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Quenched central limit theorem rates of convergence for one-dimensional random walks in random environments

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Unlike classical simple random walks, one-dimensional random walks in random environments (RWRE) are known to have a wide array of potential limiting distributions. Under certain assumptions, however, it is known that CLT-like limiting distributions hold for the walk under both the quenched and averaged measures. We give upper bounds on the rates of convergence for the quenched central limit theorems for both the hitting time and position of the RWRE with polynomial rates of convergence that depend on the distribution on environments.

Keywords: quenched central limit theorem; rates of convergence

1. Introduction

If $\{\xi_k\}_{k\geq 1}$ is an i.i.d. sequence of zero mean random variables with finite variance $\sigma^2 = E[\xi_1^2]$, then the central limit theorem implies that the rescaled sum $Z_n = \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^n \xi_k$ converges in distribution to a standard Gaussian random variable. That is, $F_n(x) = P(Z_n \leq x) \to \Phi(x)$ where Φ is the c.d.f. of the standard normal distribution. The central limit theorem, however, offers no quantitative bounds on the rate of convergence of F_n to Φ and in fact additional moment assumptions are needed to obtain such rates of convergence. The classical Berry–Esseen theorem (Berry [4], Esseen [9]) states that there is a universal constant $A_1 < \infty$ such that if ξ_1 has finite third moment then

$$\|F_n - \Phi\|_\infty = \sup_{x\in\mathbb{R}} \big\| F_n(x) - \Phi(x) \big\| \leq \frac{A_1 E[|\xi_1|^3]}{\sigma^3\sqrt{n}}, \qquad \forall n \geq 1.$$

More generally, one can obtain slower rates of convergence under weaker moment assumptions. In particular, it follows from Katz [16] that for any $\delta \in (0,1]$ there exists a universal constant $A_\delta < \infty$ such that if ξ_k has finite $(2+\delta)$th moment then

$$\|F_n - \Phi\|_\infty \leq \frac{A_\delta E[|\xi_1|^{2+\delta}]}{\sigma^{2+\delta} n^{\delta/2}}, \qquad \forall n \geq 1.$$

In this paper, we will be concerned with obtaining Berry–Esseen like rates of convergence for central limit theorems arising in one-dimensional random walks in random environments.

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A random walk in a random environment (RWRE) is a simple model for random motion in a non-homogeneous environment. The class of models that may be considered RWRE is quite large, but we will be concerned here with the case of (nearest-neighbor) one-dimensional RWRE. In this model, a *random environment* is a random sequence $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$ which can be used to determine the transition probabilities for a Markov chain on $\mathbb{Z}$ with steps of size ± 1. In particular, given an environment ω and a starting point $x \in \mathbb{Z}$ we will denote by P_ω^x the law of a Markov chain $\{X_n\}_{n\ge 0}$ defined by $P_\omega^x(X_0 = x) = 1$ and

$$P_\omega^x(X_{n+1} = y+1 \mid X_n = y) = 1 - P_\omega^x(X_{n+1} = y-1 \mid X_n = y) = \omega_y.$$

The distribution P_ω^x of the walk in a fixed environment is called the *quenched* law of the RWRE. If P denotes the probability distribution of the environment ω, then by averaging the quenched P_ω^x law with respect to P we obtain the *averaged* (or *annealed*) law of the RWRE:

$$\mathbb{P}^x(\cdot) = E\big[P_\omega^x(\cdot)\big].$$

Expectations with respect to the quenched and averaged laws of the walk are denoted by E_ω^x and $\mathbb{E}^x$, respectively. Usually the walk will be started at $X_0 = 0$ and we will use P_ω and $\mathbb{P}$ to denote the quenched and averaged laws in this case and corresponding expectations by E_ω and $\mathbb{E}$, respectively. Finally, variances under the quenched measure P_ω will be denoted by Var_ω; that is $\mathrm{Var}_\omega(Z) = E_\omega[Z^2] - E_\omega[Z]^2$.

While RWREs are a rather simple generalization of classical simple random walks, the behaviors of RWREs can be quite different than what is known for simple random walks. For instance, if the distribution on environments is such that the walk is recurrent then (under rather tame additional assumptions) the position of the walk converges in distribution to a non-Gaussian distribution when scaled by $(\log n)^2$ rather than the diffusive $\sqrt{n}$ scaling in classical simple random walks (Sinaĭ [27]). Transient RWREs can also exhibit a variety of non-Gaussian limiting distributions under non-diffusive scalings (Kesten, Kozlov and Spitzer [17], Mayer-Wolf, Roitershtein and Zeitouni [18]), but in this paper we will be assuming conditions under which it is known that CLT-like limiting distributions hold.

The first assumption that we will be making in this paper is that the environments are i.i.d.

Assumption 1. The distribution P on environments is such that $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$ is i.i.d.

For our second main assumption, we will need to introduce some additional notation. First, let

$$\rho_x = \frac{1-\omega_x}{\omega_x}, \qquad x \in \mathbb{Z}.$$

Many of the known results for RWREs can be stated in terms of the distribution of this ratio of transition probabilities. For instance, under Assumption 1 the RWRE is transient to the right if $E[\log \rho_0] < 0$ and the limiting speed $\mathrm{v}_0 = \lim_{n\to\infty} X_n/n$ is positive if and only if $E[\rho_0] < 1$ (Solomon [28]). In this paper we will be making the following assumption regarding the moments of the random variable ρ_0.

Assumption 2. $\kappa := \sup\{p > 0 : E[\rho_0^p] < 1\} > 2$ (or equivalently $E[\rho_0^{2+\delta}] < 1$ for some $\delta > 0$).

Since $t \mapsto E[\rho_0^t] = E[e^{t \log \rho_0}]$ is the moment generating function of $\log \rho_0$ and is therefore a convex function in t, it follows from Assumption 2 that $E[\log \rho_0] < 0$ (that is the walk is transient to the right) and that

$$r_p := E\big[\rho_0^p\big] < 1 \qquad \text{for all } p \in (0, \kappa). \tag{1}$$

In particular, this implies that $r_1 = E[\rho_0] < 1$ so that the speed v_0 of the walk is positive.

It should be noted that under rather mild additional assumptions it holds that

$$E\big[\rho_0^\kappa\big] = 1. \tag{2}$$

In fact, in a number of previous results in RWRE the parameter κ is defined as the unique positive solution to equation (2). For instance, the parameter κ defined this way is used in studying limiting distributions of transient RWRE in Kesten, Kozlov and Spitzer [17], Peterson and Zeitouni [24], Peterson [21], Peterson and Samorodnitsky [23], Peterson and Samorodnitsky [22], Enriquez et al. [8], Dolgopyat and Goldsheid [7], identifying the subexponential rate of decay of certain large deviation probabilities in Dembo, Peres and Zeitouni [6], Gantert and Zeitouni [12], Ahn and Peterson [1], and identifying the maximal displacement of large "bridges" of RWRE in Gantert and Peterson [10]. A number of these results assume additional technical conditions (e.g., $E[\rho_0^\kappa \log \rho_0] < \infty$ and the distribution of $\log \rho_0$ is non-lattice) to obtain certain precise tail asymptotics, but we will not need these conditions nor the slightly more restrictive definition of κ in (2).

The relevance of the parameter κ to the limiting distributions of transient RWRE comes from the fact that κ determines what moments of the hitting times of the RWRE are finite (c.f. Lemma 2.1 below); in particular, hitting times have finite second moment if $\kappa > 2$. The limiting distributions under the averaged measure $\mathbb{P}$ for transient RWRE in Kesten, Kozlov and Spitzer [17] show that CLT-like limiting distributions hold only when $\kappa > 2$. In particular, when $\kappa \in (0, 2)$ the limiting distributions are non-Gaussian with non-diffusive scaling and when $\kappa = 2$ the limiting distribution is Gaussian but with scaling $\sqrt{n \log n}$. However, when $\kappa > 2$ we have the following CLT for both the position X_n of the walk and the hitting times

$$T_n = \inf\{k \geq 0 : X_k = n\}, \qquad n \in \mathbb{Z}.$$

Theorem 1.1 (Kesten, Kozlov and Spitzer [17], Zeitouni [31]). *If Assumptions 1 and 2 hold, then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - \frac{n}{\mathrm{v}_0}}{\sigma_0 \sqrt{n}} \leq x\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}_0}{\sigma_0 \mathrm{v}_0^{3/2} \sqrt{n}} \leq x\right) = \Phi(x), \qquad \forall x \in \mathbb{R},$$

where

$$\sigma_0^2 = E\big[\mathrm{Var}_\omega(T_1)\big] + \mathrm{Var}\big(E_\omega[T_1]\big) + 2\sum_{k=1}^{\infty} \mathrm{Cov}\big(E_\omega[T_1], E_\omega^k[T_{k+1}]\big) < \infty.$$

Theorem 1.1 gives CLTs for the RWRE under the averaged measure. However, in this paper, we will be primarily interested with CLTs under the quenched measure.

Theorem 1.2 (Alili [2], Goldsheid [15], Peterson [25]). *If Assumptions 1 and 2 hold, then*

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega[T_n]}{\sigma\sqrt{n}} \le x\right) = \lim_{n\to\infty} P_\omega\left(\frac{X_n - n\mathrm{v}_0 + Z_n(\omega)}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}} \le x\right) = \Phi(x), \qquad P\text{-a.s.},$$

for all $x \in \mathbb{R}$ where

$$\sigma^2 = E\big[\mathrm{Var}_\omega(T_1)\big] < \infty \quad \text{and} \quad Z_n(\omega) = \mathrm{v}_0\left(E_\omega[T_{\lfloor n\mathrm{v}_0\rfloor}] - \frac{\lfloor n\mathrm{v}_0\rfloor}{\mathrm{v}_0}\right).$$

Before continuing, some important differences between the quenched and averaged CLTs in Theorems 1.1 and 1.2 should be noted.

