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Low-frequency estimation of continuous-time moving average Lévy processes

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In this paper, we study the problem of statistical inference for a continuous-time moving average Lévy process of the form

$$Z_t = \int_{\mathbb{R}} \mathcal{K}(t-s)\, dL_s, \qquad t \in \mathbb{R},$$

with a deterministic kernel $\mathcal{K}$ and a Lévy process L. Especially the estimation of the Lévy measure ν of L from low-frequency observations of the process Z is considered. We construct a consistent estimator, derive its convergence rates and illustrate its performance by a numerical example. On the mathematical level, we establish some new results on exponential mixing for continuous-time moving average Lévy processes.

Keywords: low-frequency estimation; Mellin transform; moving average

1. Introduction

Stochastic integrals of the type

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(s,t)\, dL_s, \tag{1}$$

where $\mathcal{K}$ is a deterministic kernel and $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process, build a large class of stochastic processes including semimartingales and non-semimartingales, cf. Basse and Pedersen [3], Basse-O'Connor and Rosińsky [5], Bender, Lindner and Schicks [9], as well as long-memory processes. Starting point was the paper by Rajput and Rosiński [16] providing conditions on the interplay between $\mathcal{K}$ and L such that Z is well defined. Continuous-time Lévy-driven moving average processes provide a unifying approach to many popular stochastic models like Lévy-driven Ornstein–Uhlenbeck processes, fractional Lévy processes and CARMA processes. Furthermore, they are the building blocks of more general models such as Lévy semistationary processes and ambit fields, cf. Barndorff-Nielsen, Benth and Veraart [1], Podolskij [15].

Statistical inference for Ornstein–Uhlenbeck processes and CARMA processes is already well studied in the literature due to the special structure of the processes, for an overview see Brockwell and Lindner [10]. Nevertheless, for general continuous-time Lévy-driven moving average processes only partial results are known so far mainly concerning parametric estimation of the kernel function, cf. Cohen and Lindner [11] for an approach via empirical moments or Zhang, Lin and Zhang [18] for a least squares approach. Further results concern limit theorems for the power variation processes, cf. Glaser [12], Basse-O'Connor, Lachieze-Rey and Podolskij [4], which may be used for statistical inference based on high-frequency data.

In this paper, we consider a special case of stationary continuous-time Lévy-driven moving average processes of the form $Z_t = \int_{-\infty}^{\infty} \mathcal{K}(s-t)\, dL_s$ and aim to infer on the unknown parameters of the driving Lévy process from its low-frequency observations. Our setting especially includes the case of Gamma-kernels of the form $\mathcal{K}(t) = t^{\alpha} e^{-\lambda t} 1_{[0,\infty)}(t)$ with $\lambda > 0$ and $\alpha > -1/2$, which serves as a popular kernel for applications in finance and turbulence, cf. Barndorff-Nielsen and Schmiegel [2]. The special symmetric case of the well-balanced Ornstein–Uhlenbeck process has been discussed in Schnurr and Woerner [17].

The considered statistical problem is rather challenging for several reasons. On the one hand, the set of parameters, that is, the so-called Lévy triplet of the driving Lévy process, contains, in general, an infinite dimensional object (a Lévy measure) making the statistical problem nonparametric. On the other hand, the relation between the parameters of the underlying Lévy process (L_t) and those of the resulting moving average process (Z_t) is rather nonlinear and implicit, pointing out to a nonlinear ill-posed statistical problem. It turns out that in Fourier domain this relation becomes exponentially linear and has a form of multiplicative convolution. This observation underlies our estimation procedure, which basically consists of three steps. First, we estimate the marginal characteristic function of the Lévy-driven moving average process (Z_t). Then we estimate the Mellin transform of the second derivative of the log-transform of the characteristic function. Finally, an inverse Mellin transform technique is used to reconstruct the Lévy density of the underlying Lévy process. In order to obtain the convergence rates of the resulting estimator, we need to analyse the mixing properties of the process (Z_t). The existing results of this kind are very sparse and basically cover only the case of the exponential kernel $\mathcal{K}$. Here, we derive the exponential mixing of the process (Z_t) under rather general assumptions.

The paper is organized as follows. In the next session, we explain our setup and discuss some basic properties of the considered model. The estimation procedure is presented in Section 3. Our main theoretical results related to the rates of convergence of the estimates are given in Section 4. Next, in Section 5, we provide a numerical example, which shows the performance of our procedure. All proofs are collected in the Appendix.

2. Setup

In this paper, we study a stationary continuous-time moving average (MA) Lévy process $(Z_t)_{t\in\mathbb{R}}$ of the form:

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(t-s)\, dL_s, \qquad t \in \mathbb{R}, \tag{2}$$

where $\mathcal{K} : \mathbb{R} \to \mathbb{R}_+$ is a measurable function and $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with the triplet $\mathcal{T} = (\gamma, \sigma^2, \nu)$. Throughout the paper, we assume that

$$\mathcal{K} \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}), \tag{3}$$

and the Lévy measure ν satisfies

$$\int x^2 \nu(dx) < \infty, \tag{4}$$

that is, the Lévy process L has finite second moment. As it is shown in Lemma 1 (see Appendix A), these conditions guarantee that the stochastic integral Z_t exists for any $t \in \mathbb{R}$, that is, there exists a sequence of step functions converging to $\mathcal{K}$, and the limit of corresponding integrals doesn't depend on the choice of this sequence. Moreover, under these assumptions, the process $(Z_t)_{t\in\mathbb{R}}$ is strictly stationary with the characteristic function of the form

$$\Phi(u) := \mathbb{E}\bigl[e^{\mathrm{i}uZ_t}\bigr] = \exp\bigl(\Psi(u)\bigr), \tag{5}$$

where

$$\Psi(u) := \int_{\mathbb{R}} \psi\bigl(u\mathcal{K}(s)\bigr)\, ds$$

and $\psi(u)$ is a characteristic exponent of the process L. Due to the Lévy–Khintchine formula, $\psi(u)$ can be represented as

$$\psi(u) := \mathrm{i}u\gamma - \sigma^2 u^2/2 + \int_{\mathbb{R}} \bigl(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux 1_{\{|x|\le 1\}}\bigr)\nu(dx).$$

In what follows, we will assume that ν is absolutely continuous w.r.t. to the Lebesgue measure on $\mathbb{R}_+$. With a slight abuse of notation, we denote by $\nu(x)$ also the density of the Lévy measure ν. Our main goal is the estimation of the Lévy density $\nu(x)$ from low-frequency observations of the process (Z_t) given that the function $\mathcal{K}$ is known.

3. Mellin transform approach

3.1. Main idea

For the sake of clarity, we first assume that σ is known. Set

$$\Psi_\sigma(u) := \Psi(u) + \frac{\sigma^2 u^2}{2} \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx. \tag{6}$$

It follows then

$$\begin{aligned}\Psi''_\sigma(u) &= \int_{\mathbb{R}} \psi''\bigl(u\mathcal{K}(x)\bigr) \cdot \mathcal{K}^2(x)\, dx + \sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx \\ &= -\int_{\mathbb{R}} \mathcal{F}[\tilde{\nu}]\bigl(u\mathcal{K}(x)\bigr) \cdot \mathcal{K}^2(x)\, dx,\end{aligned}$$

where $\tilde{\nu}(x) := x^2\nu(x)$, and $\mathcal{F}[\tilde{\nu}]$ stands for the Fourier transform of $\tilde{\nu}$. Next, let us compute the Mellin transform of Ψ''_σ:

$$\begin{aligned}
\mathcal{M}\big[\Psi''_\sigma\big](z) &= -\int_{\mathbb{R}_+}\bigg[\int_{\mathbb{R}} \mathcal{F}[\tilde{\nu}]\big(u\mathcal{K}(x)\big)\cdot \mathcal{K}^2(x)\,dx\bigg]u^{z-1}\,du \\
&= -\int_{\mathbb{R}}\bigg[\int_{\mathbb{R}_+} \mathcal{F}[\tilde{\nu}]\big(u\mathcal{K}(x)\big)\cdot u^{z-1}\,du\bigg]\mathcal{K}^2(x)\,dx \\
&= -\mathcal{M}\big[\mathcal{F}[\tilde{\nu}]\big](z)\cdot\int_{\mathbb{R}}\big(\mathcal{K}(x)\big)^{2-z}\,dx,
\end{aligned} \tag{7}$$

for all z such that $\int_{\mathbb{R}}(\mathcal{K}(x))^{2-\mathrm{Re}(z)}\,dx < \infty$ and $\int_{\mathbb{R}_+}|\mathcal{F}[\tilde{\nu}](v)|\cdot v^{\mathrm{Re}(z)-1}\,dv < \infty$. Since $\tilde{\nu} \in \mathcal{L}^1(\mathbb{R}_+)$, it holds

$$\begin{aligned}
\mathcal{M}\big[\mathcal{F}[\tilde{\nu}]\big](z) &= \int_0^\infty v^{z-1}\bigg[\int_{-\infty}^\infty e^{\mathrm{i}xv}\tilde{\nu}(x)\,dx\bigg]dv \\
&= \mathcal{M}\big[e^{\mathrm{i}\cdot}\big](z)\cdot\mathcal{M}[\tilde{\nu}_+](1-z) + \mathcal{M}\big[e^{-\mathrm{i}\cdot}\big](z)\cdot\mathcal{M}[\tilde{\nu}_-](1-z),
\end{aligned}$$

where

$$\tilde{\nu}_+(x) := \tilde{\nu}(x)\cdot 1\{x \ge 0\}, \qquad \tilde{\nu}_-(x) := \tilde{\nu}(-x)\cdot 1\{x \ge 0\}.$$

