basis $\{e_1, e_2\}$. Put

$$D_{xy} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \tag{1.6}$$

notice

$$|D_{xy}| = x_1 y_2 - x_2 y_1 \neq 0$$

since x and y are linearly independent. Define by P_{xy} the projection parallel to y from X to the subspace $\{\lambda x; \lambda \in \mathbb{R}\}$. Then P_{xy} depends only on the vectors x and y, and has the following presentation under the basis $\{e_1, e_2\}$:

$$P_{xy} = D_{xy} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot D_{xy}{}^{-1} = \frac{1}{|D_{xy}|} \begin{bmatrix} x_1 y_2 & -x_1 y_1 \\ x_2 y_2 & -x_2 y_1 \end{bmatrix}. \tag{1.7}$$

It is clear for any two linearly independent vectors x and y in X,

$$1 \leq \|P_{xy}\| < +\infty.$$

Furthermore, denote

$$p(x,y) = \begin{cases} 0 & x \text{ and } y \text{ are linearly dependent;} \\ \|P_{xy}\|^{-1} & x \text{ and } y \text{ are linearly independent.} \end{cases} \tag{1.8}$$

For any $x, y \in X$, the p-angle between x and y is defined by

$$A_p(x,y) = \arcsin(p(x,y)). \tag{1.9}$$

In an inner product space $(X, \langle ., . \rangle)$, obviously

$$p(x,y) = \frac{\langle x, y\rangle}{\|x\|\|y\|}, \tag{1.10}$$

and consequently, the p-angle is identical with the usual angle.

For more details, the interested reader is referred to [2].

2 Main results

Let X be a linear space with dimension n. Suppose that

$$x_k = (x_{k1}, \ldots, x_{kn})^T \quad , \quad k = 1, \ldots, n$$

are n linearly independent vectors in X. Put

$$D_{x_1,\ldots,x_n} = \begin{bmatrix} x_{11} & \ldots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \ldots & x_{nn} \end{bmatrix}$$

since $x_1, \ldots, x_n$ are linearly independent, we have

$$|D_{x_1,\ldots,x_n}| \neq 0.$$

Moreover, suppose that

$$x = (x_1, \ldots, x_n)^T \quad , \quad y = (y_1, \ldots, y_n)^T$$

are two linearly independent vectors in X. Extend x, y to a basis for X by adding $n-2$ vectors as

$$z_k = (z_{k1}, \ldots, z_{kn})^T \quad , \quad k = 1, \ldots, n-2.$$

Denote by $P_{x,z_1,\ldots,z_{n-2},y}$ the projection parallel to y from X to the subspace generated by $x, z_1, \ldots, z_{n-2}$. Since the vectors $x, z_1, \ldots, z_{n-2}, y$ are the eigenvectors of $P_{x,z_1,\ldots,z_{n-2},y}$, it turn implies that $P_{x,z_1,\ldots,z_{n-2},y}$ is similar to the following

$$\begin{bmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 0 \end{bmatrix}.$$

In fact, $P_{x,z_1,\ldots,z_{n-2},y}$ has a representation as follows

$$P_{x,z_1,\ldots,z_{n-2},y} = D_{x,z_1,\ldots,z_{n-2},y} \cdot \begin{bmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 0 \end{bmatrix} \cdot D^{-1}_{x,z_1,\ldots,z_{n-2},y}.$$

It is not difficult to see that for any two linearly independent vectors x and y in X,

$$1 \leq \|P_{x,z_1,\ldots,z_{n-2},y}\| < +\infty$$

in other words, $P_{x,z_1,\ldots,z_{n-2},y}$ is a bounded operator.

Furthermore, denote

$$p_{z_1,\ldots,z_{n-2}}(x,y) = \|P_{x,z_1,\ldots,z_{n-2},y}\|^{-1}$$

and let

$$p(x,y) = \sup\{p_{z_1,\ldots,z_{n-2}}(x,y) \,:\, z_1,\ldots,z_{n-2} \in X\}.$$

It is obvious that

$$p(x,y) = \max\{\|P_{x,z_1,\ldots,z_{n-2},y}\|^{-1} \,:\, z_1,\ldots,z_{n-2} \in X\ ,\ \|z_1\| = 1,\ldots,\|z_{n-2}\| = 1\}.$$

Definition 2.1. For any linearly independent x, y in X , the *p-angle* between x, y is defined by

$$A_p(x,y) = \arcsin(p(x,y)).$$

Note that p-angle is not depending on selected vectors $z_1,\ldots,z_{n-2}$.