- The quenched CLTs in Theorem 1.2 require a random (depending on the environment) centering. Indeed, when Assumptions 1 and 2 hold it follows from a CLT for sums of ergodic sequences that $\frac{E_\omega[T_n]-n/\mathrm{v}_0}{\sqrt{n}}$ converges in distribution to a centered Gaussian (see Zeitouni [31] for details) and therefore one cannot have a quenched CLT for either T_n or X_n with determinstic centering.
- The quenched CLTs are much stronger statements than the averaged CLTs. Indeed, since the quenched probabilities are random variables (randomness coming from the environment ω), the limits in the quenched CLTs are required to hold for P-a.e. environment ω. Moreover, the quenched CLTs in Theorem 1.2 together with the CLT for $\frac{E_\omega[T_n]-n/\mathrm{v}_0}{\sqrt{n}}$ can be used to obtain the averaged CLTs in Theorem 1.1.
- Both the quenched and averaged CLTs are known to hold under somewhat more general assumptions than we have used here. In particular, CLTs have been proved for RWRE in ergodic environments with certain mixing conditions in Zeitouni [31], Goldsheid [15], Peterson [25], though in these cases the parameter κ needs to be defined differently than in Assumption 2 or (2).

1.1. Main results

The main results of the present paper concern the rates of convergence in the quenched CLT results in Theorem 1.2. Rates of convergence for the averaged CLT are also of interest, but require different methods and will be studied in a future paper.

Our approach to the quenched CLTs in this paper will be to follow the approach first used by Alili [2] in which one first proves a CLT for the hitting times and then uses this to deduce the CLT for the position of the walk. Therefore, our first two main results concern the rates of convergence for the quenched CLT for hitting times. Note that while the centering in Theorem 1.2 needs to be random, the scaling is deterministic. The following two theorems however show that the rate of convergence in the quenched CLT can be improved by using an environment-dependent scaling as well.

Theorem 1.3. *Let $\overline{F}_{n,\omega}(x) = P_\omega(\frac{T_n - E_\omega[T_n]}{\sqrt{\mathrm{Var}_\omega(T_n)}} \le x)$ be the normalized quenched distribution of T_n.*

- *If $\kappa > 3$, then there exists a constant $C \in (0,\infty)$ such that*

$$\limsup_{n\to\infty} \sqrt{n}\|\overline{F}_{n,\omega} - \Phi\|_\infty \le C, \qquad P\text{-a.s.}$$

- *If $\kappa \in (2,3]$, then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} n^{\frac{3}{2}-\frac{3}{\kappa}-\varepsilon}\|\overline{F}_{n,\omega} - \Phi\|_\infty = 0, \qquad P\text{-a.s.}$$

Theorem 1.4. *Let $F_{n,\omega}(x) = P_\omega(\frac{T_n - E_\omega[T_n]}{\sigma\sqrt{n}} \le x)$ be the quenched distribution of T_n with random (environment dependent) centering and deterministic scaling.*

- *If $\kappa > 4$, then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} n^{\frac{1}{2}-\varepsilon}\|F_{n,\omega} - \Phi\|_\infty = 0, \qquad P\text{-a.s.}$$

- *If $\kappa \in (2,4]$, then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} n^{1-\frac{2}{\kappa}-\varepsilon}\|F_{n,\omega} - \Phi\|_\infty = 0, \qquad P\text{-a.s.}$$

Our final main result is the following bounds on the rates of convergence in the quenched CLT for X_n. Note that the results in this theorem give different almost sure and in probability rates of convergence for the quenched CLT.

Theorem 1.5. *Let $G_{n,\omega}(x) = P_\omega(\frac{X_n - n\mathrm{v}_0 + Z_n(\omega)}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}} \le x)$. If $\kappa > 2$, then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} n^{\frac{1}{4}-\frac{1}{2\kappa}-\varepsilon}\|G_{n,\omega} - \Phi\|_\infty = 0, \qquad P\text{-a.s.}$$

Moreover, by relaxing the mode of convergence to that of in probability, then the following stronger rates of convergence can be obtained.

- *If $\kappa \in (2, \frac{12}{5})$ then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} n^{\frac{3}{2}-\frac{3}{\kappa}-\varepsilon}\|G_{n,\omega} - \Phi\|_\infty = 0, \qquad \text{in } P\text{-probability.} \tag{3}$$

- *If $\kappa \ge \frac{12}{5}$ then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} n^{\frac{1}{4}-\varepsilon}\|G_{n,\omega} - \Phi\|_\infty = 0, \qquad \text{in } P\text{-probability.} \tag{4}$$

A comparison of the different rates of convergence for the quenched CLTs in Theorems 1.3-1.5 can be seen in Figure 1.

An outline of the proofs of the main results is as follows. Sections 2 and 3 contain analysis of quenched moments of hitting times that will be used later in the proofs of the main results.

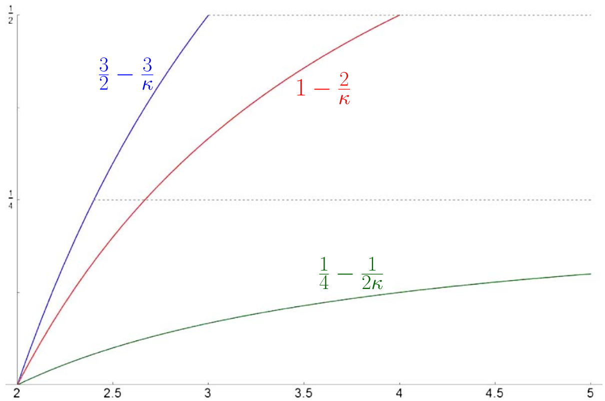

Figure 1. A comparison of the different polynomial exponents that appear in Theorems 1.3–1.5. The dotted lines are at height $1/4$ and $1/2$.

In particular, in Section 2 we show that $E[(E_\omega[T_1^m])^p] < \infty$ if $p \in (0, \kappa/m)$, and in Section 3 we control the fluctuations of $E_\omega[T_n]$ and $\mathrm{Var}_\omega(T_n)$ (Section 3 is the most technical and difficult part of the paper). The proofs of Theorems 1.3 and 1.4 are then given in Section 4. If we let

$$\tau_k = T_k - T_{k-1}, \qquad k \geq 1,$$

then under the quenched measure P_ω the random variables $\{\tau_k\}_{k\geq 1}$ are independent (but not identically distributed). Therefore, applying known results for sums of independent random variables gives a bound of $\|\overline{F}_{n,\omega} - \Phi\|_\infty$ in terms of the centered quenched moments of the crossing times τ_k. Control of these quenched moments then follows from results obtained in Section 2 and gives the rates of convergence in Theorem 1.3. Since the quenched distributions $\overline{F}_{n,\omega}$ and $F_{n,\omega}$ differ only in the choice of scaling, Theorem 1.4 then follows from Theorem 1.3 and control of the fluctuations of $\mathrm{Var}_\omega(T_n) - \sigma^2 n$ which were obtained in Section 3. Finally, in Section 5 the quenched rates of convergence in Theorem 1.5 are obtained from Theorem 1.3 in much the same way as the renewal process CLT is obtained from the standard CLT. It is here that the need for the quenched centering in the quenched CLTs presents a real difficulty, and in fact the control of the fluctuations of $E_\omega[T_n] - n/\mathrm{v}_0$ obtained in Section 3 are the main contributor to the almost sure rates of convergence in Theorem 1.5.

1.2. Discussion of main results and future work

Central limit theorems for random motion in random media are closely related to problems in stochastic homogenezation, a connection going back at least to Papanicolaou and Varadhan [20]. Results in quantitative stochastic homogenization (that is, results which give bounds on the rate of convergence of the solution of a PDE with random coefficients to the solution of the deterministic homogenized PDE) were first obtained by Yurinskiĭ [29,30], but recently there have been a number of important breakthroughs (Caffarelli and Souganidis [5], Gloria and Otto [14], Armstrong and Smart [3], Gloria, Neukamm and Otto [13]). However, the only results of which we are aware of giving quantitative rates of convergence for central limit theorems for RWRE are in Mourrat [19]. Mourrat's results differ from those in the present paper as they are for the random conductance model of RWRE rather than for RWRE in i.i.d. and he proves quantitative rates of convergence for the averaged CLT rather than the quenched CLT. Mourrat also gives rates of convergence for the random conductance model in any dimension $d \geq 1$, while our methods are restricted to one dimension. It should also be noted that the martingale method that Mourrat uses is limited to proving at best rates of convergence of $n^{-1/5}$, while the rates of convergence in Theorems 1.3–1.5 are in many cases faster than $n^{-1/5}$.

A natural question regarding the main results of this paper is the optimality of the rates of convergence obtained. The quenched rates in Theorem 1.3 when $\kappa > 3$ are clearly optimal, though it is not clear if the other rates in Theorems 1.3 and 1.4 are optimal. However, we conjecture that they are optimal in the sense that no better almost sure polynomial rate can be obtained. In particular, if one sets $\varepsilon = 0$ in these results we conjecture that the limits do not exist. For instance, if $\kappa \in (2, 3)$ we conjecture that

$$\liminf_{n\to\infty} n^{\frac{3}{2}-\frac{3}{\kappa}} \|\overline{F}_{n,\omega} - \Phi\|_\infty = 0 \quad \text{and} \quad \limsup_{n\to\infty} n^{\frac{3}{2}-\frac{3}{\kappa}} \|\overline{F}_{n,\omega} - \Phi\|_\infty = \infty, \qquad P\text{-a.s.}$$

It is less clear to us if the rates of convergence in Theorems 1.5 are optimal or not. In particular, one wonders if a different method of proof of the quenched CLT for X_n would lead to an improved rate of convergence. Zeitouni outlines in [31] how the quenched CLT for X_n can be obtained from a martingale CLT via the "harmonic corrector" approach. However, since the fluctuations of the harmonic corrector are given in this case by the fluctuations of $E_\omega[T_n] - n/\mathrm{v}_0$ it seems that this approach will not yield any better results than that of the approach in the current paper. Also, given that Theorems 1.3 and 1.4 show that the choice of normalization can affect the rates of convergence, one wonders if the rates of convergence in Theorem 1.5 can be improved by using a different centering or an environment-dependent scaling. In particular, one might suspect that better rates of convergence can be obtained for $\frac{X_n - E_\omega[X_n]}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}}$ or $\frac{X_n - E_\omega[X_n]}{\sqrt{\mathrm{Var}_\omega(X_n)}}$. Unfortunately, as of now we are not aware of any proofs of the quenched central limit theorem that work for these normalizations directly. (Of course one might be able to obtain a quenched CLT for $\frac{X_n - E_\omega[X_n]}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}}$ indirectly by using the quenched CLT for $\frac{X_n - n\mathrm{v}_0 + Z_n(\omega)}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}}$ and then proving that $\frac{E_\omega[X_n] - n\mathrm{v}_0 + Z_n(\omega)}{\sqrt{n}} \to 0$, P-a.s., but this would not lead to any possible improvement in the rate of convergence for the quenched CLT.)

