Uniqueness results related to L-functions and certain differential polynomials

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Abstract

In this paper, using the idea of weighted sharing we investigate the uniqueness problem of a meromorphic function and an L-function when certain differential polynomials generated by them share a nonzero finite value or have the same fixed points. Our results improve the recent results due to Liu-Li-Yi [Proc. Japan Acad. Ser. A, 93 (2017), 41-46].

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1 Introduction, definitions and results

L-functions are Dirichlet series with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as a prototype and are important objects in number theory. The value distribution of L-functions concerns distribution of zeros of L-functions and more generally, the c-points of L, that is, the zeros of the function $L(s) - c$, or the values in the set of pre-images

$$L^{-1} = \{s \in \mathbb{C} : L(s) = c\},$$

where and in what follows, s denotes complex variables and c denotes a value in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In connection to meromorphic functions, Nevanlinna's uniqueness theorem states that a nonconstant meromorphic function f in $\mathbb{C}$ is completely determined by five such pre-images (cf. [3], [23] and [25]). It is to be noted that an L-function can be analytically continued as meromorphic function in $\mathbb{C}$.

Let f and g be two meromorphic functions in $\mathbb{C}$ and let $c \in \overline{\mathbb{C}}$. Then f and g are said to share the value c IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. f and g are said to share the value c CM (counting multiplicities) if $f(s) - c$ and $g(s) - c$ have the same zeros with the same multiplicities. In the paper by an L-function we shall always mean an L-function L in the Selberg class S that includes the Riemann zeta function ζ and essentially those Dirichlet series where one might expect a Riemann hypothesis. An L-function belonging to S is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ satisfying the following axioms (see [15]):

(i) Ramanujan hypothesis: $a(n) << n^{\varepsilon}$ for each $\varepsilon > 0$;

(ii) Analytic continuation: There is a nonnegative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;

(iii) Functional equation: L satisfies a functional equation of the type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1 - \overline{s})},$$

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where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\mathrm{Re}\nu_j \geq 0$ and $|\omega| = 1$;

(iv) Euler product hypothesis: $L(s) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ such that $b(p^k) << p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

The degree d of an L-function L is defined to be

$$d = 2\sum_{j=1}^{K} \lambda_j,$$

where K and λ_j are respectively the positive integer and the positive real number defined in axiom (iii) of the definition of L-function.

In the recent times, the theory of L-functions along with the families of partial zeta type functions, q- zeta type functions, $(q-)L$-functions has become a prominent branch of the analytic number theory. In fact, many important investigations have been done on the unified presentations of such functions (see [16]-[18]). However, in this paper, we shall be mainly concerned with the value sharing of L-functions related to some meromorphic functions. During the last decade the value distribution of L-functions has been studied extensively (see the monograph [19] and also [5], [10], [11]). The uniqueness property related to L-functions was first studied by Steuding ([19], p. 152), as seen from the following result.

Theorem A. If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value c $(\neq \infty)$ CM, then $L_1 = L_2$.

Since L-functions are analytically continued as meromorphic functions, it becomes an interesting question that to which extent an L-function can share values with an arbitrary meromorphic function. In this direction, Li [10] proved the following uniqueness result.

Theorem B. Let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane and let a and b be any two distinct finite complex values. If f and a nonconstant L-function L share the values a CM and b IM, then $L = f$.

In 1997, it was asked by Lahiri [6]: What can be said about the relationship between two meromorphic functions f and g when two differential polynomials generated by them share some nonzero complex value? Some of the works in this direction can be found in [1, 2, 12, 22]. The following results are due to Yang-Hua [22] and Fang [2] respectively.

Theorem C. Let f and g be two nonconstant meromorphic functions and $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c_1,\ c_2$ and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t satisfying $t^{n+1} = 1$.

Theorem D. Let f and g be two nonconstant entire functions and $n,\ k$ be positive integers such that $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c_1,\ c_2$ and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t satisfying $t^n = 1$.

In connection to Theorems A-D, it is natural to ask, what can be said about the relationship between a meromorphic function f and an L-function L when $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share the value 1 CM, or when $(f^n)^{(k)}$ and $(L^n)^{(k)}$ have same fixed points, where $n,\ k$ are positive integers? Recently Liu, Li and Yi [13] proved the following results in this direction.