Moreover, we say that x, y are *p-orthogonal* if

$$A_p(x,y) = \frac{\pi}{2}.$$

Equivalently, x, y are *p-orthogonal* if there exist suitable vectors $z_1,\ldots,z_{n-2}$ such that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| = 1.$$

Theorem 2.2. The p-angle has the following properties;

(a) Homogeneity property; i.e.

$$A_p(ax,by) = A_p(x,y)$$

for every x, y in X and nonzero a, b in $\mathbb{R}$;

(b) Continuity property; i.e. If $x_n \to x$ and $y_n \to y$ (in norm), then

$$A_p(x_n,y_n) \to A_p(x,y).$$

Proof. (a) The homogeneity of A_p is concluded easily from the definition. In fact,

$$P_{ax,z_1,\dots,z_{n-2},by} = P_{x,z_1,\dots,z_{n-2},y}$$

for all scalars a, b. Therefore

$$\|P_{ax,z_1,\dots,z_{n-2},by}\| = \|P_{x,z_1,\dots,z_{n-2},y}\|$$

and consequently

$$A_p(ax, by) = A_p(x, y).$$

(b) For continuity of A_p , note that the entries of the matrix $P_{x,z_1,\dots,z_{n-2},y}$ are of the following form

$$\frac{1}{|D_{x,z_1,\dots,z_{n-2},y}|} f(x, z_1, \dots, z_{n-2}, y)$$

where f is a polynomial of $x, z_1, \dots, z_{n-2}, y$. So $P_{x,z_1,\dots,z_{n-2},y}$ is continuous on x, y, i.e.

$$\|P_{x_m,z_1,\dots,z_{n-2},y_m} - P_{x,z_1,\dots,z_{n-2},y}\| \longrightarrow 0$$

when $m \longrightarrow \infty$.

On the other hand,

$$\|P_{x_m,z'_1,\dots,z'_{n-2},y_m}\| \leq \|P_{x_m,z'_1,\dots,z'_{n-2},y_m} - P_{x,z_1,\dots,z_{n-2},y}\| + \|P_{x,z_1,\dots,z_{n-2},y}\|.$$

Therefore taking infimum over $z'_1, \dots, z'_{n-2}$ we conclude

$$\begin{aligned} \inf_{z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m}\| \leq & \\ & \inf_{z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m} - P_{x,z_1,\dots,z_{n-2},y}\| \\ & + \|P_{x,z_1,\dots,z_{n-2},y}\|, \end{aligned}$$

again taking infimum over $z_1, \dots, z_{n-2}$ we conclude

$$\begin{aligned} \inf_{z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m}\| \leq & \\ & \inf_{z_1,\dots,z_{n-2},z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m} - P_{x,z_1,\dots,z_{n-2},y}\| \\ & + \inf_{z_1,\dots,z_{n-2}} \|P_{x,z_1,\dots,z_{n-2},y}\|, \end{aligned}$$

therefore

$$\begin{aligned} & \inf_{z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m}\| - \inf_{z_1,\dots,z_{n-2}} \|P_{x,z_1,\dots,z_{n-2},y}\| \\ & \quad \leq \inf_{z_1,\dots,z_{n-2},z'_1,\dots,z'_{n-2}} \|P_{x_m,z'_1,\dots,z'_{n-2},y_m} - P_{x,z_1,\dots,z_{n-2},y}\|. \end{aligned}$$

On the other hand

$$\inf_{z_1,\ldots,z_{n-2},z'_1,\ldots,z'_{n-2}} \|P_{x_m,z'_1,\ldots,z'_{n-2},y_m} - P_{x,z_1,\ldots,z_{n-2},y}\|$$
$$\leq \inf_{z_1,\ldots,z_{n-2}} \|P_{x_m,z_1,\ldots,z_{n-2},y_m} - P_{x,z_1,\ldots,z_{n-2},y}\|.$$

It turn implies that the function

$$f(x,y) = \inf_{z_1,\ldots,z_{n-2}} \|P_{x,z_1,\ldots,z_{n-2},y}\|$$

is continuous. So the inverse of it is also continuous, i.e. the function

$$p(x,y) = (f(x,y))^{-1} = \sup_{z_1,\ldots,z_{n-2}} \|P_{x,z_1,\ldots,z_{n-2},y}\|^{-1}.$$

Finally, the function

$$A(x,y) = \arcsin(p(x,y))$$

is also continuous and it completes the proof. Q.E.D.

Lemma 2.3. Let x, y be two linearly independent vectors in X. Extending x, y to a basis as $\{x, z_1, \ldots, z_{n-2}, y\}$, every z in X can be written as

$$z = ax + c_1 z_1 + \cdots + c_{n-2} z_{n-2} + by.$$

Moreover

$$P_{x,z_1,\ldots,z_{n-2},y}(z) = ax + c_1 z_1 + \cdots + c_{n-2} z_{n-2}.$$

Proof. It is easy to obtain by definition of linearly independence and $P_{x,z_1,\ldots,z_{n-2},y}$. Q.E.D.

Theorem 2.4. The concept of p-orthogonality is compatible with the usual orthogonality in the inner product spaces.