2. Quenched moments of hitting times

In this section, we will collect some facts about quenched moments of hitting times that will be useful later. The main result is the following L^p estimate for the quenched moments of hitting times.

Lemma 2.1. *If $E[\log \rho_0] < 0$ and $\kappa > 0$, then for any integer $m \geq 1$, $E[E_\omega[\tau_1^m]^p] < \infty$ for all $p \in (0, \frac{\kappa}{m})$.*

Remark 2.2. We will only need Lemma 2.1 for $m \le 3$ and $\kappa > 2$ in the present paper. Nevertheless, since the proof generalizes easily to all $m \ge 1$ we give the more general proof here.

Remark 2.3. If the parameter κ satisfies the slightly stronger definition (2), and if the technical conditions $E[\rho_0^\kappa \log \rho_0] < \infty$ and the distribution of $\log \rho_0$ is non-lattice are also satisfied, then it is known that $P(E_\omega[\tau_1] > x) \sim C x^{-\kappa}$ as $x \to \infty$ which is a stronger statement than the L^p bounds in the statement of Lemma 2.1 for the case $m = 1$. We conjecture that under these stronger assumptions that similar tail asymptotics hold for $E_\omega[\tau_1^m]$ also; that is, we conjecture that for any $m \ge 1$ there exists a constant $C_m > 0$ such that $P(E_\omega[\tau_1^m] > x) \sim C_m x^{-\kappa/m}$ as $x \to \infty$. However, since such precise tail asymptotics are not needed for our purposes in this paper we content ourselves with the weaker L^p bounds given here.

Proof. We begin by computing recursive formulas for $E_\omega[\tau_1^m]$. To this end, it is helpful to introduce the natural left shift operator θ on the space of environments. That is, for any $k \in \mathbb{Z}$, $\theta^k \omega$ is the environment with $(\theta^k \omega)_x = \omega_{x+k}$ for every $x \in \mathbb{Z}$. With this notation, by conditioning on the first step of the walk,

$$\begin{aligned}
E_\omega\bigl[\tau_1^m\bigr] &= \omega_0 + (1-\omega_0) E_\omega^{-1}\bigl[(1+T_1)^m\bigr] \\
&= \omega_0 + (1-\omega_0) E_{\theta^{-1}\omega}\bigl[(1+\tau_1+\tau_2)^m\bigr] \\
&= \omega_0 + (1-\omega_0) \sum_{\substack{0 \le k_1, k_2 < m \\ k_1+k_2 \le m}} \binom{m}{k_1, k_2, m-k_1-k_2} E_{\theta^{-1}\omega}\bigl[\tau_1^{k_1}\bigr] E_\omega\bigl[\tau_1^{k_2}\bigr] \\
&\quad + (1-\omega_0) E_{\theta^{-1}\omega}\bigl[\tau_1^m\bigr] + (1-\omega_0) E_\omega\bigl[\tau_1^m\bigr].
\end{aligned}$$

Assuming for the moment that all of the above quenched expectations are finite we can solve this for $E_\omega[\tau_1^m]$ to obtain

$$\begin{aligned}
E_\omega\bigl[\tau_1^m\bigr] &= 1 + \rho_0 \sum_{\substack{0 \le k_1, k_2 < m \\ k_1+k_2 \le m}} \binom{m}{k_1, k_2, m-k_1-k_2} E_{\theta^{-1}\omega}\bigl[\tau_1^{k_1}\bigr] E_\omega\bigl[\tau_1^{k_2}\bigr] + \rho_0 E_{\theta^{-1}\omega}\bigl[\tau_1^m\bigr] \\
&=: f_m(\omega) + \rho_0 E_{\theta^{-1}\omega}\bigl[\tau_1^m\bigr]
\end{aligned} \tag{5}$$

(where the last equality gives the definition of $f_m(\omega)$), and iterating this we obtain

$$E_\omega\bigl[\tau_1^m\bigr] = f_m(\omega) + \sum_{k=1}^{n-1} \Pi_{-k+1,0} f_m\bigl(\theta^{-k}\omega\bigr) + \Pi_{-n+1,0} E_{\theta^{-n}\omega}\bigl[\tau_1^m\bigr],$$

where

$$\Pi_{i,j} = \prod_{x=i}^{j} \rho_x \qquad \text{for any } i \le j.$$

In the argument thus far, we have been assuming that all the quenched expectations are finite which may not necessarily be true. To account for this, we can modify the environment by adding a reflection to the right at a point to the left of the origin. In particular, for any $n \geq 1$ let $\omega(n) = \{\omega(n)_x\}_{x\in\mathbb{Z}}$ be the environment such with a reflection added at $x = -n$. That is,

$$\omega(n)_x = \begin{cases} \omega_x & \text{if } x \neq -n, \\ 1 & \text{if } x = -n. \end{cases}$$

The added reflection makes it so that τ_1 has exponential tails under the measure $P_{\omega(n)}$ so that in particular $E_{\theta^x\omega(n)}[\tau_1^m] < \infty$ for any $x \geq -n$. Therefore, repeating the above recursive argument in the environment $\omega(n)$ gives

$$E_{\omega(n)}\big[\tau_1^m\big] = f_m\big(\omega(n)\big) + \sum_{k=1}^{n-1} \Pi_{-k+1,0} f_m\big(\theta^{-k}\omega(n)\big) + \Pi_{-n+1,0}.$$

We wish to then take $n \to \infty$ in the above to obtain a formula for $E_\omega[\tau_1^m]$. Since $E[\log \rho_0] < 0$ the last term on the right vanishes almost surely as $n \to \infty$. For the other terms, by coupling the path of the walk in the environment ω to the paths in $\omega(n)$ up to the stopping time T_{-n} we see that $E_{\theta^x\omega(n)}[\tau_1^\ell] \nearrow E_{\theta^x\omega}[\tau_1^\ell]$ as $n \nearrow \infty$ for any fixed x and ℓ. In particular, this implies that $f_m(\theta^{-k}\omega(n))$ is non-decreasing in n and so the monotone convergence theorem implies that

$$E_\omega\big[\tau_1^m\big] = f_m(\omega) + \sum_{k=1}^{\infty} \Pi_{-k+1,0} f_m\big(\theta^{-k}\omega\big). \tag{6}$$

We will now use (6) to prove the moment bounds for $E_\omega[\tau_1^m]$. A key tool that we will use in the proof is the following simple lemma which follows from Minkowski's inequality when $p \geq 1$ and the sub-additivity of $x \mapsto x^p$ when $p \in (0, 1)$.

Lemma 2.4. *Let $Y_1, Y_2, \ldots$ be non-negative random variables and let $Z = \sum_{k=0}^\infty Y_k$.*

- *If $p < 1$ and $\sum_{k=0}^\infty E[Y_k^p] < \infty$, then $E[Z^p] < \infty$.*
- *If $p \geq 1$ and $\sum_{k=0}^\infty E[Y_k^p]^{1/p} < \infty$, then $E[Z^p] < \infty$.*

By this lemma and (6) it will be enough to show that $E[(\Pi_{-k+1,0} f_m(\theta^{-k}\omega))^p]$ is decreasing exponentially fast if $p \in (0, \frac{\kappa}{m})$. To prove this, first note $f_m(\omega)$ depends only on the environment to the left of the origin. Therefore, since the environment is i.i.d. we have that

$$E\big[\big(\Pi_{-k+1,0} f_m\big(\theta^{-k}\omega\big)\big)^p\big] = E\big[(\Pi_{-k+1,0})^p\big] E\big[f_m(\omega)^p\big] = (r_p)^k E\big[f_m(\omega)^p\big].$$

Since it follows from (1) that $r_p < 1$ we have thus reduced ourselves to proving

$$E\big[f_m(\omega)^p\big] < \infty, \qquad \text{for all } p \in \left(0, \frac{\kappa}{m}\right), m \geq 1. \tag{7}$$

We will prove (7) by induction on $m \geq 1$. In the case $m = 1$, we have that $f_1(\omega) = 1 + \rho_0$ and so $E[f_1(\omega)^p] = E[(1+\rho_0)^p] < \infty$ for $p \in (0, \kappa)$ holds. Next, we will assume that (7) holds up to $m-1$; that is, we will assume that $E[E_\omega[\tau_1^k]^p] < \infty$ for any $p \in (0, \frac{\kappa}{k})$ and $k \leq m-1$. Under this assumption, if $0 \leq k_1, k_2 < m$ and $k_1 + k_2 \leq m$ then Hölder's inequality implies that

$$
\begin{aligned}
&E\big[\big(E_{\theta^{-1}\omega}\big[\tau_1^{k_1}\big]E_\omega\big[\tau_1^{k_2}\big]\big)^p\big] \\
&\quad \leq \big(E\big[E_\omega\big[\tau_1^{k_1}\big]^{\frac{mp}{k_1}}\big]\big)^{\frac{k_1}{m}} \big(E\big[E_\omega\big[\tau_1^{k_2}\big]^{\frac{mp}{m-k_1}}\big]\big)^{\frac{m-k_1}{m}} < \infty \qquad \text{if } p \in \left(0, \frac{\kappa}{m}\right),
\end{aligned}
$$

where the expectations on the right are finite by the induction assumption since $\frac{mp}{k_1} < \frac{\kappa}{k_1}$ and $\frac{mp}{m-k_1} \leq \frac{mp}{k_2} < \frac{\kappa}{k_2}$. This is enough to conclude that (7) holds for m as well, and by induction for all $m \geq 1$. □

We close this section by noting some additional consequences of the recursive formula for $E_\omega[\tau_1^m]$ that will be useful later in the paper. For ease of notation, we will introduce the following notation for the quenched mean and variance of hitting times that will be used throughout the paper.