Theorem E. Let f be a nonconstant meromorphic function, L be an L-function and let $n,\ k$ be two positive integers such that $n > 3k+6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem F. Let f be a nonconstant meromorphic function, L be an L-function and let $n,\ k$ be two positive integers such that $n > 3k+6$. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

Regarding Theorems E and F it is quite natural to ask the following question.

Question 1. Is it possible to relax the nature of sharing the value in Theorems E and F?

In 2001, an idea of gradation of sharing of values known as weighted sharing of values was introduced in [8] which measures how close a shared value being shared IM or to being shared CM. The notion is as follows.

Definition 1. [8] Let $a \in \overline{\mathbb{C}}$ and l be nonnegative integer or infinity. We denote by $E_l(a; f)$ the set of all a-points of f where an a-point of multiplicity p is counted p times if $p \leq l$ and $l+1$ times if $p > l$. If $E_l(a; f) = E_l(a; g)$, we say that $f,\ g$ share the value a with weight l.

The definition implies that if f, g share some value a with weight l, then z_0 is an a-point of f with multiplicity $p(\leq l)$ if and only if it is an a-point of g with multiplicity $p(\leq l)$ and z_0 is an a-point of f with multiplicity $p(> l)$ if and only if it is an a-point of g with multiplicity $q(> l)$, where p is not necessarily equal to q.

We write $f,\ g$ share (a, l) to mean that $f,\ g$ share the value a with weight l. Clearly if $f,\ g$ share (a, l), then $f,\ g$ share (a, l_1) for any integer l_1 where $0 \leq l_1 < l$. Also we note that $f,\ g$ share the value a CM or IM if and only if $f,\ g$ share (a, ∞) or $(a, 0)$ respectively.

In the paper, with the aid of weighted sharing we shall find out the possible answers of the above question. We shall prove the following two theorems that improve Theorems E and F respectively by relaxing the nature of sharing of values. The main results of the paper are as follows.

Theorem 1. Let f be a nonconstant meromorphic function, L be an L-function, and $n,\ k$ be positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(1, l)$ and one of the following conditions is satisfied: (i) $l \geq 2$ and $n > 3k+6$, (ii) $l = 1$ and $n > \frac{7k}{2} + \frac{13}{2}$, (iii) $l = 0$ and $n > 7k+11$, then $f = tL$ for some constant t satisfying $t^n = 1$.

Theorem 2. Let f be a nonconstant meromorphic function, L be an L-function, and $n,\ k$ be positive integers. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share $(0, l)$ and one of the following conditions is satisfied: (i) $l \geq 2$ and $n > 3k+6$, (ii) $l = 1$ and $n > \frac{7k}{2} + \frac{13}{2}$, (iii) $l = 0$ and $n > 7k+11$, then $f = tL$ for some constant t satisfying $t^n = 1$.

We apply Nevanlinna value distribution theory to prove our main results. It is assumed that the reader is familiar with the standard notations such as $m(r, f),\ N(r, f),\ \overline{N}(r, f),\ N(r, a; f),$ $\overline{N}(r, a; f)$, $T(r, f)$ etc. and the fundamental results of Nevanlinna theory (see [3], [9], [23] and [25]).

For a nonconstant meromorphic function f in the complex plane we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of r with finite linear measure. For a function of finite order, $O(\log r)$ and $S(r,f)$ means the same quantity. Moreover, we shall use the following definitions of the order $\rho(f)$ and the lower order $\mu(f)$ of a meromorphic function f (see [3, 23, 25]):

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r},\ \ \mu(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

Also, a meromorphic function α is said to be a small function of f provided that $T(r,\alpha)=S(r,f)$.

We now explain the following definitions and notations that have been used in the paper.

Definition 2. [7] For $a\in\overline{\mathbb{C}}$, we denote by $N(r,a;f\mid=1)$ the counting function of simple a-points of f. For a positive integer p we denote by $N(r,a;f\mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f\mid\leq p)$ we denote the corresponding reduced counting function.

Analogously we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.

Definition 3. [8] Let p be positive integer or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$. Then

$$N_p(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+...+\overline{N}(r,a;f\mid\geq p).$$

Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 4. Let a be any value in the extended complex plane and let k be an arbitrary non-negative integer. We define

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}$$

and

$$\delta_k(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N_k(r,a;f)}{T(r,f)}.$$

Remark 1. From the definitions of $\Theta(a,f)$ and $\delta_k(a,f)$, it is clear that

$$0\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)\leq\delta_1(a,f)\leq\Theta(a,f)\leq 1.$$

2 Lemmas

In this section, we present some lemmas that will be needed in the sequel.