Analogously

$$\mathcal{M}\big[\overline{\mathcal{F}[\tilde{\nu}]}\big](z) = \mathcal{M}\big[e^{-\mathrm{i}\cdot}\big](z)\cdot\mathcal{M}[\tilde{\nu}_+](1-z) + \mathcal{M}\big[e^{\mathrm{i}\cdot}\big](z)\cdot\mathcal{M}[\tilde{\nu}_-](1-z).$$

Note that the Mellin transforms $\mathcal{M}[\tilde{\nu}_\pm](1-z)$ are defined for all z with $\mathrm{Re}(z) \in (0, 1)$, provided $\tilde{\nu}_\pm$ are bounded at 0. Next, using the fact that

$$\mathcal{M}\big[e^{\mathrm{i}\cdot}\big](z) = \Gamma(z)e^{\mathrm{i}\pi z/2}, \qquad \mathcal{M}\big[e^{-\mathrm{i}\cdot}\big](z) = \Gamma(z)e^{-\mathrm{i}\pi z/2}$$

for all z with $\mathrm{Re}(z) \in (0, 1)$ (see [14], 5.1–5.2), we get

$$\begin{aligned}
\mathcal{M}\big[\Psi''_\sigma\big](z) &= -\big(\mathcal{M}[\tilde{\nu}_+](1-z)e^{\mathrm{i}\pi z/2} + \mathcal{M}[\tilde{\nu}_-](1-z)e^{-\mathrm{i}\pi z/2}\big)\Gamma(z)\int_{\mathbb{R}}\big(\mathcal{K}(x)\big)^{2-z}\,dx, \\
\mathcal{M}\big[\overline{\Psi''_\sigma}\big](z) &= -\big(\mathcal{M}[\tilde{\nu}_+](1-z)e^{-\mathrm{i}\pi z/2} + \mathcal{M}[\tilde{\nu}_-](1-z)e^{\mathrm{i}\pi z/2}\big)\Gamma(z)\int_{\mathbb{R}}\big(\mathcal{K}(x)\big)^{2-z}\,dx.
\end{aligned}$$

From these equations, it follows

$$\begin{aligned}
\mathcal{M}[\tilde{\nu}_+](1-z) &= \frac{\mathcal{M}[\Psi''_\sigma](z)}{Q_1(z)} - \frac{\mathcal{M}[\overline{\Psi''_\sigma}](z)}{Q_2(z)}, \\
\mathcal{M}[\tilde{\nu}_-](1-z) &= \frac{\mathcal{M}[\overline{\Psi''_\sigma}](z)}{Q_1(z)} - \frac{\mathcal{M}[\Psi''_\sigma](z)}{Q_2(z)}
\end{aligned} \tag{8}$$

with

$$
\begin{aligned}
Q_1(z) &:= -\frac{e^{\mathrm{i}\pi z} - e^{-\mathrm{i}\pi z}}{e^{\mathrm{i}\pi z/2}} \Gamma(z) \int_{\mathbb{R}} \bigl(\mathcal{K}(x)\bigr)^{2-z}\, dx, \\
Q_2(z) &:= -\frac{e^{\mathrm{i}\pi z} - e^{-\mathrm{i}\pi z}}{e^{-\mathrm{i}\pi z/2}} \Gamma(z) \int_{\mathbb{R}} \bigl(\mathcal{K}(x)\bigr)^{2-z}\, dx.
\end{aligned} \tag{9}
$$

The relations (8) form a basis of our estimation procedure, as they relate the measures ν_+ and ν_- to the function Ψ_σ, which can be directly estimated from data.

3.2. Estimation procedure

Assume that the process Z is observed on the equidistant time grid $\{\Delta, 2\Delta, \ldots, n\Delta\}$. Our aim is to estimate the Lévy density ν of the process L. First, we estimate the Mellin transforms $\mathcal{M}[\Psi_\sigma'']$ and $\mathcal{M}[\overline{\Psi_\sigma''}]$ by

$$
\begin{aligned}
\mathcal{M}_n\bigl[\Psi_\sigma''\bigr](1-z) &:= \int_0^{U_n} \left[\frac{\Phi_n''(u)}{\Phi_n(u)} - \left(\frac{\Phi'_n(u)}{\Phi_n(u)}\right)^2 + \sigma^2 \|\mathcal{K}\|_{L^2}^2\right] u^{-z}\, du, \\
\mathcal{M}_n\bigl[\overline{\Psi_\sigma''}\bigr](1-z) &:= \int_0^{U_n} \overline{\left[\frac{\Phi_n''(u)}{\Phi_n(u)} - \left(\frac{\Phi_n'(u)}{\Phi_n(u)}\right)^2\right.} \left. + \sigma^2 \|\mathcal{K}\|_{L^2}^2\right] u^{-z}\, du,
\end{aligned} \tag{10}
$$

respectively, where U_n is a sequence of cut-offs tending to infinity as $n \to \infty$ and

$$
\Phi_n(u) := \frac{1}{n} \sum_{j=1}^{n} e^{\mathrm{i} u Z_{j\Delta}}.
$$

Second, we recover the measures $\tilde{\nu}_+$ and $\tilde{\nu}_-$ by applying the inverse Mellin techniques:

$$
\begin{aligned}
\tilde{\nu}_{n+}(x) &:= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \left(\frac{\mathcal{M}_n[\Psi_\sigma''](z)}{Q_1(z)} - \frac{\mathcal{M}_n[\overline{\Psi_\sigma''}](z)}{Q_2(z)}\right) x^{-z}\, dz, \\
\tilde{\nu}_{n-}(x) &:= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \left(\frac{\mathcal{M}_n[\overline{\Psi_\sigma''}](z)}{Q_1(z)} - \frac{\mathcal{M}_n[\Psi_\sigma''](z)}{Q_2(z)}\right) x^{-z}\, dz,
\end{aligned} \tag{11}
$$

where $V_n \to \infty$ is another sequence of regularising parameters. Finally, we define the estimator for the function $\tilde{\nu}(x) = x^2 \nu(x)$ as

$$
\tilde{\nu}_n(x) := \tilde{\nu}_{n+}(x) + \tilde{\nu}_{n-}(-x). \tag{12}
$$

3.3. Simplified version of the estimation procedure

In practice the estimation procedure described in previous section can be significantly simplified under some additional assumptions. For example, if the Lèvy process L is a subordinator with finite activity of jumps and zero drift, then one can consider the first derivative of the function $\Psi_\sigma(u)$ instead of the second, and get that

$$\mathcal{M}\big[\Psi_\sigma'\big](z) = \breve{Q}(z) \cdot \mathcal{M}[\breve{\nu}](1-z), \qquad \operatorname{Re}(z) \in (0,1),$$

where $\breve{\nu}(x) = x\nu(x)$, and

$$\breve{Q}(z) = \mathrm{i}\Gamma(z)\exp\{\mathrm{i}\pi z/2\} \int_{\mathbb{R}} \big(K(x)\big)^{1-z}\,dx. \tag{13}$$

The estimation scheme remains the same: we first estimate the Mellin transform of the function Ψ_σ', and then infer on the Lévy measure ν by applying the Mellin transform techniques. More precisely, in the first step we construct the estimate

$$\mathcal{M}_n\big[\Psi_\sigma'\big](1-z) := \int_0^{U_n} \left[\frac{\Phi_n'(u)}{\Phi_n(u)} + \sigma^2 u \|\mathcal{K}\|_{L^2}^2\right] u^{-z}\,du, \tag{14}$$

where $U_n \to \infty$. In the second step, we define the estimate of $\breve{\nu}$ via

$$\breve{\nu}_n(x) := \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \frac{\mathcal{M}_n[\Psi_\sigma'](1-z)}{\breve{Q}(1-z)} x^{-z}\,dz \tag{15}$$

with some $c \in (0,1)$ and a sequence $V_n \to \infty$.

3.4. Case of unknown σ

In this subsection, we suggest how our procedure can be adapted to the case of unknown σ. First, note that for a properly chosen bounded kernel w with $\operatorname{supp}(w) \subseteq [1,2]$ and $\int_0^\infty w(u)\,du = 1$,

$$\begin{aligned}\int_{\mathbb{R}_+} w_n(u)\Psi''(u)\,du = -\sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(x)\,dx \\ - \int_{\mathbb{R}}\int_{\mathbb{R}_+} w(u)\mathcal{F}[\tilde{\nu}]\big(uU_n\mathcal{K}(x)\big)\mathcal{K}^2(x)\,du\,dx\end{aligned}$$

with $w_n(u) := U_n^{-1} w(u/U_n)$ and some sequence $U_n \to \infty$. Suppose that $|\mathcal{F}[\tilde{\nu}](u)| \le C(1+u)^{-\alpha}$ for all $u \ge 0$ and some constants $\alpha > 0$, $C > 0$, then

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}_+} w(u)\mathcal{F}[\tilde{\nu}]\big(uU_n\mathcal{K}(x)\big)\mathcal{K}^2(x)\,du\,dx\right| \le \|w\|_\infty \int_{\mathbb{R}} \frac{\mathcal{K}^2(x)}{(1+U_n\mathcal{K}(x))^\alpha}\,dx \to 0$$

as $n \to \infty$. For example, in the case of a one-sided exponential kernel $\mathcal{K}(x) = e^{-x}\mathbb{I}(x \ge 0)$, we derive

$$\int_{\mathbb{R}} \frac{\mathcal{K}^2(x)}{(1+U_n\mathcal{K}(x))^\alpha}\,dx = \frac{1}{U_n^{-2}} \int_0^{U_n} \frac{z}{(1+z)^\alpha}\,dz \lesssim \begin{cases} U_n^{-\alpha}, & \alpha < 2, \\ U_n^{-2}\log(U_n), & \alpha = 2, \\ U_n^{-2}, & \alpha > 2, \end{cases}$$

as $n \to \infty$. Hence, the quantity

$$-\left[\int_{\mathbb{R}} \mathcal{K}^2(x)\,dx\right]^{-1} \int_{\mathbb{R}_+} w_n(u)\left[\frac{\Phi''_n(u)}{\Phi_n(u)} - \left(\frac{\Phi'{}_n(u)}{\Phi_n(u)}\right)^2\right] du$$

can be used to estimate σ^2.