$$
\mu_k = E_{\theta^k\omega}[\tau_1] \quad \text{and} \quad V_k = \mathrm{Var}_{\theta^k\omega}(\tau_1).
$$

When $m = 1, 2$, the recursive formula (5) (applied to the shifted environment $\theta^k\omega$) yields

$$
\mu_k = 1 + \rho_k + \rho_k \mu_{k-1}, \tag{8}
$$

and

$$
E_{\theta^k\omega}\big[\tau_1^2\big] = 1 + \rho_k\big(1 + 2\mu_{k-1} + 2\mu_k + 2\mu_{k-1}\mu_k + E_{\theta^{k-1}\omega}\big[\tau_1^2\big]\big).
$$

Inserting the first formula into the second and then simplifying yields

$$
\begin{aligned}
E_{\theta^k\omega}\big[\tau_1^2\big] &= 1 + \rho_k\big(1 + 2\mu_{k-1} + 2\big\{(1+\rho_k) + (1+2\rho_k)\mu_{k-1} + \rho_k\mu_{k-1}^2\big\} + E_{\theta^{k-1}\omega}\big[\tau_1^2\big]\big) \\
&= 1 + \rho_k\big(1 + 2(1+\rho_k) + 4(1+\rho_k)\mu_{k-1} + 2\rho_k\mu_{k-1}^2 + E_{\theta^{k-1}\omega}\big[\tau_1^2\big]\big) \\
&= (1+\rho_k)(1+2\rho_k) + 4\rho_k(1+\rho_k)\mu_{k-1} + 2\rho_k^2\mu_{k-1}^2 + \rho_k E_{\theta^{k-1}\omega}\big[\tau_1^2\big].
\end{aligned} \tag{9}
$$

Combining (8) and (9) then yields the following recursive formula for the quenched variance.

$$
\begin{aligned}
V_k &= E_{\theta^k\omega}\big[\tau_1^2\big] - \mu_k^2 \\
&= (1+\rho_k)(1+2\rho_k) + 4\rho_k(1+\rho_k)\mu_{k-1} + 2\rho_k^2\mu_{k-1}^2 + \rho_k E_{\theta^{k-1}\omega}\big[\tau_1^2\big] \\
&\quad - (1+\rho_k)^2 - 2\rho_k(1+\rho_k)\mu_{k-1} - \rho_k^2\mu_{k-1}^2 \\
&= \rho_k(1+\rho_k) + 2\rho_k(1+\rho_k)\mu_{k-1} + \rho_k^2\mu_{k-1}^2 + \rho_k E_{\theta^{k-1}\omega}\big[\tau_1^2\big] \\
&= \rho_k(1+\rho_k) + 2\rho_k(1+\rho_k)\mu_{k-1} + \rho_k(1+\rho_k)\mu_{k-1}^2 + \rho_k V_{k-1} \\
&= \rho_k(1+\rho_k)(1+\mu_{k-1})^2 + \rho_k V_{k-1}.
\end{aligned} \tag{10}
$$

Finally, we note that since μ_{k-1} is independent of ρ_k, one can take expectations of both sides of (8) (or square both sides and then take expectations) to obtain the following explicit formulas for the first two moments of the quenched hitting times.

$$\frac{1}{\mathrm{v}_0} = \mathbb{E}[\tau_1] = E[\mu_0] = \frac{1+r_1}{1-r_1}, \qquad E\bigl[\mu_0^2\bigr] = \frac{1+3r_1+3r_2+r_1r_2}{(1-r_1)(1-r_2)}. \tag{11}$$

This formula for $\mathbb{E}[\tau_1]$ is well known and in fact was originally obtained in this manner in the seminal paper of Solomon [28]. Similarly, taking expectations of both sides of (10) and using the formulas in (11) and the fact that V_{k-1} is independent of ρ_k one can obtain

$$\sigma^2 = E\bigl[\mathrm{Var}_\omega(\tau_1)\bigr] = E[V_0] = \frac{4(1+r_1)(r_1+r_2)}{(1-r_2)(1-r_1)^2}. \tag{12}$$

We will briefly provide the details of this argument for (12) since the formula here corrects for a small typo in the formula given in Goldsheid [15].

$$\begin{aligned} E[V_0] &= (r_1+r_2)\bigl(1+2E[\mu_0]+E\bigl[\mu_0^2\bigr]\bigr)+r_1E[V_0] \\ &= (r_1+r_2)\left(1+2\frac{1+r_1}{1-r_1}+\frac{1+3r_1+3r_2+r_1r_2}{(1-r_1)(1-r_2)}\right)+r_1E[V_0] \\ &= (r_1+r_2)\left(\frac{4(1+r_1)}{(1-r_1)(1-r_2)}\right)+r_1E[V_0]. \end{aligned}$$

Solving this for $E[V_0]$ we obtain the formula in (12).

3. Asymptotics of the quenched mean and variance of the hitting times

Since $E_\omega[T_n] = \sum_{k=0}^{n-1}\mu_k$ and $\mathrm{Var}_\omega(T_n) = \sum_{k=0}^{n-1}V_k$ and since $\{\mu_k\}_{k\in\mathbb{Z}}$ and $\{V_k\}_{k\in\mathbb{Z}}$ are ergodic sequences, it follows that $E_\omega[T_n]/n \to E[\mu_0] = \mathbb{E}[\tau_1] = \frac{1}{\mathrm{v}_0}$ and $\mathrm{Var}_\omega(T_n)/n \to E[V_0] = \sigma^2$, almost surely as $n\to\infty$. However, for the proofs of our main results we will need control on the fluctuations of $E_\omega[T_n]$ and $\mathrm{Var}_\omega(T_n)$ from these deterministic limits. The first such result we need is the following lemma which was proved by Goldsheid.

Lemma 3.1 (Lemma 4 in Goldsheid [15]). *If $\kappa > 2$, then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{E_\omega[T_n]-\frac{n}{\mathrm{v}_0}}{n^{1/2+\varepsilon}} = 0, \qquad P\text{-}a.s.$$

The main results of this section are the following two propositions, the first of which gives an improvement to Lemma 3.1 by controlling the fluctuations the quenched mean of hitting times of nearby locations and the second of which which controls the fluctuations of the quenched variance of hitting times.

Proposition 3.2. *For any $n \geq 1$ and $\varepsilon > 0$ denote $I_{\varepsilon,n} = [n\mathrm{v}_0 - n^{1/2+\varepsilon}, n\mathrm{v}_0 + n^{1/2+\varepsilon}]$. If $\kappa > 2$, then*

$$\lim_{n\to\infty} \max_{k,\ell\in I_{\varepsilon,n}} \frac{|E_\omega[T_k] - E_\omega[T_\ell] - \frac{k-\ell}{\mathrm{v}_0}|}{n^{1/4+\varepsilon/2+\varepsilon'}} = 0, \qquad \textit{in } P\textit{-probability, for any } \varepsilon' > 0, \tag{13}$$

and

$$\lim_{n\to\infty} \max_{k,\ell\in I_{\varepsilon,n}} \frac{|E_\omega[T_k] - E_\omega[T_\ell] - \frac{k-\ell}{\mathrm{v}_0}|}{n^{\frac{1}{4}+\frac{1}{2\kappa}+\varepsilon(\frac{1}{2}-\frac{1}{\kappa})+\varepsilon'}} = 0, \qquad P\textit{-a.s., for any } \varepsilon' > 0. \tag{14}$$

Proposition 3.3. *If $\kappa > 2$, then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\mathrm{Var}_\omega(T_n) - \sigma^2 n}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0, \qquad P\textit{-a.s.}$$

Remark 3.4. It was shown in Goldsheid [15], Lemma 5, that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{k,\ell\in I_{\varepsilon,n}} \frac{|E_\omega[T_k] - E_\omega[T_\ell] - \frac{k-\ell}{\mathrm{v}_0}|}{\sqrt{n}} = 0, \qquad P\text{-a.s.}$$

Thus, for $\varepsilon \in (0, 1/2)$, Proposition 3.2 is an improvement on the results in Goldsheid [15].

Remark 3.5. The change in the magnitude of the fluctuations of $\mathrm{Var}_\omega(T_n)$ at $\kappa = 4$ in Proposition 3.3 is due to the fact that $\mathrm{Var}_\omega(T_n)$ has finite second moment when $\kappa > 4$. In fact, though we will not need this here, it can be shown that if $\kappa > 4$ then $\frac{\mathrm{Var}_\omega(T_n)-\sigma^2 n}{\sqrt{n}}$ converges in distribution to a zero mean Gaussian random variable. We also suspect that under additional regularity assumptions ($E[\rho_0^\kappa] = 1$, $E[\rho_0^\kappa \log\rho_0] < \infty$ and the distribution of $\log\rho_0$ is non-lattice) that if $\kappa \in (2, 4)$ then $\frac{\mathrm{Var}_\omega(T_n)-\sigma^2 n}{n^{2/\kappa}}$ converges in distribution to a $\kappa/2$-stable random variable.

The main idea of the proofs of both Propositions 3.2 and 3.3 is that $E_\omega[T_n] - n/\mathrm{v}_0$ and $\mathrm{Var}_\omega(T_n) - \sigma^2 n$ can be approximated by martingales which are sums of stationary ergodic sequences. To this end, it will be helpful to first state and prove the following general lemma.

Lemma 3.6. *Let that $\{Z_k\}_{k\in\mathbb{Z}}$ be a stationary ergodic sequence and let $\{W_n\}_{n\geq 0}$ be the martingale defined by $W_0 = 0$ and*

$$W_n = \sum_{k=0}^{n-1} \big(Z_k - E[Z_k|\mathcal{F}_{k-1}]\big), \qquad \textit{where } \mathcal{F}_k = \sigma(Z_j : j \leq k),$$

and let $W_n^* = \max_{k\leq n} |W_k|$. If $E[|Z_1|^p] < \infty$ for all $1 \leq p < \alpha$, then for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{W_n^*}{n^{\frac{1}{\alpha\wedge 2}+\varepsilon}} = 0, \qquad P\textit{-a.s.}$$

Additionally, if $\alpha > 2$ then $E[|W_n^*|^p] = \mathcal{O}(n^{p/2})$ for all $p \in [2, \alpha)$.