Lemma 1. [21] Suppose that f is a nonconstant meromorphic function and let $a_0,a_1,\ldots,a_n$ be finite complex numbers such that $a_n\neq 0$. Then

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2. [23] Let f be a nonconstant meromorphic function and k be a positive integer. Then

$$N\left(r,0;f^{(k)}\right) \leq N(r,0;f) + k\overline{N}(r,f) + S(r,f),$$

as $r \to \infty$, except possibly outside a set of finite linear measure.

Lemma 3. [3] Let f be a nonconstant meromorphic function, k be a positive integer and let c be a nonzero finite complex number. Then

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N(r,0;f) + N\left(r,c;f^{(k)}\right) - N\left(r,0;f^{(k+1)}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}(r,0;f) + \overline{N}\left(r,c;f^{(k)}\right) - N_0\left(r,0;f^{(k+1)}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r,0;f^{(k+1)}\right)$ is the counting function of those zeros of $f^{(k+1)}$ in $|z| < r$ which are not zeros of $f(f^{(k)} - c)$ in $|z| < r$.

Lemma 4. [4] Let f be a transcendental meromorphic function in the complex plane. Then corresponding to each $\Lambda > 1$, there exists a set $M(\Lambda) \subset (0, +\infty)$, with lower logarithmic density not exceeding the value $d(\Lambda) = 1 - (2e^{\Lambda-1} - 1)^{-1} > 0$, i.e.,

$$\underline{\log dens} M(\Lambda) = \liminf_{r \longrightarrow \infty} \frac{1}{\log r} \int_{M(\Lambda)\cap[1,r]} \frac{dt}{t} \leq d(\Lambda),$$

provided that for all $r \notin M(\Lambda)$ and for each positive integer k,

$$\limsup_{r \longrightarrow \infty} \frac{T(r,f)}{T(r,f^{(k)})} \leq 3e\Lambda.$$

Lemma 5. [24] Let f be a nonconstant meromorphic function, $\alpha(\not\equiv 0, \infty)$ be a small function of f. Then

$$T(r,f) \leq \overline{N}(r,f) + N(r,0;f) + N\left(r,0;f^{(k)} - \alpha\right) - N\left(r,0;\left(\frac{f^{(k)}}{\alpha}\right)'\right) + S(r,f).$$

Lemma 6. [26] Let $E \subset (0, +\infty)$ be a set of finite linear measure and let $f_1,\ f_2$ be two nonconstant meromorphic functions such that $\overline{N}(r,f_j) + \overline{N}(r,0;f_j) = S(r),\ (j = 1,2)$. Then either $\overline{N}_0(r,1;f_1,f_2) = S(r)$ or there exist two integers p and q satisfying $|p|+|q| > 0$ such that $f_1^p f_2^q = 1$. Here $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2 in $|z| < r$, $T(r) = T(r,f_1) + T(r,f_2)$ and $S(r) = o\{T(r)\}$ as $r \to \infty$ and $r \notin E$.

Lemma 7. [14] Let F and G be two transcendental meromorphic functions and let $k(\geq 1),\ l(\geq 0)$ be two integers. Suppose that $F^{(k)} - P$ and $G^{(k)} - P$ share $(0,l)$, where $P \not\equiv 0$ is a polynomial. Then either $F^{(k)}G^{(k)} = P^2$ or $F = G$, whenever F and G satisfies one of the following conditions:
(i) $l \geq 2$ and $\Delta_{11} = 2\Theta(\infty,F) + (k+2)\Theta(\infty,G) + \delta_{k+2}(0,F) + \delta_{k+2}(0,G) > k+5$
and $\Delta_{12} = 2\Theta(\infty,G) + (k+2)\Theta(\infty,F) + \delta_{k+2}(0,G) + \delta_{k+2}(0,F) > k+5$;
(ii) $l = 1$ and
$\Delta_{21} = \left(\frac{k}{2} + \frac{5}{2}\right)\Theta(\infty,F) + (k+2)\Theta(\infty,G) + \frac{1}{2}\delta_{k+1}(0,F) + \delta_{k+2}(0,F) + \delta_{k+2}(0,G) > \frac{3k}{2} + 6$