4. Convergence

Assume that the following condition holds.

(A1) The Lévy density ν fulfills $\int_{-1}^{1} |x|\nu(x)\,dx < \infty$, and moreover, for some $A > 0$, $\alpha \in (0, 1)$, $\beta_+ > 0$, $\beta_- > 0$, $c \in (0, 1)$, it holds

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \bigl(1 + |y|\bigr)^{\alpha} \bigl|\mathcal{F}[\tilde{\nu}](y)\bigr|\,dy \le A, \\ \displaystyle\int_{\mathbb{R}} e^{\beta_\pm |u|} \bigl|\mathcal{M}[\tilde{\nu}_\pm](c + \mathrm{i}u)\bigr|\,du \le A. \end{cases}$$

Note that without loss of generality we can also assume that the constant A is such that $\int_{\mathbb{R}} (|x| \vee x^2)\nu(x)\,dx \le A$, since the integral in the left-hand side is bounded.

Example 1. Consider a class of Lévy processes (L_t) with Lévy measure $\nu(x) = \nu_+(x) + \nu_-(-x)$, such that

$$\nu_\pm(x) = \sum_{j=1}^{J^{(\pm)}} a_j^{(\pm)} x^{-\eta_j^{(\pm)} - 1} e^{-\lambda_j^{(\pm)} x} \cdot \mathbb{I}\{x \ge 0\},$$

where $J^{(+)}, J^{(-)} \in \mathbb{N} \cup 0$, $a_j^{(+)}, a_j^{(-)} > 0$, $\eta_j^{(+)}, \eta_j^{(-)} < 1$, $\lambda_j^{(+)}, \lambda_j^{(-)} > 0$ for all j. Note that this class includes the tempered stable processes, corresponding to the case $J^{(+)} = 1$, $J^{(-)} = 0$ and $\eta_1^{(+)} \in (0, 1)$. Since

$$\begin{aligned} \mathcal{F}[\tilde{\nu}](y) &= \sum_{j=1}^{J^{(+)}} a_j^{(+)} \bigl(\lambda_j^{(+)} - \mathrm{i}y\bigr)^{\eta_j^{(+)} - 2} \Gamma\bigl(2 - \eta_j^{(+)}\bigr) \\ &\quad + \sum_{j=1}^{J^{(-)}} a_j^{(-)} \bigl(\lambda_j^{(-)} + \mathrm{i}y\bigr)^{\eta_j^{(-)} - 2} \Gamma\bigl(2 - \eta_j^{(-)}\bigr), \end{aligned}$$

$$\mathcal{M}[\tilde{\nu}_\pm](z) = \sum_{j=1}^{J^{(\pm)}} a_j^{(\pm)} \bigl(\lambda_j^{(\pm)}\bigr)^{\eta_j^{(\pm)} - z - 1} \Gamma\bigl(z - \eta_j^{(\pm)} + 1\bigr), \qquad \forall \operatorname{Re}(z) > \max_{j=1,\dots,J^{(\pm)}} \bigl(\eta_j^{(\pm)}\bigr) - 1,$$

we derive that (A1) holds with any

$$\alpha \in \bigl(0, 1 - \max\bigl(\eta_1^{(+)}, \ldots, \eta_{J^{(+)}}^{(+)}, \eta_1^{(-)}, \ldots, \eta_{J^{(-)}}^{(-)}\bigr)\bigr) \quad \text{and} \quad \beta_-, \beta_+ \in (0, \pi/2).$$

Example 2. Many other examples can be constructed from the compound process $\sum_{k=1}^{N_t} \xi_k$, where $\xi_1, \xi_2, \ldots$ is a sequence of i.i.d. r.v.'s with absolutely continuous distribution, and $(N_t)_{t \geq 0}$ is a Poisson process independent of $\xi_1, \xi_2, \ldots$. Since the Lévy measure in this case is proportional to the probability density $p(\cdot)$ of ξ_1, the condition (A1) is in fact a condition on $p(\cdot)$. In particular, Example 5 from [6] yields that (A1) is satisfied for the CPP such that ξ_1 has a half-normal distribution with density

$$p(x) = \sqrt{\frac{2}{\pi}} \frac{1}{v} \exp\left\{-\frac{x^2}{2v^2}\right\} \cdot \mathbb{I}\{x \geq 0\},$$

where $v > 0$.

The next theorem gives a general upper bound for the difference between $\tilde{\nu}_n(x)$ and $\tilde{\nu}(x)$, which depends on behaviour of the weighted empirical processes

$$D_j(u) := \frac{\Phi_n^{(j)}(u) - \Phi^{(j)}(u)}{\Phi(u)}, \qquad j = 0, 1, 2.$$

Theorem 1. *Consider the estimate $\tilde{\nu}_n(x)$ of $\tilde{\nu}(x)$ constructed by (10), (11), (12) with some sequences $U_n \to \infty$, $V_n \to \infty$ and $c \in (0, 1)$. Assume that (A1) holds with the same $c \in (0, 1)$ and some $A > 0$, $\alpha \in (0, 1)$, $\beta_+ > 0$, $\beta_- > 0$. Fix some $K > 0$ and denote*

$$\mathcal{A}_K := \Bigl\{ \max_{j=0,1,2} \|D_j\|_{U_n} \geq K \varepsilon_n \Bigr\}, \qquad K \geq 0,$$

where for any real valued function f on $\mathbb{R}$, $\|f\|_{U_n} := \sup_{u \in [-U_n, U_n]} |f(u)|$, and ε_n is a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$, and

$$K \varepsilon_n \bigl(1 + \|\Psi'_\sigma\|_{U_n}\bigr) \leq 1/2.$$

Then on the set $\mathcal{A}_K^{\mathcal{C}}$ (complementary set to $\mathcal{A}_K$), the estimate $\tilde{\nu}_n(x)$ satisfies

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \bigl\{ |x|^c \bigl|\tilde{\nu}_n(x) - \tilde{\nu}(x)\bigr| \bigr\} \\ &\quad \leq \frac{1}{2\pi} \int_{\{|v| \leq V_n\}} \frac{\Omega_n}{\min(|Q_1(1 - c - \mathrm{i}v)|, |Q_2(1 - c - \mathrm{i}v)|)} \, dv + \frac{A}{2\pi} e^{-\beta V_n}, \end{aligned} \tag{16}$$

where $Q_1(\cdot)$, $Q_2(\cdot)$ are defined in (9),

$$\Omega_n := \frac{2K}{1-c} \varepsilon_n W_n U_n^{1-c} + \left(A + \frac{2^\alpha A}{1-c}\right) \int_{\mathbb{R}} \bigl[\mathcal{K}(x)\bigr]^{c+1} \bigl[1 + U_n \mathcal{K}(x)\bigr]^{-\alpha} \, dx,$$

with

$$W_n := 2 + \|\Psi''_\sigma\|_{U_n} + \|\Psi'_\sigma\|^2_{U_n} + 3\|\Psi'_\sigma\|_{U_n}. \tag{17}$$

Proof. The proof is given in Appendix B. □

Remark 1. *Similarly, in the setup of Section 3.3, we can establish analogous result for the estimate $\breve{\nu}_n(x)$ of $\breve{\nu}(x)$. Namely, we can show (see p. 919), that if $\nu(x)$ satisfies the assumption (A1) (where $\tilde{\nu}$ is changed to $\breve{\nu}$ everywhere), then the estimate $\breve{\nu}_n(x)$ satisfies*

$$\sup_{x\in\mathbb{R}_+}\left\{x^c\left|\breve{\nu}_n(x)-\breve{\nu}(x)\right|\right\} \le \frac{1}{2\pi}\int_{\{|v|\le V_n\}} \frac{\widetilde{\Omega}_n}{|\breve{Q}(1-c-\mathrm{i}v)|}\,dv + \frac{A}{2\pi}e^{-\beta V_n}, \tag{18}$$

where $\breve{Q}(\cdot)$ as in (13),

$$\widetilde{\Omega}_n := \frac{2K}{1-c}\varepsilon_n \widetilde{W}_n U_n^{1-c} + \left(A + \frac{2^\alpha A}{1-c}\right)\int_{\mathbb{R}}\left[\mathcal{K}(x)\right]^c\left[1+U_n\mathcal{K}(x)\right]^{-\alpha}dx,$$

with $\widetilde{W}_n := 1 + \|\Psi'_\sigma\|_{U_n}$.