Proof. We will divide the proof into two cases: $\alpha > 2$ and $\alpha \in (1, 2]$. In both cases, however we will use that

$$E\left[\sum_{k=1}^{n} |W_k - W_{k-1}|^p\right] = \sum_{k=1}^{n} E\left[\left|Z_{k-1} - E[Z_{k-1} \mid \mathcal{F}_{k-2}]\right|^p\right]$$
$$= nE\left[\left|Z_1 - E[Z_1 \mid \mathcal{F}_0]\right|^p\right] = \mathcal{O}(n), \qquad \text{for } p < \alpha.$$

Case I: $\alpha > 2$. If $p \in [2, \alpha)$, it follows from the Burkholder–Davis–Gundy inequality and then Jensen's inequality that there exists a constant $C_p > 0$ depending only on p such that

$$E\left[\left|W_n^*\right|^p\right] \le C_p E\left[\left(\sum_{k=1}^{n}(W_k - W_{k-1})^2\right)^{p/2}\right]$$
$$\le C_p n^{p/2-1} E\left[\sum_{k=1}^{n} |W_k - W_{k-1}|^p\right] = \mathcal{O}\left(n^{p/2}\right).$$

From this it follows that $P(W_n^* > \delta n^{1/2+\varepsilon}) = \mathcal{O}(n^{-\varepsilon p})$, and so if we let $n_k = \lceil k^{2/(\varepsilon p)} \rceil$ it follows from the Borel–Cantelli lemma that

$$\lim_{k\to\infty} \frac{W_{n_k}^*}{n_k^{1/2+\varepsilon}} = 0, \qquad P\text{-a.s.}$$

Finally, since W_n^* is non-decreasing in n and $n_{k+1}/n_k \to 1$ as $k \to \infty$ the conclusion of the lemma follows easily.

Case II: $\alpha \in (1, 2]$. If $p \in [1, \alpha)$ then the Burkholder–Davis–Gundy inequality implies that

$$E\left[\left|W_n^*\right|^p\right] \le C_p E\left[\left(\sum_{k=1}^{n}(W_k - W_{k-1})^2\right)^{p/2}\right]$$
$$\le C_p E\left[\sum_{k=1}^{n} |W_k - W_{k-1}|^p\right] = \mathcal{O}(n),$$

where in the second inequality we used that $p/2 < 1$. Therefore, if $\max\{1, \frac{1+\varepsilon\alpha/2}{1/\alpha+\varepsilon}\} \le p < \alpha$, then

$$P\left(W_n^* > \delta n^{1/\alpha+\varepsilon}\right) = \mathcal{O}\left(n^{1-p(\frac{1}{\alpha}+\varepsilon)}\right) = \mathcal{O}\left(n^{-\varepsilon\alpha/2}\right),$$

where the last equality follows from $1 - p(\frac{1}{\alpha} + \varepsilon) \le 1 - \frac{1+\varepsilon\alpha/2}{1/\alpha+\varepsilon}(\frac{1}{\alpha} + \varepsilon) = -\frac{\varepsilon\alpha}{2}$. Letting $n_k = \lceil k^{4/(\varepsilon\alpha)} \rceil$, it follows from the Borel–Cantelli lemma that

$$\lim_{k\to\infty} \frac{W_{n_k}^*}{n_k^{1/\alpha+\varepsilon}} = 0, \qquad P\text{-a.s.}$$

As in Case I, the conclusion of the lemma follows easily from this since $n_{k+1}/n_k \to 1$ as $k \to \infty$. $\square$

We are now ready to give the proofs of the main results of this section.

Proof of Proposition 3.2. Consider the martingale defined by $M_0 = 0$ and

$$M_n = \sum_{k=0}^{n-1} \bigl(\mu_k - E[\mu_k \mid \mathcal{F}_{k-1}]\bigr), \qquad n \geq 1, \text{where } \mathcal{F}_m = \sigma(\omega_x : x \leq m).$$

To see the relevance of this martingale, note that it follows from the recursion for μ_k in (8) and the fact that ρ_k is independent of $\mathcal{F}_{k-1}$ that $E[\mu_k \mid \mathcal{F}_{k-1}] = 1 + r_1 + r_1\mu_{k-1}$. Using this we can re-write the martingale as

$$\begin{aligned}
M_n &= \sum_{k=0}^{n-1} \mu_k - (1+r_1)n - r_1 \sum_{k=-1}^{n-2} \mu_k \\
&= (1-r_1) \sum_{k=0}^{n-1} \mu_k - (1+r_1)n + r_1(\mu_{n-1} - \mu_{-1}) \\
&= (1-r_1)\left(E_\omega[T_n] - \frac{n}{\mathrm{v}_0}\right) + r_1(\mu_{n-1} - \mu_{-1}),
\end{aligned}$$

where in the last equality we used the explicit formula for v_0 in (11). It follows from this representation of the martingale that

$$\max_{k,\ell \in I_{\varepsilon,n}} \left| E_\omega[T_k] - E_\omega[T_\ell] - \frac{k-\ell}{\mathrm{v}_0} \right| \leq \max_{k,\ell \in I_{\varepsilon,n}} \frac{|M_k - M_\ell|}{1-r_1} + \frac{2r_1}{1-r_1} \max_{k \in I_{\varepsilon,n}} \mu_{k-1}. \tag{15}$$

To control the first term on the right in (15), it follows from Lemmas 2.1 and 3.6 imply that for any $p \in [2, \kappa)$ there exists a constant $C > 0$ such that

$$E\Bigl[\max_{\ell \in [k,k+n]} |M_\ell - M_k|^p\Bigr] \leq C n^{p/2}, \qquad \forall k \geq 0,$$

and thus

$$\begin{aligned}
&P\Bigl(\max_{k,\ell \in I_{\varepsilon,n}} |M_k - M_\ell| \geq \delta n^{1/4+\varepsilon/2+\varepsilon'}\Bigr) \\
&\quad\leq P\left(\max_{\ell \in I_{\varepsilon,n}} |M_\ell - M_{\lceil n\mathrm{v}_0 - n^{1/2+\varepsilon}\rceil}| \geq \frac{\delta}{2} n^{1/4+\varepsilon/2+\varepsilon'}\right) \\
&\quad\leq \frac{E[\max_{\ell \in I_{\varepsilon,n}} |M_\ell - M_{\lceil n\mathrm{v}_0 - n^{1/2+\varepsilon}\rceil}|^p]}{(\delta/2)^p n^{p(1/4+\varepsilon/2+\varepsilon')}} \\
&\quad= \mathcal{O}\bigl(n^{-p\varepsilon'}\bigr).
\end{aligned} \tag{16}$$

To control the second term on the right in (15), note that it follows from Lemma 2.1 and a pth moment bound for $p \in [2,\kappa)$ that

$$\begin{aligned} P\Big(\max_{k\in I_{\varepsilon,n}} \mu_{k-1} > \delta n^{\frac14+\frac{\varepsilon}{2}+\varepsilon'}\Big) &\le |I_{\varepsilon,n}| P\big(\mu_0 > \delta n^{\frac14+\frac{\varepsilon}{2}+\varepsilon'}\big) \\ &= \mathcal{O}\big(n^{\frac12+\varepsilon-p(\frac14+\frac{\varepsilon}{2}+\varepsilon')}\big) = \mathcal{O}\big(n^{-p\varepsilon'}\big). \end{aligned} \tag{17}$$

Applying (16) and (17) to (15) proves the convergence in probability statement in (13).

For the proof of the almost sure convergence in (14), we will use the bounds in (16) and (17) but we will need to restrict ourselves to $\varepsilon' > \frac{1/2-\varepsilon}{\kappa}$. For any such ε', fix p such that $\max\{2, \frac{1/2-\varepsilon}{\varepsilon'}\} < p < \kappa$ and then $\gamma > 0$ such that $\frac12 - \varepsilon < \frac{1}{\gamma} < p\varepsilon'$. If we let $n_k = \lfloor k^\gamma \rfloor$ then since $\gamma p \varepsilon' > 1$ it follows from (16) and (17) applied to (15) that

$$\lim_{k\to\infty} \max_{\ell,m\in I_{\varepsilon,n_k}} \frac{|E_\omega[T_m] - E_\omega[T_\ell] - \frac{m-\ell}{\mathrm{v}_0}|}{n_k^{1/4+\varepsilon/2+\varepsilon'}} = 0, \qquad P\text{-a.s.} \tag{18}$$

Next, since $\gamma(1/2-\varepsilon) < 1$ it follows that $n_{k+1}\mathrm{v}_0 - n_{k+1}^{1/2+\varepsilon} < n_k \mathrm{v}_0 + n_k^{1/2+\varepsilon}$ for k large, so that $I_{\varepsilon,n_k} \cap I_{\varepsilon,n_{k+1}} \neq \varnothing$ for all k large. If $n_k \le n < n_{k+1}$ and $I_{\varepsilon,n_k} \cap I_{\varepsilon,n_{k+1}}$, then it follows that

$$\max_{\ell,m\in I_{\varepsilon,n}} \frac{|M_m - M_\ell|}{n^{1/4+\varepsilon/2+\varepsilon'}} \le \max_{\ell,m\in I_{\varepsilon,n_k}\cup I_{\varepsilon,n_{k+1}}} \frac{|E_\omega[T_m] - E_\omega[T_\ell] - \frac{m-\ell}{\mathrm{v}_0}|}{n_k^{1/4+\varepsilon/2+\varepsilon'}},$$

and using (18) and the fact that $n_{k+1}/n_k \to 1$ as $k \to \infty$ the right-hand side vanishes almost surely as $k \to \infty$. Thus, we have shown that

$$\lim_{n\to\infty} \max_{k,\ell\in I_{\varepsilon,n}} \frac{|E_\omega[T_k] - E_\omega[T_\ell] - \frac{k-\ell}{\mathrm{v}_0}|}{n^{1/4+\varepsilon/2+\varepsilon'}} = 0, \qquad P\text{-a.s., for any } \varepsilon' > \frac{1/2-\varepsilon}{\kappa}.$$

Note that by taking ε' arbitrarily close to $\frac{1/2-\varepsilon}{\kappa}$ this is equivalent to the statement (14) we are trying to prove. □