Proof. Let X be an inner product space. First, assume that x, y are orthogonal. We shall show that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| = 1$$

for suitable choice of $z_1, \ldots, z_{n-2}$. To this end, extending x, y to a basis as $\{x, z_1, \ldots, z_{n-2}, y\}$ to an orthogonal basis for X, we show that

$$\|P\| = 1$$

where $P = P_{x,z_1,\ldots,z_{n-2},y}$ is the orthogonal projection associated with the subspace generated by $\{x, z_1, \ldots, z_{n-2}\}$.

Since

$$y \in [span\{x, z_1, \ldots, z_{n-2}\}]^{\perp}$$

and for any z in X, we have

$$Pz \in span\{x, z_1, \ldots, z_{n-2}\}$$

we conclude that

$$y \perp Pz$$

and we have

$$\|z\|^2 = \|z - Pz + Pz\|^2 = \|z - Pz\|^2 + \|Pz\|^2 \geq \|Pz\|^2$$

therefore

$$\|p\| \leq 1,$$

now, taking $z = x$, we have

$$Px = x$$

so

$$\|Px\| = \|x\|$$

hence

$$\|P\| = 1.$$

Next, assume that x, y are not orthogonal. We shall show that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| > 1$$

for all choices of $z_1, \ldots, z_{n-2}$. To this end, assume that $z_1, \ldots, z_{n-2}$ are arbitrary in X. We show that

$$\|P\| > 1$$

where $P = P_{x,z_1,\ldots,z_{n-2},y}$ is the parallel projection associated with the subspace generated by $\{x, z_1, \ldots, z_{n-2}\}$.

Since

$$\{y\}^\perp \neq span\{x, z_1, \ldots, z_{n-2}\}$$

there exists a nonzero vector z in $\{y\}^\perp$ that does not belong to $span\{x, z_1, \ldots, z_{n-2}\}$. For this z we have

$$Pz - z \perp z.$$

We conclude that

$$\|Pz\|^2 = \|Pz - z + z\|^2 = \|Pz - z\|^2 + \|z\|^2 > \|z\|^2$$

therefore

$$\|P\| > 1$$

as claimed. Q.E.D.

Theorem 2.5. Let x, y be two linearly independent vectors in X. If

$$A_p(x, y) = \frac{\pi}{2}$$

then x is Birkhoff orthogonal to y.

Proof. Suppose that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| = 1,$$

then we have

$$\|x\| = \|P_{x,z_1,\ldots,z_{n-2},y}(x, by)\| \leq \|P_{x,z_1,\ldots,z_{n-2},y}\|\|(x+by)\| = \|(x+by)\|$$

Thus, x is Birkhoff orthogonal to y. This completes the proof. Q.E.D.

Example 2.6. To find the angle between $x = (1,0,0)$ and $y = (0,1,0)$ in l_3^p ,where $1 \leq p < \infty$, we consider the following matrix

$$P = \begin{bmatrix} 1 & a & 0 \\ 0 & b & 1 \\ 0 & c & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & b & 1 \\ 0 & c & 0 \end{bmatrix}^{-1}$$

then we have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & b/c \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho(P^T P) = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x - [(\frac{b}{c})^2 + 1] \end{vmatrix} = (x-1)[x(x - ((\frac{b}{c})^2 + 1))].$$

Therefor maximum of eigenvalue of $P^T P$ is $(\frac{b}{c})^2 + 1$ and then by definition

$$p(x, y) = 1$$

i.e.,

$$A_p(x, y) = \frac{\pi}{2}.$$

Similarly, if $x = (1,0,0)$ and $y = (0,0,1)$ then $A_p(x,y) = \frac{\pi}{2}$.

References

[1] G. Birkhoff, *Orthogonality in normed linear spaces*, Duke Math. J., 1(1935), 169-172.

[2] Chen Zhi-zhi, Lin Wei and Luo Lu-lin: *Projections, Birkhoff Orthogonality and Angles in Normed Spaces*, Communications in Mathematical Research, 27(4)(2011), 378-384.

[3] C. R. Diminnie, E. Z. Andalafte and R. W. Freese, *Generalized angles and a characterization of inner product spaces*. Houston J. Math., 14(1988), 475-480.

[4] R. C. James, : *Orthogonality in normed linear spacess*, Duke Math. J., 12(1945), 291-301.

[5] R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc.Trans. , 61(1947), 265-292.

[6] D. H. Ji, and S. L. Wu, *Quantitative characterization of the difference between Birkhoff orthogonality and Isosceles orthogonality*, J. Math. Ana. Appl., 323(2006), 1-7.

[7] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces — A survey, Part I*, Exposition. Math., 19(2001), 97-142.

[8] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces — A survey, Part II*, Exposition. Math., 22(2004), 93-144.

[9] P. M. Milicic, *Sur le g-angle dans un espace norme*, Mat. Vesnik, 45(1993), 43-48.

[10] P. M. Milicic, *On the B-angle and g-angle in normed spaces*, ournal of inequalities in pure and applied mathematics, Volume 8 (2007), Issue 3, Article 99.