Theorem 1 implies that $\sup_{x\in\mathbb{R}}\{|x|^c|\tilde{\nu}_n(x)-\tilde{\nu}(x)|\}$ converges to 0 on the set $\mathcal{A}_K^{\mathcal{C}}$ as long as $\varepsilon_n W_n U_n^{1-c} \to 0$ and $U_n \to \infty$. It would be a worth mentioning that W_n can be uniformly bounded, see Lemma 2 from Appendix C.

Let us now estimate the probability of the event $\mathcal{A}_K$. This probability crucially depends on the mixing properties of the process (Z_t).

Theorem 2. *Suppose that the following assumptions are fulfilled.*

(A2) *The kernel $\mathcal{K}$ satisfies*

$$\sum_{j=-\infty}^{\infty}\left|\mathcal{F}[\mathcal{K}]\left(2\pi\frac{j}{\Delta}\right)\right| \le K^*, \tag{19}$$

$$(\mathcal{K}\star\mathcal{K})(\Delta j) \le \kappa_0|j|^{\kappa_1}e^{-\kappa_2|j|}, \qquad \forall j\in\mathbb{Z} \tag{20}$$

for some positive constants K^*, κ_0, κ_1 and κ_2, and moreover all eigenvalues of the matrix

$$\mathcal{M} = \big((\mathcal{K}\star\mathcal{K})\big(\Delta(j-k)\big)\big)_{k,j\in\mathbb{Z}}$$

are bounded from below and above by two finite positive constants.

(A3) *The Lévy measure ν satisfies*

$$\int_{|x|>1} e^{R|x|}\nu(x)\,dx \le A_R \tag{21}$$

for some $R>0$ and $A_R>0$. Moreover, the process (L_t) has a non-zero Gaussian part, that is, $\sigma>0$.

Then under the choice

$$\varepsilon_n = \sqrt{\frac{\log(n)}{n}} \cdot \exp\left(\frac{A}{2}\sigma^2 U_n^2 \int \left(\mathcal{K}(x)\right)^2 dx\right)$$

with a constant A such that $\int_{\mathbb{R}} x^2 \nu(dx) \le A$ (see remarks after (A1)), *it holds for n large enough and any $K > 0$*

$$\mathbb{P}(\mathcal{A}_K) \le \frac{C_1}{\sqrt{K}} \frac{\sqrt{U_n} n^{(1/4)-C_2 K^2}}{\log^{1/4}(n)},$$

where the positive constants C_1, C_2 may depend on K^*, A_R and κ_i, $i = 0, 1, 2$. Hence by an appropriate choice of K we can ensure that $\mathbb{P}(\mathcal{A}_K) \to 0$ as $n \to \infty$.

Proof. The proof of Theorem 2 is based on some kind of exponential mixing for the general Lévy-driven moving average processes of the form (2), and is provided in Appendix D. In fact, such mixing properties were previously established in the literature only for the processes Z corresponding to the exponential kernel function $\mathcal{K}$. □

The assumption (A2) of Theorem 2 may seem to be strong, but as shown below, is fulfilled for a large family of kernels (22).

Example 3. Consider the class of symmetric kernels of the form

$$\mathcal{K}(x) = \left(\sum_{r=0}^{R} b_r |x|^{k_r}\right) e^{-\rho|x|}, \tag{22}$$

where $\rho > 0$, $b_r \ge 0$ for all $r = 0, \dots, R$, and $0 \le k_0 < \cdots < k_R$. Note that the assumption $k_0 \ge 0$ guarantees that $\mathcal{K} \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$. Since

$$\mathcal{F}[\mathcal{K}](u) = \sum_{r=0}^{R} b_r \Gamma(k_r + 1)\left[\frac{1}{(\rho - \mathrm{i}u)^{k_r+1}} + \frac{1}{(\rho + \mathrm{i}u)^{k_r+1}}\right],$$

we conclude that

$$\sum_{j=-\infty}^{\infty} \left|\mathcal{F}[\mathcal{K}]\left(2\pi \frac{j}{\Delta}\right)\right| \le 2 \sum_{r=0}^{R} b_r \Gamma(k_r + 1)\left[\sum_{j=-\infty}^{\infty} \left(\rho^2 + \left(\frac{2\pi j}{\Delta}\right)^2\right)^{-(k_r+1)/2}\right] < \infty,$$

that is, (19) is fulfilled. Assumption (20) and assumption on the matrix $\mathcal{M}$ are proved in Lemma 4 (see p. 925), and therefore (A2) holds. Next, assume that the Lévy measure ν is supported on $\mathbb{R}_+$ and satisfies the assumptions (A1) and (A3). For instance, it can be chosen as a sum of a Brownian motion with drift and a jump process with a Lévy measure considered in Example 1. Then, as it is shown in Appendix E,

$$\Omega_n \lesssim K \varepsilon_n U_n^{1-c} + U_n^{-\alpha}, \qquad n \to \infty$$

and

$$\int_{\{|v|\le V_n\}} \frac{1}{|Q(1-c-\mathrm{i}v)|}\,dv \lesssim \begin{cases} V_n^{c+3/2}, & b_0>0, R=0,\\ V_n^{c+1}, & \text{otherwise.}\end{cases}$$

As a result we have on $\mathcal{A}_K^{\mathcal{C}}$

$$\sup_{x\in\mathbb{R}}\bigl\{|x|^c\bigl|\tilde{\nu}_n(x)-\tilde{\nu}(x)\bigr|\bigr\} \lesssim V_n^{\zeta}\bigl(\varepsilon_n U_n^{(1-c)}+U_n^{-\alpha}\bigr)+e^{-\beta V_n}$$

with $\beta=\beta_+$ and $\zeta=c+1+\mathbb{I}\{R=0\}/2$. By taking $V_n=\varkappa\log(U_n)$ with $\varkappa>\alpha/\beta$ and $U_n=\theta\log^{1/2}(n)$ for any $\theta<(A\int(\mathcal{K}(x))^2\,dx)^{-1/2}$, we conclude that

$$\sup_{x\in\mathbb{R}}\bigl\{|x|^c\bigl|\tilde{\nu}_n(x)-\tilde{\nu}(x)\bigr|\bigr\} \lesssim \log^{-\alpha/2}(n), \qquad n\to\infty,$$

where $\lesssim$ stands for an inequality with some positive finite constant depending on the parameters of the corresponding class.

5. Numerical example

5.1. Simulation

Consider the integral $Z_t := \int_{\mathbb{R}} \mathcal{K}(t-s)\,dL_s$ with the kernel $\mathcal{K}(x)=e^{-|x|}$ and the Lévy process

$$L_t = L_t^{(1)}\mathbb{I}\{t>0\} - L_{-t}^{(2)}\mathbb{I}\{t<0\},$$

constructed from the independent compound Poisson processes

$$L_t^{(1)} \stackrel{d}{=} L_t^{(2)} \stackrel{d}{=} \sum_{k=1}^{N_t}\xi_k,$$

where N_t is a Poisson process with intensity λ, and $\xi_1,\xi_2,\dots$ are independent r.v.'s with standard exponential distribution. Note that the Lévy density of the process $L_t^{(1)}$ is $\nu(x)=\lambda e^{-x}$.

For $k=1,2$, denote the jump times of $L_t^{(k)}$ by $s_1^{(k)}, s_2^{(k)},\dots$ and the corresponding jump sizes by $\xi_1^{(k)},\xi_2^{(k)},\dots$. Then

$$Z_t = \sum_{j=0}^{\infty}\mathcal{K}\bigl(t-s_j^{(1)}\bigr)\xi_j^{(1)} - \sum_{j=0}^{\infty}\mathcal{K}\bigl(t+s_j^{(2)}\bigr)\xi_j^{(2)}.$$

In practice, we truncate both series in the last representation by finding a value $x_{\max} := \max_{x\in\mathbb{R}_+}\{\mathcal{K}(x)>\alpha\}$ for a given level α. Let

$$\tilde{Z}_t = \sum_{k\in K^{(1)}}\mathcal{K}\bigl(t-s_j^{(1)}\bigr)\xi_j^{(1)} - \sum_{k\in K^{(2)}}\mathcal{K}\bigl(t+s_j^{(2)}\bigr)\xi_j^{(2)},$$

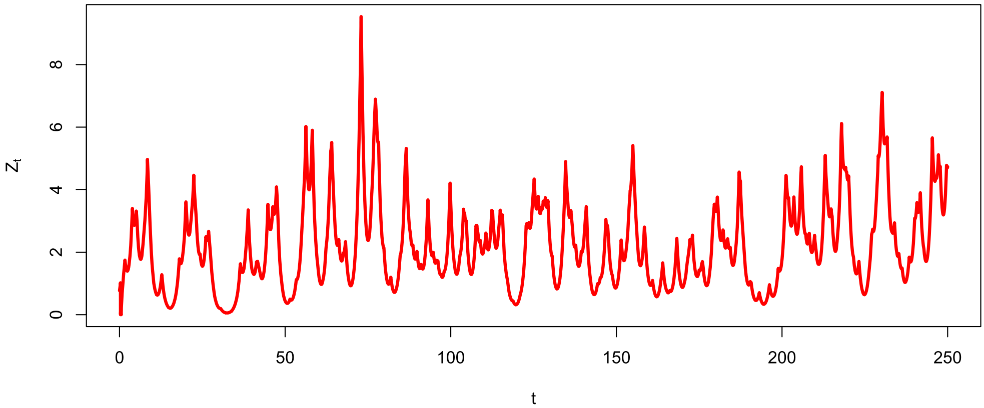

Figure 1. Typical trajectory of the process Z_t constructed from the compound Poisson process with positive jumps.

where

$$K^{(1)} := \{k : \max(0, t - x_{\max}) < s_k^{(1)} < t + x_{\max}\},$$
$$K^{(2)} := \{k : 0 < s_k^{(2)} < \max(0, -t + x_{\max})\}.$$

For simulation study, we take $\lambda = 1$, $\alpha = 0.01$ (and therefore $x_{\max} = 6.908$). Typical trajectory of the process $\tilde{Z}_t$ is presented on Figure 1.