and $\Delta_{22}=\left(\frac{k}{2}+\frac{5}{2}\right)\Theta(\infty,G)+(k+2)\Theta(\infty,F)+\frac{1}{2}\delta_{k+1}(0,G)+\delta_{k+2}(0,G)+\delta_{k+2}(0,F)>\frac{3k}{2}+6$;
(iii) $l=0$ and
$\Delta_{31}=(2k+4)\Theta(\infty,F)+(2k+3)\Theta(\infty,G)+2\delta_{k+1}(0,F)+\delta_{k+1}(0,G)+\delta_{k+2}(0,F)+\delta_{k+2}(0,G)>4k+11$
and $\Delta_{32}=(2k+4)\Theta(\infty,G)+(2k+3)\Theta(\infty,F)+2\delta_{k+1}(0,G)+\delta_{k+1}(0,F)+\delta_{k+2}(0,G)+\delta_{k+2}(0,F)>4k+11$.

3 Proof of the Theorems

[Proof of Theorem 2] Let d be the degree of the L-function L. Then by Steuding ([19], p.150), we have

$$T(r,L)=\frac{d}{\pi}r\log r+O(r). \tag{3.1}$$

We see that any zero z_0 of L of multiplicity q_0 is a zero of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ with multiplicity at least nq_0-k-2. Also any zero z_1 of $\frac{(L^n)^{(k)}}{z}-1$ of multiplicity q_1 is a zero of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ of multiplicity q_1-1. Since an L-function can have at most one pole at $z=1$ in the complex plane, using (3.1) and Lemmas 1 and 5 we get

$$\begin{aligned}
T(r,L^n) &= nT(r,L)+S(r,f)\\
&\leq N(r,0;L^n)+N\left(r,0;\frac{(L^n)^{(k)}}{z}-1\right)-N\left(r,0;\left(\frac{(L^n)^{(k)}}{z}\right)'\right)+S(r,f)\\
&\leq (k+2)\overline{N}(r,0;L)+\overline{N}\left(r,0;\frac{(L^n)^{(k)}}{z}-1\right)-N_0\left(r,0;\left(\frac{(L^n)^{(k)}}{z}\right)'\right)+S(r,f)\\
&\leq (k+2)T(r,L)+\overline{N}\left(r,0;\frac{(f^n)^{(k)}}{z}-1\right)+S(r,f)\\
&\leq (k+2)T(r,L)+T\left(r,(f^n)^{(k)}\right)+S(r,f),
\end{aligned}$$

where $N_0\left(r,0;\left(\frac{(L^n)^{(k)}}{z}\right)'\right)$ is the counting function of those zeros of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ in $|z|<r$ which are not the zeros of L and $\frac{(L^n)^{(k)}}{z}-1$ in $|z|<r$.
This implies

$$(n-k-2)T(r,L)\leq T\left(r,(f^n)^{(k)}\right)+S(r,f). \tag{3.2}$$

From (3.1) it is clear that L is a transcendental meromorphic function. Now combining this with (3.2), Theorem 1.5 [23] and the assumption of the lower bound of n, we obtain that $(f^n)^{(k)}$ and so f is a transcendental meromorphic function. Using Lemma 1, we have

$$\begin{aligned}
\Theta(\infty,f^n) &= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,f^n)}{T(r,f^n)}\\
&= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,f)}{nT(r,f)+O(1)}\geq 1-\frac{1}{n},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}\delta_{k+2}(0,f^n) &= 1-\limsup_{r\longrightarrow\infty}\frac{N_{k+2}(r,0;f^n)}{T(r,f^n)}\\ &= 1-\limsup_{r\longrightarrow\infty}\frac{(k+2)\overline{N}(r,0;f)}{nT(r,f)+O(1)}\geq 1-\frac{k+2}{n},\end{aligned}\tag{3.4}$$

and similarly

$$\delta_{k+2}(0,L^n)\geq 1-\frac{k+2}{n},\tag{3.5}$$

$$\delta_{k+1}(0,f^n)\geq 1-\frac{k+1}{n},\tag{3.6}$$

$$\delta_{k+1}(0,L^n)\geq 1-\frac{k+1}{n}.\tag{3.7}$$

Since an L-function has at most one pole $z=1$ in the complex plane, we have

$$N(r,L)\leq \log r+O(1).$$

So using (3.1) we deduce that

$$\Theta(\infty,L^n)=1.\tag{3.8}$$

Considering $F=f^n$, $G=L^n$ in Lemma 7 we now get the following three cases.