Proof of Proposition 3.3. Consider the martingale $\{L_n\}_{n\ge0}$ defined by $L_0 = 0$ and

$$L_n = \mathrm{Var}_\omega(T_n) - \sum_{k=0}^{n-1} E[V_k \mid \mathcal{F}_{k-1}] = \sum_{k=0}^{n-1}\big(V_k - E[V_k \mid \mathcal{F}_{k-1}]\big).$$

It follows from Lemma 2.1 that $E[|V_0|^p] \le E[|E_\omega[\tau_1^2]|^p] < \infty$ for any $p < \kappa/2$, and thus Lemma 3.6 implies that

$$\lim_{n\to\infty} \frac{L_n}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0, \qquad P\text{-a.s., for any } \varepsilon > 0. \tag{19}$$

To compare L_n to $\mathrm{Var}_\omega(T_n) - \sigma^2 n$ we need to give a different representation of L_n. To this end, it follows from the recursive formula for the quenched variance in (10) that

$$E[V_k \mid \mathcal{F}_{k-1}] = E\big[\big(\rho_k + \rho_k^2\big)(1+\mu_{k-1})^2 + \rho_k V_{k-1} | \mathcal{F}_{k-1}\big] = (r_1 + r_2)(1+\mu_{k-1})^2 + r_1 V_{k-1},$$

and thus

$$\begin{aligned} L_n &= \mathrm{Var}_\omega(T_n) - \sum_{k=0}^{n-1} \big\{(r_1+r_2)(1+\mu_{k-1})^2 + r_1 V_{k-1}\big\} \\ &= \mathrm{Var}_\omega(T_n) - (r_1+r_2)\left\{ n + 2\sum_{k=-1}^{n-2} \mu_k + \sum_{k=-1}^{n-2} \mu_k^2 \right\} - r_1 \sum_{k=-1}^{n-2} V_k \\ &= (1-r_1)\,\mathrm{Var}_\omega(T_n) - (r_1+r_2)\left\{ n + 2E_\omega[T_n] + \sum_{k=0}^{n-1} \mu_k^2 \right\} \\ &\quad - (r_1+r_2)\big(2\mu_{n-1} + \mu_{n-1}^2 - 2\mu_{-1} - \mu_{-1}^2\big) + r_1(V_{n-1} - V_{-1}). \end{aligned} \tag{20}$$

To further simplify this, note that it follows from (11) and (12) that

$$\begin{aligned} &(1-r_1)\sigma^2 - (r_1+r_2)\big(1 + 2E[\mu_0] + E\big[\mu_0^2\big]\big) \\ &\quad = \frac{4(r_1+r_2)(1+r_1)}{(1-r_1)(1-r_2)} - (r_1+r_2)\left(1 + \frac{2(1+r_1)}{1-r_1} + \frac{1+3r_1+3r_2+r_1r_2}{(1-r_1)(1-r_2)}\right) \\ &\quad = \frac{4(r_1+r_2)(1+r_1)}{(1-r_1)(1-r_2)} - \frac{4(r_1+r_2)(1+r_1)}{(1-r_1)(1-r_2)} = 0. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} L_n &= (1-r_1)\big(\mathrm{Var}_\omega(T_n) - \sigma^2\big) - (r_1+r_2)\left\{ 2\left(E_\omega[T_n] - \frac{n}{\mathrm{v}_0}\right) + \sum_{k=0}^{n-1}\big(\mu_k^2 - E\big[\mu_0^2\big]\big) \right\} \\ &\quad - (r_1+r_2)\big(2\mu_{n-1} + \mu_{n-1}^2 - 2\mu_{-1} - \mu_{-1}^2\big) + r_1(V_{n-1} - V_{-1}). \end{aligned}$$

From this representation of L_n, by (19) and Lemma 3.1 we see that to finish the proof of Proposition 3.3 it is enough to show that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1}(\mu_k^2 - E[\mu_0^2])}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0, \qquad P\text{-a.s.} \tag{21}$$

and

$$\lim_{n\to\infty} \frac{\mu_{n-1}^2 + V_{n-1}}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = \lim_{n\to\infty} \frac{E_{\theta^{n-1}\omega}[\tau_1^2]}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0, \qquad P\text{-a.s.} \tag{22}$$

To prove (22), note that it follows from Lemma 2.1 that

$$P\left(E_{\theta^{n-1}\omega}\left[\tau_1^2\right] \geq \delta n^{\frac{2}{4\wedge\kappa}+\varepsilon}\right) = \mathcal{O}\left(n^{-\frac{\kappa}{4\wedge\kappa}-\frac{\varepsilon\kappa}{4}}\right) = \mathcal{O}\left(n^{-1-\frac{\varepsilon\kappa}{4}}\right),$$

and then (22) follows from the Borel–Cantelli lemma.

It remains only to prove (21), and to do this we will use another martingale. Define $H_n = 0$ and

$$H_n = \sum_{k=0}^{n-1} \left\{\mu_k^2 - E\left[\mu_k^2 \mid \mathcal{F}_{k-1}\right]\right\}, \qquad n \geq 1.$$

Note that Lemmas 2.1 and 3.6 imply that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{H_n}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0, \qquad P\text{-a.s.} \tag{23}$$

To use this to prove (21), we need to give a different representation of H_n. Using the recursive formula for μ_k in (8), it follows that

$$\begin{aligned} E\left[\mu_k^2 \mid \mathcal{F}_{k-1}\right] &= E\left[(1+\rho_k)^2 + 2\rho_k(1+\rho_k)\mu_{k-1} + \rho_k^2\mu_{k-1}^2 \mid \mathcal{F}_{k-1}\right] \\ &= 1 + 2r_1 + r_2 + 2(r_1+r_2)\mu_{k-1} + r_2\mu_{k-1}^2. \end{aligned}$$

Consequently, the martingale H_n can be re-written as

$$\begin{aligned} H_n &= \sum_{k=0}^{n-1} \mu_k^2 - (1+2r_1+r_2)n - 2(r_1+r_2)\sum_{k=-1}^{n-2} \mu_k - r_2 \sum_{k=-1}^{n-2} \mu_k^2 \\ &= (1-r_2)\sum_{k=0}^{n-1} \mu_k^2 - (1+2r_1+r_2)n - 2(r_1+r_2)E_\omega[T_n] \\ &\quad + 2(r_1+r_2)(\mu_{n-1} - \mu_{-1}) + r_2\left(\mu_{n-1}^2 - \mu_{-1}^2\right). \end{aligned}$$

Since the explicit formulas for $E[\mu_0]$ and $E[\mu_0^2]$ in (11) imply that

$$\begin{aligned} (1-r_2)E\left[\mu_0^2\right] - 2(r_1+r_2)E[\mu_0] &= \frac{1+3r_1+3r_2+r_1r_2}{1-r_1} - 2(r_1+r_2)\frac{1+r_1}{1-r_1} \\ &= \frac{1+r_1+r_2-2r_1^2-r_1r_2}{1-r_1} \\ &= 1+2r_1+r_2, \end{aligned}$$

we can further simplify the expression for H_n as

$$H_n = (1-r_2)\sum_{k=0}^{n-1}\left(\mu_k^2 - E\left[\mu_0^2\right]\right) - 2(r_1+r_2)\left(E_\omega[T_n] - \frac{n}{\mathrm{v}_0}\right) \\ + 2(r_1+r_2)(\mu_{n-1}-\mu_{-1}) + r_2\left(\mu_{n-1}^2 - \mu_{-1}^2\right). \tag{24}$$

An argument similar to the proof of (22) shows that $\lim_{n\to\infty}\frac{\mu_{n-1}^2}{n^{\frac{2}{4\wedge\kappa}+\varepsilon}} = 0$, P-a.s., and thus the proof of (21) follows from applying (23) and Lemma 3.1 to (24). $\square$

4. Quenched CLT rates of convergence for hitting times

Since the hitting times $T_n = \sum_{k=1}^n \tau_k$ are the sum of random variables that are independent under the quenched measure, a key element in our proof of Theorems 1.3 and 1.4 will be the following generalization of the Berry–Esseen estimates.

Theorem 4.1 (Theorem V.3.6 in Petrov [26]). *Let $S_n = \sum_{k=1}^n \xi_i$ be the sum of independent zero mean random variable with finite variance. For any $\delta \in (0,1]$ there exists a universal constant $A_\delta > 0$ such that*

$$\sup_x\left|P\left(\frac{S_n}{\sqrt{\mathrm{Var}(S_n)}} \le x\right) - \Phi(x)\right| \le \frac{A_\delta}{\mathrm{Var}(S_n)^{1+\frac{\delta}{2}}}\sum_{k=1}^n E\left[|\xi_i|^{2+\delta}\right].$$

Proof of Theorem 1.3. Since under the quenched measure $T_n - E_\omega[T_n] = \sum_{k=1}^n(\tau_k - E_\omega[\tau_k])$ is the sum of independent zero mean random variables, it follows immediately from Theorem 4.1 (with $\delta = 1$) that

$$\sup_x\left|\overline{F}_{n,\omega}(x) - \Phi(x)\right| \le \frac{A_1}{\mathrm{Var}_\omega(T_n)^{3/2}}\sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right]. \tag{25}$$

Since $\mathrm{Var}_\omega(T_n)/n \to \sigma^2$ almost surely as $n\to\infty$ we need only to consider the asymptotics of the last sum on the right. The analysis is different in the cases $\kappa > 3$ and $\kappa \in (2,3]$.