5.2. Estimation procedure

We will use the simplified version of the estimation procedure described in Section 3.3.

Estimation of the Mellin transform of $\Psi'(\cdot)$

The most natural estimate is

$$\mathcal{M}_n[\Psi'](1-z) := \mathrm{i} \int_0^{U_n} \frac{\mathrm{mean}(Z_{k\Delta} e^{\mathrm{i}uZ_{k\Delta}})}{\mathrm{mean}(e^{\mathrm{i}uZ_{k\Delta}})} u^{-z}\, du, \tag{23}$$

where

$$\mathrm{mean}\big(e^{\mathrm{i}uZ_{k\Delta}}\big) := \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}uZ_{k\Delta}},$$
$$\mathrm{mean}\big(Z_{k\Delta} e^{\mathrm{i}uZ_{k\Delta}}\big) = \frac{1}{n}\sum_{k=1}^{n} Z_{k\Delta} e^{\mathrm{i}uZ_{k\Delta}}.$$

In order to improve the numerical rates of convergence of the integral involved in (23), we slightly modify this estimate:

$$\mathcal{M}_n\big[\Psi'\big](1-z) := \mathrm{i}\int_0^{U_n}\left[\frac{\mathrm{mean}(Z_{k\Delta}e^{\mathrm{i}uZ_{k\Delta}})}{\mathrm{mean}(e^{\mathrm{i}uZ_{k\Delta}})} - \mathrm{mean}(Z)e^{\mathrm{i}u}\right]u^{-z}\,du + 2\mathrm{i}\lambda\Gamma(1-z)\exp\big\{\mathrm{i}\pi(1-z)/2\big\}.$$

Note that $\mathcal{M}_n[\Psi'](1-z)$ is also a consistent estimate of $\mathcal{M}[\Psi'](1-z)$ (since $\mathrm{mean}(Z) \to 2\lambda$), but involves the integral with better convergence properties. In our case $\mathcal{M}[\tilde{\nu}](z) = \lambda\Gamma(1+z)$, and therefore the Mellin transform of the function Ψ' is equal to

$$\mathcal{M}\big[\Psi'\big](1-z) = \breve{Q}(1-z)\cdot\mathcal{M}[\breve{\nu}](z) = 2\mathrm{i}\lambda\frac{\Gamma(1-z)\Gamma(1+z)}{z}e^{\mathrm{i}\pi(1-z)/2}.$$

We estimate $\mathcal{M}[\Psi'](1-z)$ for $z = c + \mathrm{i}v_k$, where c is fixed and $v_k, k = 1, \dots, K$, are taken on the equidistant grid from $(-V_n)$ to V_n with step $\delta = 2V_n/K$. Typical behavior of the the Mellin transform $\mathcal{M}[\Psi'](1-z)$ and its estimate $\mathcal{M}_n[\Psi'](1-z)$ is illustrated by Figure 2.

Estimation of $\nu(x)$

Finally, we estimate the Lévy density $\nu(x)$ by

$$\hat{\nu}_n(x) := \frac{\delta}{2\pi x}\sum_{k=1}^{K}\mathrm{Re}\left\{\frac{\mathcal{M}_n[\Psi'](1-c-\mathrm{i}v_k)}{\breve{Q}(1-c-\mathrm{i}v)}\cdot x^{-(c+\mathrm{i}v_k)}\right\}$$

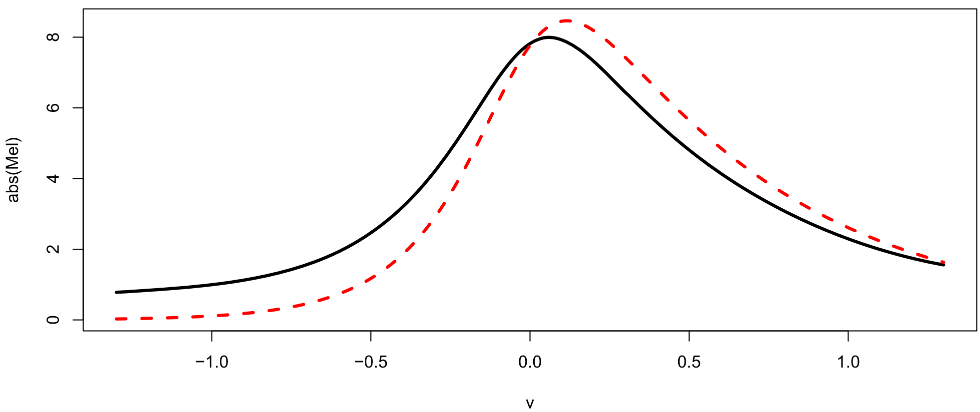

Figure 2. Absolute values of the empirical (solid curve) and theoretical (dashed curve) Mellin transforms of the function $\Psi'(\cdot)$ depending on the imaginary part of the argument.

Table 1. The results of the optimization procedure

n	U_n	V_n	mean($\mathcal{R}(\hat{\nu}_n)$)	Var($\mathcal{R}(\hat{\nu}_n)$)
1000	0.4	1.1	0.0109	1.62×10^{-5}
5000	0.4	1.2	0.0079	9.07×10^{-6}
10 000	0.5	1.3	0.0063	6.56×10^{-6}
20 000	0.3	1.3	0.0023	9.15×10^{-7}

and measure the quality of this estimate by the L^2-norm on the interval $[1, 3]$:

$$\mathcal{R}(\hat{\nu}_n) = \int_1^3 \big(\hat{\nu}_n(x) - \nu(x)\big)^2 \, dx.$$

To show the convergence of this estimate, we made simulations with different values of n. The parameters U_n and V_n are chosen by numerical optimization of $\mathcal{R}(\tilde{\nu}_n)$. The results of this optimization, for different values of n, as well as the means and variances of the estimate $\tilde{\nu}_n$ based on 20 simulation runs, are given in Table 1.

The boxplots of this estimate based on 20 simulation runs are presented on Figure 3.

Appendix A: Existence of Z_t

Lemma 1. *The conditions* (3) *and* (4) *guarantee that the stochastic integral Z_t exists.*

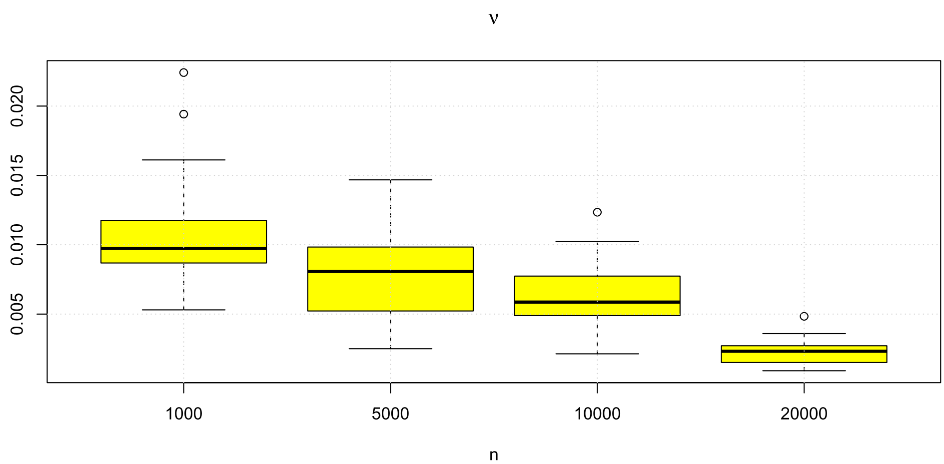

Figure 3. Boxplot of the estimate $\mathcal{R}(\hat{\nu}_n)$ based on 20 simulation runs.