Case 1. Let $l\geq 2$. Then using (3.3)-(3.5) and (3.8) we have $\Delta_{11}\geq k+6-\frac{2k+6}{n}$ and $\Delta_{12}\geq k+6-\frac{3k+6}{n}$. Since $n>3k+6$, by (i) of Lemma 7 we have two possibilities, either $(f^n)^{(k)}(L^n)^{(k)}=z^2$ or $f^n=L^n$.

If $f^n=L^n$, we have nothing to prove as the conclusion of the theorem follows immediately. Therefore we assume that $(f^n)^{(k)}(L^n)^{(k)}=z^2$. We claim that 0 is a Picard exceptional value of both f and L. If not, let $z_2(\neq 0)\in\mathbb{C}$ be a zero of f with multiplicity $p_2(\geq 1)$. Therefore from the assumption that $(f^n)^{(k)}(L^n)^{(k)}=z^2$ it follows that $z_2=1$ is a pole of L with multiplicity $q_2(\geq 1)$ such that $np_2-k=nq_2+k$, i.e., $n(p_2-q_2)=2k$, and so $n\leq 2k$. This is a contradiction to the lower bound of n in Theorem 2 and hence proves our claim for the function f. Similarly we can prove the claim for L. Again, using (3.1), Lemma 1, Theorem 1.15 [23], a result of Whittaker [20], the definition of the order of meromorphic function and also by the assumption that $\frac{(f^n)^{(k)}}{z}\frac{(L^n)^{(k)}}{z}=1$ we get

$$\rho(f)=\rho(f^n)=\rho\left(\frac{(f^n)^{(k)}}{z}\right)=\rho\left(\frac{(L^n)^{(k)}}{z}\right)=\rho(L^n)=\rho(L)=1.\tag{3.9}$$

Now from (3.9), Lemma 2 and $\frac{(f^n)^{(k)}}{z}\frac{(L^n)^{(k)}}{z}=1$ and the fact that $z=1$ is the only possible pole of L in $\mathbb{C}$, we obtain that

$$\begin{aligned}(n+k)\overline{N}(r,f) &\leq N(r,(f^n)^{(k)}) \leq N\left(r,\frac{(f^n)^{(k)}}{z}\right)+O(1)\\ &\leq N\left(r,0;\frac{(L^n)^{(k)}}{z}\right)+O(1)\\ &\leq N\left(r,0;(L^n)^{(k)}\right)+O(1)\\ &\leq N(r,0;L^n)+k\overline{N}(r,L^n)+S(r,f)\\ &\leq S(r,f). \end{aligned}\tag{3.10}$$

Since $z=1$ is the only possible pole of L in $\mathbb{C}$, using (3.10) it follows that

$$\overline{N}(r,f)+\overline{N}(r,L)\leq S(r,f).\tag{3.11}$$

We set

$$\Gamma_1=\frac{F_1}{G_1},\ \Gamma_2=\frac{F_1-1}{G_1-1},\tag{3.12}$$

where $F_1=\frac{(f^n)^{(k)}}{z}$ and $G_1=\frac{(L^n)^{(k)}}{z}$.

Since f and L are transcendental meromorphic functions, we get from (3.12) that $\Gamma_1\not\equiv 0$ and $\Gamma_2\not\equiv 0$. Now suppose that at least one of Γ_1 and Γ_2 is a nonzero constant. Then, from (3.12) we see that F_1 and G_1 share ∞ CM. Combining this with the fact that $F_1G_1=1$ we find that ∞ is a Picard exceptional value of both f and L. Next we assume that each of Γ_1 and Γ_2 is a nonconstant meromorphic function.