Case I: $\kappa > 3$. In this case it follows from Lemma 2.1 that $\mathbb{E}[|\tau_1 - E_\omega[\tau_1]|^3] < \infty$. Therefore, Birkhoff's Ergodic theorem implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right] = \mathbb{E}\left[\left|\tau_1 - E_\omega[\tau_1]\right|^3\right].$$

Applying this to (25), we obtain that

$$\limsup_{n\to\infty} \sqrt{n} \sup_x \left|\overline{F}_{n,\omega}(x) - \Phi(x)\right| \le \lim_{n\to\infty} \frac{A_1\sqrt{n}}{\operatorname{Var}_\omega(T_n)^{3/2}} \sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right]$$
$$= \frac{A_1 \mathbb{E}[|\tau_1 - E_\omega[\tau_1]|^3]}{\sigma^3}.$$

Case II: $\kappa \in (2,3]$. It follows from Lemma 2.1 that for any $p < \kappa/3$,

$$E\left[\left(E_\omega\left[\left|\tau_1 - E_\omega[\tau_1]\right|^3\right]\right)^p\right] \le 4^p E\left[\left(E_\omega\left[\tau_1^3\right] + \left(E_\omega[\tau_1]\right)^3\right)^p\right]$$
$$\le 4^p 2^{p-1} E\left[E_\omega\left[\tau_1^3\right]^p + E_\omega[\tau_1]^{3p}\right] < \infty.$$

Since the quenched expectations $E_\omega[|\tau_k - E_\omega[\tau_k]|^3]$ are an ergodic sequence in k, it follows that if $p < \frac{\kappa}{3} \le 1$ then

$$\limsup_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right] = \limsup_{n\to\infty} \left\{ \frac{1}{n} \left(\sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right] \right)^p \right\}^{1/p}$$
$$\le \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{k=1}^n \left(E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right]\right)^p \right\}^{1/p}$$
$$= \left\{ E\left[\left(E_\omega\left[\left|\tau_1 - E_\omega[\tau_1]\right|^3\right]\right)^p\right]\right\}^{1/p} < \infty, \qquad P\text{-a.s.}$$

By taking p arbitrarily close to $\kappa/3$, we can therefore conclude that

$$\lim_{n\to\infty} \frac{1}{n^{\frac{3}{\kappa}+\varepsilon}} \sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right] = 0, \qquad P\text{-a.s.}$$

Applying this to (25) we obtain that for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} n^{\frac{3}{2}-\frac{3}{\kappa}-\varepsilon} \sup_x \left|\overline{F}_{n,\omega}(x) - \Phi(x)\right|$$
$$\le A_1 \left(\frac{n}{\operatorname{Var}_\omega(T_n)}\right)^{3/2} \frac{1}{n^{\frac{3}{\kappa}+\varepsilon}} \sum_{k=1}^n E_\omega\left[\left|\tau_k - E_\omega[\tau_k]\right|^3\right] = 0, \qquad P\text{-a.s.}$$

□

Remark 4.2. In the case of $\kappa \in (2,3]$ one might wonder if better rates of convergence could be obtained by applying Theorem 4.1 with $2 + \delta < \kappa$. However, it's easy to see that this only gives $n^{\frac{\kappa}{2}-1-\varepsilon}\|\overline{F}_{n,\omega} - \Phi\|_\infty \to 0$ for any $\varepsilon > 0$, and since $\frac{\kappa}{2} - 1 < \frac{3}{2} - \frac{3}{\kappa}$ when $\kappa \in (2,3)$ the bounds in the statement of Theorem 1.3 are better.

Proof of Theorem 1.4. Since

$$F_{n,\omega}(x) = P_\omega\left(\frac{T_n - E_\omega[T_n]}{\sqrt{\mathrm{Var}_\omega(T_n)}} \le x\sqrt{\frac{\sigma^2 n}{\mathrm{Var}_\omega(T_n)}}\right) = \overline{F}_{n,\omega}\left(x\sqrt{\frac{\sigma^2 n}{\mathrm{Var}_\omega(T_n)}}\right),$$

we note that

$$\sup_x\left|F_{n,\omega}(x) - \Phi(x)\right| \le \sup_x\left|\overline{F}_{n,\omega}(x) - \Phi(x)\right| + \sup_x\left|\Phi\left(x\sqrt{\frac{\sigma^2 n}{\mathrm{Var}_\omega(T_n)}}\right) - \Phi(x)\right|. \tag{26}$$

The first term on the right can be controlled by Theorem 1.3, while for the second term on the right we note (see, for instance, Petrov [26], Section V.3, equation (3.3)) that

$$\sup_x\left|\Phi(x) - \Phi(ax)\right| \le \begin{cases} \dfrac{1}{\sqrt{2\pi e}}\dfrac{1-a}{a} & \text{if } a \in (0,1), \\ \dfrac{1}{\sqrt{2\pi e}}(a-1) & \text{if } a \ge 1. \end{cases}$$

It follows from Proposition 3.3 that for any $\varepsilon > 0$, P-a.e. environment ω,

$$\sqrt{\frac{\sigma^2 n}{\mathrm{Var}_\omega(T_n)}} = 1 + o\left(n^{\frac{2}{4\wedge\kappa}+\varepsilon-1}\right), \qquad \text{for } P\text{-a.e. environment } \omega,$$

and therefore

$$\lim_{n\to\infty} n^{1-\frac{2}{4\wedge\kappa}-\varepsilon}\sup_x\left|\Phi\left(x\sqrt{\frac{\sigma^2 n}{\mathrm{Var}_\omega(T_n)}}\right) - \Phi(x)\right| = 0, \qquad P\text{-a.s.}$$

Since in all cases the rate of decay of the first term on the right in (25) given by Theorem 1.3 decays faster than $n^{-1+\frac{2}{4\wedge\kappa}+\varepsilon}$ this completes the proof of Theorem 1.4. □

5. Quenched CLT rates of convergence for the walk

As noted in the introduction, we will obtain rates of convergence for the quenched CLT for X_n from the rates of the quenched CLT for T_n in Theorem 1.3. The transfer of limiting distributions from hitting times to the position of the walk hinges on the fact that $P_\omega(T_k > n) = P_\omega(X_n^* < k)$ where $X_n^* = \max_{k\le n} X_k$ is the running maximum of the walk up to time n. In preparation for the proof of Theorem 1.5, we will first prove the following lemma which will allow us to compare the distribution of X_n^* and X_n.

Lemma 5.1. *If $\kappa > 0$, then there exists a constant $B > 0$ such that $P_\omega(X_n^* - X_n \ge B\log n) \le \frac{1}{\sqrt{n}}$ for P-a.e. environment ω and for all n sufficiently large.*

Proof. It was shown in Gantert and Shi [11] that if $\kappa > 0$ then $\mathbb{P}(T_{-m} < \infty) \le C_1 e^{-C_2 m}$ for some constants $C_1, C_2 > 0$ and all $m \ge 1$. It follows from this that

$$\begin{aligned}\mathbb{P}\big(X_n^* - X_n \ge m\big) &\le \sum_{k=0}^{n-1} \mathbb{P}\Big(\inf_{i > T_k} X_i \le k - m\Big) \\ &= \sum_{k=0}^{n-1} \mathbb{P}^k(T_{k-m} < \infty) = n\mathbb{P}(T_{-m} < \infty) \le C_1 n e^{-C_2 m}.\end{aligned}$$

Therefore, by Chebychev's inequality, we have

$$\begin{aligned}P\bigg(P_\omega\big(X_n^* - X_n \ge B \log n\big) > \frac{1}{\sqrt{n}}\bigg) &\le \sqrt{n}\mathbb{P}\big(X_n^* - X_n \ge B \log n\big) \\ &\le C_1 n^{3/2} e^{-C_2 B \log n}.\end{aligned}$$

If $B > \frac{5}{2C_2}$, then this bound is summable and the conclusion of the lemma follows from the Borel–Cantelli lemma. □

Proof of Theorem 1.5. Since the distribution function $\Phi(x)$ is continuous, rates of convergence for $G_{n,\omega}$ are equivalent to rates of convergence for

$$G_{n,\omega}^\circ(x) = \lim_{\varepsilon \to 0^+} G_{n,\omega}(x + \varepsilon) = P_\omega\bigg(\frac{X_n - n\mathrm{v}_0 + Z_n(\omega)}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}} < x\bigg).$$

Since it is more convenient for the proof, we will prove rates of convergence for $G_{n,\omega}^\circ$. In fact, the strategy of the proof will be to first prove rates of convergence for

$$G_{n,\omega}^*(x) = P_\omega\bigg(\frac{X_n^* - n\mathrm{v}_0 + Z_n(\omega)}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}} < x\bigg)$$

and then use Lemma 5.1 to obtain corresponding rates of convergence for $G_{n,\omega}^\circ$. Indeed, since

$$\begin{aligned}&\big|G_{n,\omega}^\circ(x) - \Phi(x)\big| \\ &\quad \le \bigg|G_{n,\omega}^\circ(x) - G_{n,\omega}^*\bigg(x + \frac{B \log n}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}}\bigg)\bigg| + \sup_{y \in \mathbb{R}}\big|G_{n,\omega}^*(y) - \Phi(y)\big| \\ &\qquad + \bigg|\Phi\bigg(x + \frac{B \log n}{\sigma \mathrm{v}_0^{3/2}\sqrt{n}}\bigg) - \Phi(x)\bigg| \\ &\quad \le P_\omega\big(X_n^* - X_n \ge B \log n\big) + \sup_{y \in \mathbb{R}}\big|G_{n,\omega}^*(y) - \Phi(y)\big| + \frac{B \log n}{\sigma \mathrm{v}_0^{3/2}\sqrt{2\pi n}},\end{aligned}$$

it follows from Lemma 5.1 that to prove the almost sure convergence rate of convergence in Theorem 1.5 we need only to show

$$\lim_{n\to\infty} n^{\frac{1}{4}-\frac{1}{2\kappa}-\varepsilon} \sup_{x\in\mathbb{R}} \left| G^*_{n,\omega}(x) - \Phi(x) \right| = 0, \qquad P\text{-a.s., for any } \varepsilon > 0. \tag{27}$$

For the proof of (27) we begin by noting that since $P_\omega(X^*_n < k) = P_\omega(T_k > n)$ for any $n, k \ge 1$ that

$$G^*_{n,\omega}(x) = P_\omega\bigl(X^*_n < n\mathrm{v}_0 - Z_n(\omega) + x\sigma \mathrm{v}_0^{3/2}\sqrt{n}\bigr) = P_\omega(T_{k(n,\omega,x)} > n) \tag{28}$$

whenever

$$k(n,\omega,x) := \bigl\lceil n\mathrm{v}_0 - Z_n(\omega) + x\sigma \mathrm{v}_0^{3/2}\sqrt{n}\bigr\rceil \ge 1.$$

Throughout the remainder of our proof, we will fix an arbitrary $\varepsilon \in (0, 1/2)$. Let $x^-_{n,\varepsilon} = x^-_{n,\varepsilon}(\omega)$ and $x^+_{n,\varepsilon} = x^+_{n,\varepsilon}(\omega)$ be such that $k(n,\omega,x^-_{n,\varepsilon}) = \lceil n\mathrm{v}_0 - n^{1/2+\varepsilon}\rceil$ and $k(n,\omega,x^+_{n,\varepsilon}) = \lfloor n\mathrm{v}_0 + n^{1/2+\varepsilon}\rfloor$. We will use (28) and Theorem 1.3 to control $|G^*_{n,\omega}(x) - \Phi(x)|$ but our analysis will be different depending on whether or not $x \in [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$.