Proof. As shown in [16], under the conditions

$$\int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \left(\left|\mathcal{K}(s)x\right|^2 \wedge 1\right) \nu(dx)\, ds < \infty, \tag{24}$$

$$\sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(s)\, ds < \infty, \tag{25}$$

$$\int_{\mathbb{R}} \left| \mathcal{K}(s) \left(\gamma + \int_{\mathbb{R}} x \left(1_{\{|x\mathcal{K}(s)| \le 1\}} - 1_{\{|x| \le 1\}}\right) \nu(dx) \right) \right| ds < \infty \tag{26}$$

the stochastic integral in (2) exists. In our case, (25) is trivial, and condition (24) directly follows from the inequality

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \left(\left|\mathcal{K}(s)x\right|^2 \wedge 1\right) \nu(dx)\, ds \\
&\quad \le \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \left|\mathcal{K}(s)x\right|^2 \nu(dx)\, ds \\
&\quad = \int_{\mathbb{R}} \left(\mathcal{K}(s)\right)^2 ds \cdot \int_{\mathbb{R}\setminus\{0\}} x^2 \nu(dx).
\end{aligned}$$

As to the condition (26), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \mathcal{K}(s) \left(\gamma - \int_{\mathbb{R}} x 1_{\{|x| \le 1\}} \nu(dx) \right) + \int_{\mathbb{R}} x \mathcal{K}(s) 1_{\{|x\mathcal{K}(s)| \le 1\}} \nu(dx) \right| ds \\
&\quad = \int_{\mathbb{R}} \left| \mathcal{K}(s) \mathbb{E}[L_1] - \int_{\mathbb{R}} x \mathcal{K}(s) 1_{\{|x\mathcal{K}(s)| > 1\}} \nu(dx) \right| ds \\
&\quad \le \left|\mathbb{E}[L_1]\right| \int_{\mathbb{R}} \mathcal{K}(s)\, ds + \int_{\mathbb{R}} \int_{\mathbb{R}} x^2 \left(\mathcal{K}(s)\right)^2 \nu(dx)\, ds,
\end{aligned}$$

where the right-hand side is finite due to our assumptions. This observation completes the proof. □

Appendix B: Proof of Theorem 1

For the sake of simplicity, we will provide the proof for the case when $\operatorname{supp}(\nu) \subset \mathbb{R}_+$. For the general case, the proof follows the same lines separately for $\tilde{\nu}_+$ and $\tilde{\nu}_-$.

Denote $G_j(u) = \Psi_{\sigma,n}^{(j)}(u) - \Psi_{\sigma}^{(j)}(u),\ j = 1, 2$, where

$$\Psi_{\sigma,n}(u) = \log \Phi_n(u) + \frac{\sigma^2 u^2}{2} \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx.$$

Then

$$G_1(u) = \frac{D_1(u) - D_0(u)\Psi'_\sigma(u)}{1 + D_0(u)}, \tag{27}$$

$$\begin{aligned} G_2(u) = {} & \frac{(-\Psi''_\sigma(u) + (\Psi'_\sigma(u))^2 + \Psi'_\sigma(u)G_1(u))D_0(u)}{1 + D_0(u)} \\ & - \frac{(2\Psi'_\sigma(u) + G_1(u))D_1(u)}{1 + D_0(u)} + \frac{D_2(u)}{1 + D_0(u)}. \end{aligned} \tag{28}$$

We have

$$\begin{aligned} \tilde{\nu}_n(x) - \tilde{\nu}(x) = {} & \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \left[\frac{\mathcal{M}_n[\Psi''_\sigma](1-z) - \mathcal{M}[\Psi''_\sigma](1-z)}{Q(1-z)} \right] x^{-z}\, dz \\ & - \frac{1}{2\pi} \int_{\{|v| \geq V_n\}} \mathcal{M}[\tilde{\nu}](c + \mathrm{i}v) x^{-(c+\mathrm{i}v)}\, dv \end{aligned}$$

and

$$\begin{aligned} x^c\big(\tilde{\nu}_n(x) - \tilde{\nu}(x)\big) = {} & \frac{1}{2\pi} \int_{\{|v| \leq V_n\}} \frac{R_1(v) + R_2(v)}{Q(1 - c - \mathrm{i}v)} x^{-\mathrm{i}v}\, dv \\ & - \frac{1}{2\pi} \int_{\{|v| \geq V_n\}} \mathcal{M}[\tilde{\nu}](c + \mathrm{i}v) x^{-\mathrm{i}v}\, dv, \end{aligned} \tag{29}$$

where $Q(\cdot) = Q_1(\cdot)$, and

$$R_1(v) := \int_0^{U_n} G_2(u) u^{-c-\mathrm{i}v}\, du,$$

$$R_2(v) := -\int_{U_n}^{\infty} \Psi''_\sigma(u) u^{-c-\mathrm{i}v}\, du.$$

We have on $\overline{\mathcal{A}_K}$, under the assumption $K\varepsilon_n(1 + \|\Psi'_\sigma\|_{U_n}) \leq 1/2$, that the denominator of the fractions in G_1 and G_2 can be lower bounded as follows:

$$\min_{u \in [-U_n, U_n]} \big|1 + D_0(u)\big| \geq 1 - \max_{u \in [-U_n, U_n]} \big|D_0(u)\big| \geq 1 - K\varepsilon_n \geq 1/2.$$

Therefore,

$$\|G_1\|_{U_n} \leq 2K\varepsilon_n\big(1 + \|\Psi'_\sigma\|_{U_n}\big) \leq 1,$$

$$\begin{aligned} \|G_2\|_{U_n} \leq 2K\varepsilon_n\big(1 & + \|\Psi''_\sigma\|_{U_n} + \|(\Psi'_\sigma)^2\|_{U_n} \\ & + \big(1 + \|\Psi'_\sigma\|_{U_n}\big)\|G_1\|_{U_n} + 2\|\Psi'_\sigma\|_{U_n}\big). \end{aligned}$$

Thus

$$\left|R_1(v)\right| \le \frac{2K}{1-c} U_n^{1-c} \varepsilon_n \left(2 + \left\|\Psi_\sigma''\right\|_{U_n} + \left\|\Psi_\sigma'\right\|_{U_n}^2 + 3\left\|\Psi_\sigma'\right\|_{U_n}\right).$$

Since

$$\Psi_\sigma''(u) = -\int_{-\infty}^{\infty} \mathcal{K}^2(x) \cdot \mathcal{F}[\tilde{\nu}]\big(u\mathcal{K}(x)\big)\, dx,$$

it holds for any $z \in \mathbb{C}$

$$\begin{aligned}\int_{U_n}^{\infty} \Psi_\sigma''(u) u^{-z}\, du &= -\int_{-\infty}^{\infty} \mathcal{K}^2(x) \left[\int_{U_n}^{\infty} \mathcal{F}[\tilde{\nu}]\big(u\mathcal{K}(x)\big) u^{-z}\, du\right] dx \\ &= -\int_{-\infty}^{\infty} \big[\mathcal{K}(x)\big]^{z+1} \left[\int_{U_n\mathcal{K}(x)}^{\infty} \mathcal{F}[\tilde{\nu}](v) v^{-z}\, dv\right] dx.\end{aligned}$$

Next, for any fixed $x \in \mathbb{R}$, we can upper bound the inner integral in the right-hand side of the last formula:

$$\left|\int_{U_n\mathcal{K}(x)}^{\infty} \mathcal{F}[\tilde{\nu}](v) v^{-z}\, dv\right| \le \big(1 + U_n\mathcal{K}(x)\big)^{-\alpha} \cdot \int_0^{\infty} v^{-\operatorname{Re}(z)} (1+v)^{\alpha} \big|\mathcal{F}[\tilde{\nu}](v)\big|\, dv.$$

Due to (24) we get that for any z with $\operatorname{Re}(z) \in (0, 1)$ it holds

$$\int_0^{\infty} v^{-\operatorname{Re}(z)} (1+v)^{\alpha} \big|\mathcal{F}[\tilde{\nu}](v)\big|\, dv < \frac{\bar{\delta}}{1 - \operatorname{Re}(z)} + A$$

with $\bar{\delta} = 2^{\alpha} \int_{\mathbb{R}_+} x^2 \nu(x)\, dx \le 2^{\alpha} A$ due to the remark after (A1). Finally, we conclude that

$$\begin{aligned}\left|R_2(v)\right| &= \left|\int_{U_n}^{\infty} \Psi_\sigma''(y) y^{-c-\mathrm{i}v}\, dy\right| \\ &\le \left(\frac{\bar{\delta}}{1-c} + A\right) \int_{\mathbb{R}} \big[\mathcal{K}(x)\big]^{c+1} \big(1 + U_n\mathcal{K}(x)\big)^{-\alpha}\, dx.\end{aligned}$$

Now an upper bound for the last term in (30) follows from the assumption on the Mellin transform of the function $\bar{\nu}$. Indeed, since (A1) is assumed, it holds

$$\begin{aligned}&\left|\int_{\{|u| \ge V_n\}} \mathcal{M}[\tilde{\nu}](c + \mathrm{i}u) x^{-\mathrm{i}u}\, du\right| \\ &\quad \le e^{-\beta V_n} \int_{\{|u| \ge V_n\}} e^{\beta V_n} \big|\mathcal{M}[\tilde{\nu}](c + \mathrm{i}u)\big|\, du \le A e^{-\beta V_n}.\end{aligned}$$

This observation completes the proof of Theorem 1.

Proof of Remark 1. The proof basically follows the same lines. Analogously to the key identity (30), we are using the following decomposition:

$$x^c\big(\breve{\nu}_n(x)-\breve{\nu}(x)\big)=\frac{1}{2\pi}\int_{\{|v|\le V_n\}}\frac{\breve{R}_1(v)+\breve{R}_2(v)}{\breve{Q}(1-c-\mathrm{i}v)}x^{-\mathrm{i}v}\,dv\\ -\frac{1}{2\pi}\int_{\{|v|\ge V_n\}}\mathcal{M}[\breve{\nu}](c+\mathrm{i}v)x^{-\mathrm{i}v}\,dv,$$

where

$$\breve{R}_1(v):=\int_0^{U_n}G_1(u)u^{-c-\mathrm{i}v}\,du,\qquad \breve{R}_2(v):=-\int_{U_n}^{\infty}\Psi'_\sigma(u)u^{-c-\mathrm{i}v}\,du.$$

Note that on the set $\overline{\mathcal{A}_K}$,

$$\big|\breve{R}_1(v)\big|\le\frac{1}{1-c}\|G_1\|_{U_n}U_n^{1-c}\le\frac{2K}{1-c}U_n^{1-c}\varepsilon_n\big(1+\|\Psi'_\sigma\|_{U_n}\big).$$

The treatment for $\breve{R}_2$ also follows the same lines as in the proof for (i),

$$\big|\breve{R}_2(v)\big|=\left|\int_{U_n}^{\infty}\Psi'_\sigma(y)y^{-c-\mathrm{i}v}\,dy\right|\le\left(\frac{\bar{\delta}}{1-c}+A\right)\int_{\mathbb{R}}\big[\mathcal{K}(x)\big]^c\big(1+U_n\mathcal{K}(x)\big)^{-\alpha}\,dx.\qquad\square$$

Appendix C: Boundedness of W_n

The following lemma holds.