From (3.12) we can deduce that

$$F_1=\frac{\Gamma_1(1-\Gamma_2)}{\Gamma_1-\Gamma_2},\ G_1=\frac{1-\Gamma_2}{\Gamma_1-\Gamma_2}.\tag{3.13}$$

Without loss of generality suppose that there exists a subset $E\subset\mathbb{R}^+$ with infinite linear measure such that $T(r,G_1)\leq T(r,F_1)$ and

$$\begin{aligned}T(r,F_1) &\leq 2\{T(r,\Gamma_1)+T(r,\Gamma_2)\}+S(r)\\ &\leq 8T(r,F_1)+S(r),\end{aligned}\tag{3.14}$$

as $r\in E$ and $r\to\infty$ where $S(r)=o\{T(r)\}$ and $T(r)=T(r,\Gamma_1)+T(r,\Gamma_2)$. Therefore using (3.9), Lemma 2 and the condition that 0 is a Picard exceptional value of both f and L, we have

$$\begin{aligned}N(r,0;F_1) &= N\left(r,0;\frac{(f^n)^{(k)}}{z}\right)\\ &\leq N\left(r,0;(f^n)^{(k)}\right)+O(1)\\ &\leq k\overline{N}(r,f)+S(r,f).\end{aligned}\tag{3.15}$$

Now by (3.10), (3.11) and (3.15) we get

$$N(r,0;F_1)+N(r,0;G_1)\leq S(r,f). \tag{3.16}$$

From the condition that $F_1G_1=1$, it is easy to see that F_1 and G_1 share 1 and -1 CM. Since F_1 and G_1 share 1 CM, using (3.11), (3.12) and (3.16) and noting that F_1 and G_1 are transcendental, we obtain

$$\overline{N}(r,\Gamma_j)+\overline{N}(r,0;\Gamma_j)=S(r),\ (j=1,2), \tag{3.17}$$

as $r\in E$ and $r\to\infty$. Now we shall show that $\overline{N}_0(r,1;\Gamma_1,\Gamma_2)=S(r)$ is not possible. Since F_1 and G_1 share -1 CM, from (3.11), (3.12), (3.15) and Nevanlinna's second fundamental theorem we have

$$\begin{aligned} T(r,F_1) &\leq \overline{N}(r,0;F_1)+\overline{N}(r,-1;F_1)+\overline{N}(r,F_1)+S(r,F_1)\\ &\leq \overline{N}(r,-1;F_1)+S(r,f)+S(r,F_1)\\ &\leq \overline{N}_0(r,1;\Gamma_1,\Gamma_2)+S(r,F_1), \end{aligned} \tag{3.18}$$

as $r\in E$ and $r\to\infty$.

If $\overline{N}_0(r,1;\Gamma_1,\Gamma_2)=S(r)$, we get from (3.14) and (3.18) that $T(r,\Gamma_1)+T(r,\Gamma_2)\leq S(r)$, a contradiction. Therefore by Lemma 6, (3.12) and (3.17) it follows that there exist two relatively prime integers p and q such that $|p|+|q|>0$ and $\Gamma_1^p\Gamma_2^q=1$. Therefore from (3.12) we get that

$$\left(\frac{F_1}{G_1}\right)^p\left(\frac{F_1-1}{G_1-1}\right)^q=1. \tag{3.19}$$

Now we discuss the following two subcases.

Subcase 1.1 Assume that $pq\geq 0$. From (3.19) we see that F_1 and G_1 share ∞ CM. Then noting that $F_1G_1=1$ i.e., $\frac{(f^n)^{(k)}}{z}\frac{(L^n)^{(k)}}{z}=1$, we obtain that ∞ is a Picard exceptional value of f and L. This together with the fact that 0 is another Picard exceptional value of f and L, and by (3.9), we can write L as

$$L(z)=e^{c_1z+c_2},$$

where $c_1(\neq 0)$ and c_2 are constants.

Therefore by the result of Hayman [[3], p. 7] we get that

$$T(r,L)=T(r,e^{c_1z+c_2})=\frac{|c_1|r}{\pi}(1+o(1)),$$

a contradiction to (3.1).

Subcase 1.2 Assume that $pq<0$. Without loss of generality let $p>0$ and $q<0$ and $q=-q^*$, for some positive integer q^*. Therefore (3.19) reduces to

$$\left(\frac{F_1}{G_1}\right)^p=\left(\frac{F_1-1}{G_1-1}\right)^{q^*}. \tag{3.20}$$

From $F_1G_1=1$ it follows that if z_3 be a pole of F_1 of some multiplicity $p_3(\geq 1)$, then z_3 is also a zero of G_1 of multiplicity p_3. Therefore from (3.20) we get $2p=q^*=-q$. This gives, $p=1$ and $q=-q^*=-2$ as p and q are prime to each other. Hence we get that $F_1(G_1-1)^2=G_1(F_1-1)^2$, which is nothing but our obtained result $F_1G_1=1$. Now we shall deduce a contradiction by using other method.