Case I: $x \in [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$. In this case, it follows from (28) that

$$\begin{aligned}
&\left| G^*_{n,\omega}(x) - \Phi(x) \right| \\
&\quad = \left| P_\omega\left(\frac{T_{k(n,\omega,x)} - E_\omega[T_{k(n,\omega,x)}]}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} > \frac{n - E_\omega[T_{k(n,\omega,x)}]}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} \right) - \Phi(x) \right| \\
&\quad = \left| \overline{F}_{k(n,\omega,x),\omega}\left(\frac{n - E_\omega[T_{k(n,\omega,x)}]}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} \right) - \Phi(-x) \right| \\
&\quad \le \sup_{t\in\mathbb{R}} \left| \overline{F}_{k(n,\omega,x),\omega}(t) - \Phi(t) \right| + \left| \Phi\left(\frac{n - E_\omega[T_{k(n,\omega,x)}]}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} \right) - \Phi(-x) \right| \\
&\quad \le \sup_{|m - n\mathrm{v}_0| \le n^{1/2+\varepsilon}} \|\overline{F}_{m,\omega} - \Phi\|_\infty + \frac{1}{\sqrt{2\pi}} \left| \frac{E_\omega[T_{k(n,\omega,x)}] - n}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} - x \right|.
\end{aligned} \tag{29}$$

The first term in (29) can be controlled by Theorem 1.3. For the second term in (29), note first of all that (recalling the definition of $Z_n(\omega)$ from the statement of Theorem 1.2)

$$\begin{aligned}
n &= E_\omega[T_{\lfloor n\mathrm{v}_0\rfloor}] - \left(E_\omega[T_{\lfloor n\mathrm{v}_0\rfloor}] - \frac{\lfloor n\mathrm{v}_0\rfloor}{\mathrm{v}_0} \right) + \frac{n\mathrm{v}_0 - \lfloor n\mathrm{v}_0\rfloor}{\mathrm{v}_0} \\
&= E_\omega[T_{\lfloor n\mathrm{v}_0\rfloor}] - \frac{Z_n(\omega)}{\mathrm{v}_0} + \mathcal{O}(1),
\end{aligned}$$

where here (and below) we will use $\mathcal{O}(1)$ to denote uniformly bounded error terms coming from integer rounding. Therefore,

$$\begin{aligned}
&E_\omega[T_{k(n,\omega,x)}] - n \\
&\quad = E_\omega[T_{k(n,\omega,x)}] - E_\omega[T_{\lfloor n\mathrm{v}_0\rfloor}] + \frac{Z_n(\omega)}{\mathrm{v}_0} + \mathcal{O}(1)
\end{aligned}$$

$$= \left(E_\omega[T_{k(n,\omega,x)}] - E_\omega[T_{\lfloor n\mathrm{v}_0 \rfloor}] - \frac{k(n,\omega,x) - n\mathrm{v}_0}{\mathrm{v}_0} \right) + \frac{k(n,\omega,x) - n\mathrm{v}_0 + Z_n(\omega)}{\mathrm{v}_0} + \mathcal{O}(1)$$
$$= \left(E_\omega[T_{k(n,\omega,x)}] - E_\omega[T_{\lfloor n\mathrm{v}_0 \rfloor}] - \frac{k(n,\omega,x) - n\mathrm{v}_0}{\mathrm{v}_0} \right) + x\sigma\sqrt{n\mathrm{v}_0} + \mathcal{O}(1),$$

where the last equality follows from the definition of $k(n,\omega,x)$. Since $x \in [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$ implies that $k(n,\omega,x) \in I_{\varepsilon,n} = [n\mathrm{v}_0 - n^{1/2+\varepsilon}, n\mathrm{v}_0 + n^{1/2+\varepsilon}]$ it follows from Proposition 3.2 that the first term in the last line is bounded (uniformly over $x \in [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$) by something that is $o(n^{\frac{1}{4}+\frac{1}{2\kappa}+\varepsilon(\frac{1}{2}-\frac{1}{\kappa})+\varepsilon'})$ for any $\varepsilon' > 0$. Finally, we claim that $\mathrm{Var}_\omega(T_{k(n,\omega,x)})$ is asymptotically close to $\sigma^2\mathrm{v}_0 n$ uniformly over $x \in [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$. Indeed, since

$$\mathrm{Var}_\omega(T_{n\mathrm{v}_0 - n^{1/2+\varepsilon}}) \le \mathrm{Var}_\omega(T_{k(n,\omega,x)}) \le \mathrm{Var}_\omega(T_{n\mathrm{v}_0 + n^{1/2+\varepsilon}}),$$

it follows from the fact that $\mathrm{Var}_\omega(T_m) \sim \sigma^2 m$ that

$$\lim_{n\to\infty} \sup_{x\in[x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]} \left| \frac{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}{\sigma^2 n\mathrm{v}_0} - 1 \right| = 0, \qquad P\text{-a.s.}$$

We have therefore shown that for any $\varepsilon' > 0$,

$$\lim_{n\to\infty} n^{\frac{1}{4}-\frac{1}{2\kappa}-\varepsilon(\frac{1}{2}-\frac{1}{\kappa})-\varepsilon'} \sup_{x\in[x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]} \left| \frac{E_\omega[T_{k(n,\omega,x)}] - n}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} - x \right| = 0, \qquad P\text{-a.s.} \tag{30}$$

Since Theorem 1.3 implies that the first term in (29) decays strictly faster than $n^{-\frac{1}{4}+\frac{1}{2\kappa}}$, we can conclude that

$$\lim_{n\to\infty} n^{\frac{1}{4}-\frac{1}{2\kappa}-\varepsilon(\frac{1}{2}-\frac{1}{\kappa})-\varepsilon'} \sup_{x\in[x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]} \left| G^*_{n,\omega}(x) - \Phi(x) \right| = 0, \qquad P\text{-a.s.} \tag{31}$$

Case II: $x \notin [x^-_{n,\varepsilon}, x^+_{n,\varepsilon}]$. Since Lemma 3.1 implies that $Z_n(\omega)/n^{1/2+\varepsilon} \to 0$, it follows that for n large enough (depending on ω) $x^-_{n,\varepsilon} < -n^{-\varepsilon/2}$ and $x^+_{n,\varepsilon} > n^{\varepsilon/2}$. Therefore, by the monotonicity of the distribution functions we have

$$\begin{aligned} \sup_{x<x^-_{n,\varepsilon}} \left| G^*_{n,\omega}(x) - \Phi(x) \right| &\le G^*_{n,\omega}(x^-_{n,\varepsilon}) + \Phi(x^-_{n,\varepsilon}) \\ &\le \left| G^*_{n,\omega}(x^-_{n,\varepsilon}) - \Phi(x^-_{n,\varepsilon}) \right| + 2\Phi(x^-_{n,\varepsilon}) \\ &\le \left| G^*_{n,\omega}(x^-_{n,\varepsilon}) - \Phi(x^-_{n,\varepsilon}) \right| + 2\Phi(-n^{\varepsilon/2}), \end{aligned} \tag{32}$$

and similarly

$$\begin{aligned}\sup_{x > x_{n,\varepsilon}^-} \left| G_{n,\omega}^*(x) - \Phi(x) \right| &= \sup_{x > x_{n,\varepsilon}^-} \left| \left(1 - G_{n,\omega}^*(x)\right) - \left(1 - \Phi(x)\right) \right| \\ &\leq 1 - G_{n,\omega}^*\left(x_{n,\varepsilon}^+\right) + 1 - \Phi\left(x_{n,\varepsilon}^+\right) \\ &\leq \left| G_{n,\omega}^*\left(x_{n,\varepsilon}^+\right) - \Phi\left(x_{n,\varepsilon}^+\right) \right| + 2\left(1 - \Phi\left(n^{\varepsilon/2}\right)\right).\end{aligned} \tag{33}$$

Since $\Phi(-n^{\varepsilon/2}) = 1 - \Phi(n^{\varepsilon/2})$ decays faster than any polynomial in n, applying (31) to (32) and (33) we obtain that

$$\lim_{n\to\infty} n^{\frac{1}{4} - \frac{1}{2\kappa} - \varepsilon(\frac{1}{2} - \frac{1}{\kappa}) - \varepsilon'} \sup_{x\in\mathbb{R}} \left| G_{n,\omega}^*(x) - \Phi(x) \right| = 0, \qquad P\text{-a.s.}$$

Finally, note that since $\varepsilon, \varepsilon' > 0$ were arbitrary this completes the proof of the almost sure rate of convergence in Theorem 1.5.

The proof of the weaker in probability rates of convergence for G_n in (3) and (4) are almost the same as the above proof of the almost sure convergence rates. The only difference is that instead of (30), the convergence in probability statement in Proposition 3.2 gives that for any $\varepsilon' > 0$

$$\lim_{n\to\infty} n^{\frac{1}{4} - \frac{\varepsilon}{2} - \varepsilon'} \sup_{x\in[x_{n,\varepsilon}^-, x_{n,\varepsilon}^+]} \left| \frac{E_\omega[T_{k(n,\omega,x)}] - n}{\sqrt{\mathrm{Var}_\omega(T_{k(n,\omega,x)})}} - x \right| = 0, \qquad \text{in } P\text{-probability}.$$

The rest of the proof is essentially the same with the exception that when $\kappa \in (2, \frac{12}{5})$ and $\varepsilon > 0$ is sufficiently small the dominant term in (29) is the first term which by Theorem 1.3 is $o(n^{-\frac{3}{2} + \frac{3}{\kappa} + \varepsilon''})$ for any $\varepsilon'' > 0$. □

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