Lemma 2. *Let A be a constant such that $\int_{\mathbb{R}}(|x|\vee x^2)\nu(x)\,dx\le A$. Then W_n defined by (17) is uniformly bounded.*

Proof. Indeed,

$$\big|\psi'(u)+\sigma^2u\big|=\left|\mathrm{i}\gamma+\int_{\mathbb{R}}\mathrm{i}xe^{\mathrm{i}ux}\nu(x)\,dx\right|\le\gamma+\int_{\mathbb{R}}|x|\nu(x)\,dx\le\gamma+A,$$

by our remark after (A1). Analogously,

$$\big|\psi''(u)+\sigma^2\big|=\left|\int_{\mathbb{R}}x^2e^{\mathrm{i}ux}\nu(x)\,dx\right|\le\int_{\mathbb{R}}x^2\nu(x)\,dx\le A.$$

Therefore

$$\|\Psi'_\sigma\|_{U_n}=\left\|\int_{\mathbb{R}}\big(\psi'\big(u\mathcal{K}(x)\big)+\sigma^2u\big)\mathcal{K}(x)\,dx\right\|_{U_n}\le(\gamma+A)\|\mathcal{K}\|_{L^1},$$

$$\|\Psi''_\sigma\|_{U_n}=\left\|\int_{\mathbb{R}}\big(\psi''\big(u\mathcal{K}(x)\big)+\sigma^2\big)\mathcal{K}^2(x)\,dx\right\|_{U_n}\le A\|\mathcal{K}\|_{L^2}^2,$$

where the integrals in the right-hand sides are bounded due to the assumption $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. $\square$

Appendix D: Mixing properties of the Lévy-based MA processes

Proposition 1. *Let (L_t) be a Lévy process with Lévy triplet (μ, σ^2, ν), where $\sigma > 0$ and* $\operatorname{supp}(\nu) \subseteq \mathbb{R}_+$. *Consider a Lévy-based moving average process of the form*

$$Z_s = \int \mathcal{K}(s-t)\, dL_t, \qquad s \geq 0$$

with a non-negative kernel $\mathcal{K}$. Fix some $\Delta > 0$ and denote

$$Z_S := (Z_{j\Delta})_{j \in S}$$

for any subset S of $\{1, \dots, n\}$. Fix two natural numbers m and p such that $m + p \leq n$. For any subsets $S \subseteq \{1, \dots, m\}$ and $S' \subseteq \{p+m, \dots, n\}$, let g and g' be two real valued functions on $\mathbb{R}^{|S|}$ and $\mathbb{R}^{|S'|}$ satisfying

$$\max\left\{ \left\| e^{-R_S^\top \cdot} g \right\|_{L^1}, \left\| e^{-R_{S'}^\top \cdot} g' \right\|_{L^1} \right\} < \infty$$

for some $R_S \in \mathbb{R}_+^{|S|}$ and $R_{S'} \in \mathbb{R}_+^{|S'|}$, and denote $C_\circ := \| e^{-R_S^\top \cdot} g \|_{L^1} \cdot \| e^{-R_{S'}^\top \cdot} g' \|_{L^1}$. Suppose that the Fourier transform $\widehat{\mathcal{K}}$ of $\mathcal{K}$ fulfils

$$K^* := \sum_{j=-\infty}^{\infty} \left| \widehat{\mathcal{K}}\left(2\pi \frac{j}{\Delta} \right) \right| < \infty$$

and

$$\int_{|x|>1} e^{R^*|x|} x^2 \nu(dx) \leq A_{R^*}$$

for $R^* = \frac{\|R_{S \cup S'}\|_\infty K^*}{\Delta}$. Then

$$\begin{aligned} \left| \operatorname{Cov}\left(g(Z_S), g'(Z_{S'}) \right) \right| \leq{}& C_R C_\circ \max_{|l|>p} (\mathcal{K} \star \mathcal{K})(l\Delta) \\ &\times \int \| u_{S \cup S'} - \mathrm{i} R_{S \cup S'} \|^2 \exp\left(-\sigma^2 \lambda_{S \cup S'}(u) \right) du_{S \cup S'}, \end{aligned} \tag{30}$$

where $\lambda_S(u) := \sum_{k,j \in S} u_k u_j (\mathcal{K} \star \mathcal{K})(\Delta(k-j))$ for any $u \in \mathbb{R}^n$ and $C_R = \exp(\sigma^2 \lambda_{S \cup S'}(R_{S \cup S'}))$.

Proof. For the sake of simplicity, we prove the result for the case $\operatorname{supp}(\nu) \in \mathbb{R}_+$. We have for any $S \subseteq \{1, \dots, n\}$

$$\begin{aligned}\Phi_S(u_S - \mathrm{i}R_S) &:= \mathbb{E}\Big[\exp\Big(i \sum_{j\in S} u_j Z_{j\Delta} + \sum_{j\in S} R_j Z_{j\Delta}\Big)\Big]\\ &= \exp\Big(\int \psi\Big(\sum_{j\in S}(u_j - \mathrm{i}R_j)\mathcal{K}(t - j\Delta)\Big)\,dt\Big),\end{aligned}$$

where $u_S := (u_j \in \mathbb{R}, j \in S)$ and $R_S := (R_j \in \mathbb{R}_+, j \in S)$, provided

$$\mathbb{E}\Big[\exp\Big(\sum_{j\in S} R_j Z_{j\Delta}\Big)\Big] < \infty.$$

Denote for any subsets $S \subseteq \{1, \dots, m\}$ and $S' \subseteq \{p + m, \dots, n\}$,

$$\begin{aligned}&D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})\\ &\quad:= \Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'}) - \Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'}),\end{aligned}$$

where it is assumed that

$$\mathbb{E}\Big[\exp\Big(\sum_{j\in S\cup S'} R_j Z_{j\Delta}\Big)\Big] < \infty.$$

Then using the elementary inequality $|e^z - e^y| \le (|e^z| \vee |e^y|)|y - z|$, $y, z \in \mathbb{C}$, we derive

$$\begin{aligned}&\big|D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})\big|\\ &\quad\le \big\{\big|\Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \vee |\Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'})\big|\big\}\\ &\qquad\times \bigg|\int \bigg\{\psi\Big(\sum_{j\in S\cup S'}(u_j - \mathrm{i}R_j)\mathcal{K}(x - j\Delta)\Big) - \psi\Big(\sum_{j\in S}(u_j - \mathrm{i}R_j)\mathcal{K}(x - j\Delta)\Big)\\ &\qquad - \psi\Big(\sum_{j\in S'}(u_j - \mathrm{i}R_j)\mathcal{K}(x - j\Delta)\Big)\bigg\}\,dx\bigg|.\end{aligned}$$

Due to Lemma 3 and the Poisson summation formula, we derive

$$\begin{aligned}&\big|D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})\big|\\ &\quad\le \big\{\big|\Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})\big| \vee \big|\Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'})\big|\big\}\\ &\qquad\times \Big[\sum_{j\in S}\sum_{l\in S'}\big|(u_l - \mathrm{i}R_l)(u_j - \mathrm{i}R_j)\big|(\mathcal{K}\star\mathcal{K})\big((j - l)\Delta\big)\Big]\\ &\qquad\times \int y^2 e^{\frac{y\|R\|_\infty K^*}{\Delta}}\,\nu(dy).\end{aligned}$$

We have

$$\begin{aligned}&\operatorname{Cov}\bigl(g(Z_S), g'(Z_{S'})\bigr)\\&\quad=\int_{\mathbb{R}_+^{|S|}}\int_{\mathbb{R}_+^{|S'|}} g(x_S)g'(x_{S'})\bigl(p_{S,S'}(x_S,x_{S'})-p_S(x_S)p_{S'}(x_{S'})\bigr)\,dx_S\,dx_{S'}\end{aligned}$$

and the Parseval's identity implies

$$\begin{aligned}\operatorname{Cov}\bigl(g(Z_S), g'(Z_{S'})\bigr)=\frac{1}{(2\pi)^{|S|+|S'|}}&\int_{\mathbb{R}^{|S|}}\int_{\mathbb{R}^{|S'|}}\widehat{g}(\mathrm{i}R_S-u_S)\widehat{g'}(\mathrm{i}R_{S'}-u_{S'}).\\&\times D(u_S-\mathrm{i}R_S,u_{S'}-\mathrm{i}R_{S'})\,du_S\,du_{S'},\end{aligned}$$

$\widehat{g}$ stands for the Fourier transform of g. Hence

$$\begin{aligned}&\bigl|\operatorname{Cov}\bigl(g(Z_S), g'(Z_{S'})\bigr)\bigr|\\&\quad\le\frac{C_R}{(2\pi)^{|S|+|S'|}}\int_{\mathbb{R}^{|S|}}\int_{\mathbb{R}^{|S'|}}\bigl|D(u_S-\mathrm{i}R_S,u_{S'}-\mathrm{i}R_{S'})\bigr|\,du_S\,du_{S'}.\end{aligned}$$