Since $z=1$ is the only possible pole of L and so of $(L^n)^{(k)}$, using (3.16) we get

$$(L^n)^{(k)}(z)=\frac{zP(z)}{(z-1)^m}e^{c_3z+c_4}, \tag{3.21}$$

where $P(z)$ is nonzero polynomial, m is a nonnegative integer and $c_3(\neq 0)$, c_4 are constants.

Now using the result of Hayman [[3], p. 7], Lemma 4 we get from (3.21) that there exists a subset $E\subset(0,+\infty)$ with logarithmic measure $\log measE=\int_E\frac{dt}{t}=\infty$ such that for any given sufficiently large number $\Lambda>1$, we have

$$\begin{aligned}T(r,L) &\leq 3e\Lambda T\left(r,(L^n)^{(k)}\right)\\ &= \frac{3e\Lambda|c_3|r}{\pi}(1+o(1))+S(r,f),\end{aligned}$$

as $r\in E$ and $r\to\infty$. This clearly contradicts with (3.1).

Case 2. Let $l=1$. Then using (3.3)-(3.8) we have $\Delta_{21}\geq\frac{3k}{2}+7-\frac{3k+7}{n}$ and $\Delta_{22}\geq\frac{3k}{2}+7-\frac{7k+13}{2n}$. Since $n>\frac{7k}{2}+\frac{13}{2}$, by (ii) of Lemma 7 we have either $(f^n)^{(k)}(L^n)^{(k)}=z^2$ or $f^n=L^n$. Therefore proceeding exactly in the similar manner as of Case 1 we can get the conclusion of the theorem.

Case 3. Let $l=0$. Then using (3.3)-(3.8) we have $\Delta_{31}\geq 4k+12-\frac{7k+11}{n}$ and $\Delta_{32}\geq 4k+12-\frac{7k+10}{n}$. Since $n>7k+11$, by (iii) of Lemma 7 we have the same possibilities, either $(f^n)^{(k)}(L^n)^{(k)}=z^2$ or $f^n=L^n$. Proceeding as in Case 1 the conclusion of the theorem follows immediately. This proves Theorem 2.

[**Proof of Theorem** 1] By Steuding ([19], p.150) we have (3.1). We see that $z=1$ is the only possible pole of L in $\mathbb{C}$. Then by Lemmas 1 and 3 and the assumption of Theorem 1, we get

$$\begin{aligned}nT(r,L) &= T(r,L^n)+S(r,f)\\ &\leq \overline{N}(r,L^n)+N_{k+1}(r,0;L^n)+\overline{N}\left(r,1;(L^n)^{(k)}\right)-N_0\left(r,0;(L^n)^{(k+1)}\right)+S(r,f)\\ &\leq \overline{N}(r,L)+(k+1)\overline{N}(r,0;L)+\overline{N}\left(r,1;(f^n)^{(k)}\right)+S(r,f)\\ &\leq (k+1)T(r,L)+T\left(r,(f^n)^{(k)}\right)+S(r,f).\end{aligned}$$

This gives

$$(n-k-1)T(r,L) \leq T\left(r,(f^n)^{(k)}\right)+S(r,f).$$

From (3.1) it follows that L is a transcendental meromorphic function. Combining this with the above inequality, Theorem 1.5 [23] and the assumption of the lower bound of n, we obtain that

$(f^n)^{(k)}$ and so f is a transcendental meromorphic function. Then proceeding similarly as in the proof of Theorem 2, we get three cases for $l \geq 2$, $l = 1$ and $l = 0$ each of which leads to the conclusion that either $(f^n)^{(k)}(L^n)^{(k)} = 1$ or $f^n = L^n$. If $f^n = L^n$, we have $f = tL$ with some t satisfying $t^n = 1$. If $(f^n)^{(k)}(L^n)^{(k)} = 1$, then considering $F_2 = (f^n)^{(k)}$ and $G_2 = (L^n)^{(k)}$ such that $F_2G_2 = 1$ and then arguing similarly as in Case 1, we get a contradiction. This completes the proof of Theorem 1.

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