Furthermore, for any set $S\subset\{1,\dots,n\}$, we have

$$\int\psi\Biggl(\sum_{j\in S}(u_j-\mathrm{i}R_j)\mathcal{K}(s-j\Delta)\Biggr)\,ds\le-\sigma^2\lambda_S(u)+\sigma^2\lambda_S(R).$$

As a result

$$\bigl|\Phi_S(u_S-\mathrm{i}R_S)\bigr|\le C_R\exp\bigl(-\sigma^2\lambda_S(u)\bigr)$$

and

$$\begin{aligned}\bigl|D(u_S-\mathrm{i}R_S,u_{S'}-\mathrm{i}R_{S'})\bigr|\le\max_{|l|>p}(\mathcal{K}\star\mathcal{K})(l\Delta)&\sum_{j\in S}\sum_{l\in S'}\bigl|(u_l-\mathrm{i}R_l)(u_j-\mathrm{i}R_j)\bigr|\\&\times C_R\exp\bigl(-\sigma^2\lambda_{S\cup S'}(u)\bigr).\end{aligned}$$

□

Lemma 3. *Set*

$$\psi(z)=\int_0^\infty\bigl(\exp(zx)-1\bigr)\nu(dx)$$

for any $z\in\mathbb{C}$, *such that the integral* $\int_{|x|>1}\exp(\operatorname{Re}(z)x)\nu(dx)$ *is finite. Then*

$$\bigl|\psi(z_1+z_2)-\psi(z_1)-\psi(z_2)\bigr|\le2|z_1||z_2|\int x^2e^{x(\operatorname{Re}(z_1)+\operatorname{Re}(z_2))}\nu(dx),$$

provided the integral $\int x^2e^{x(\operatorname{Re}(z_1)+\operatorname{Re}(z_2))}\nu(dx)$ *is finite.*

Proof. We have

$$\begin{aligned}
&\psi(z_1+z_2)-\psi(z_1)-\psi(z_2)\\
&\quad=\int_0^\infty \bigl(\exp\bigl((z_1+z_2)x\bigr)-\exp(z_1x)-\exp(z_2x)+1\bigr)\nu(dx)\\
&\quad=\int_0^\infty \bigl(\exp(z_1x)-1\bigr)\bigl(\exp(z_2x)-1\bigr)\nu(dx).
\end{aligned}$$

Since

$$\begin{aligned}
\bigl|\exp(z)-1\bigr| &= \bigl|e^{\mathrm{Re}(z)}e^{i\,\mathrm{Im}(z)}-1\bigr|\\
&= \bigl|e^{\mathrm{Re}(z)}\bigl(e^{i\,\mathrm{Im}(z)}-1\bigr)+e^{\mathrm{Re}(z)}-1\bigr|\\
&\le \bigl|\mathrm{Im}(z)\bigr|e^{\mathrm{Re}(z)}+\bigl|e^{\mathrm{Re}(z)}-1\bigr|\\
&\le \bigl(\bigl|\mathrm{Re}(z)\bigr|+\bigl|\mathrm{Im}(z)\bigr|\bigr)e^{\mathrm{Re}(z)}\\
&\le \sqrt{2}|z|e^{\mathrm{Re}(z)},
\end{aligned}$$

we get

$$\begin{aligned}
\bigl|\psi(z_1+z_2)-\psi(z_1)-\psi(z_2)\bigr| &\le \int_0^\infty \bigl|\exp(z_1x)-1\bigr|\bigl|\exp(z_2x)-1\bigr|\nu(dx)\\
&\le 2|z_1||z_2|\int x^2e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx).
\end{aligned}$$

□

Proof of Theorem 2. The rest of the proof of Theorem 2 basically follows the same lines as the proof of Proposition 3.3 from [7]. First note that

$$\max_{|u|\le U_n}\frac{|\Phi_n(u)-\Phi(u)|}{|\Phi(u)|}\le \exp\biggl\{C_1\sigma^2U_n^2\int_{\mathbb{R}}\bigl(\mathcal{K}(x)\bigr)^2\,dx\biggr\}\cdot\max_{|u|\le U_n}\bigl|\Phi_n(u)-\Phi(u)\bigr|$$

for n large enough. Next, we separately consider the real and imaginary parts of the difference between $\Phi_n(u)$ and $\Phi(u)$. Denote

$$S_n(u):=n\,\mathrm{Re}\bigl(\Phi_n(u)-\Phi(u)\bigr)=\sum_{k=1}^n\bigl[\cos(uZ_{k\Delta})-\mathbb{E}\bigl[\cos(uZ_{k\Delta})\bigr]\bigr].$$

Since $S_n(u)$ is a sum of centred real-valued random variables, bounded by 2 and satisfying (30) and (33), there exist a positive constant c_1 such that

$$\mathbb{P}\bigl\{\bigl|S_n(u)\bigr|\ge x\bigr\}\le \exp\biggl\{\frac{-c_1x^2}{2n+x\log(n)\log\log(n)}\biggr\},\qquad \forall x\ge 0, \tag{31}$$

see Theorem 1 from [13]. In order to apply now the classical chaining argument, we divide the interval $[-U_n, U_n]$ by $2J$ equidistant points $(u_j) =: \mathcal{G}$, where $u_j = U_n(-J + j)/J$, $j = 1, \dots, 2J$. Applying (31), we get for any $x \geq 0$,

$$\mathbb{P}\left\{\max_{u_j \in \mathcal{G}} \left|S_n(u_j)\right| \geq x/2\right\} \leq 2J \exp\left\{\frac{-c_1 x^2}{8n + 2x \log(n) \log\log(n)}\right\}. \tag{32}$$

Note that for any $u \in [-U_n, U_n]$ there exists a point $u^\star \in \mathcal{G}$ such that $|u - u^\star| \leq U_n/J$ and therefore for all $k \in 1, \dots, n$,

$$\left|\cos(uZ_{k\Delta}) - \cos\left(u^\star Z_{k\Delta}\right)\right| \leq |Z_{k\Delta}| \cdot \left|u - u^\star\right| \leq |Z_{k\Delta}| \cdot U_n/J.$$

Next, we get

$$\begin{aligned}
&\mathbb{P}\left\{\max_{|u| \leq U_n} \left|S_n(u)\right| \geq x\right\} \\
&\quad \leq \mathbb{P}\left\{\max_{u_j \in \mathcal{G}} \left|S_n(u_j)\right| \geq x/2\right\} + \mathbb{P}\left\{\sum_{k=1}^{n} \left(|Z_{k\Delta}| + \mathbb{E}\left[|Z_{k\Delta}|\right]\right) U_n/J \geq x/2\right\}.
\end{aligned}$$

Applying (32) and the Markov inequality, we arrive at

$$\mathbb{P}\left\{\max_{|u| \leq U_n} \left|S_n(u)\right| \geq x\right\} \leq 2J \exp\left\{\frac{-c_1 x^2}{8n + 2x \log(n) \log\log(n)}\right\} + \frac{4U_n}{xJ} n \mathbb{E}|Z_\Delta|,$$

where $\mathbb{E}[|Z_\Delta|] \leq (\mathbb{E}[|Z_\Delta|^2])^{1/2}$ is finite due to (4). The choice

$$J = \mathrm{floor}\left(\sqrt{\frac{U_n n}{x} \cdot \exp\left\{\frac{c_1 x^2}{8n + 2x \log(n) \log\log(n)}\right\}}\right),$$

where $\mathrm{floor}(\cdot)$ stands for the largest integer smaller than the argument, leads to the estimate

$$\begin{aligned}
\mathbb{P}\left\{\max_{|u| \leq U_n} \left|S_n(u)\right| \geq x\right\} &\leq c_2 \sqrt{\frac{U_n n}{x}} \exp\left\{\frac{-c_1 x^2}{16n + 4x \log(n) \log\log(n)}\right\} \\
&\leq c_2 \sqrt{\frac{U_n n}{x}} \exp\left\{\frac{-c_3 x^2}{n}\right\},
\end{aligned}$$

which holds for n large enough with $c_2 = 2(1 + \mathbb{E}[|Z_\Delta|])$, $c_3 = c_1/17$, provided $x \lesssim n^{1-\varepsilon}$ with some $\varepsilon > 0$. Finally,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{|u| \leq U_n} \left|S_n(u)\right| \geq x\right\} \\
&\quad \geq \mathbb{P}\left\{\max_{|u| \leq U_n} \frac{|\mathrm{Re}(\Phi_n(u) - \Phi(u))|}{|\Phi(u)|} \geq \frac{x}{n} \exp\left\{C_1 \sigma^2 U_n^2 \int_{\mathbb{R}} \left(\mathcal{K}(x)\right)^2 dx\right\}\right\}.
\end{aligned}$$

Therefore, the choice

$$x = Kn \exp\left\{-C_1 \sigma^2 U_n^2 \int_{\mathbb{R}} \big(\mathcal{K}(x)\big)^2 \, dx\right\} \varepsilon_n / 2 = K\sqrt{n \log(n)}/2$$

with any positive K leads to

$$\mathbb{P}\left\{\max_{|u| \le U_n} \frac{|\operatorname{Re}(\Phi_n(u) - \Phi(u))|}{|\Phi(u)|} \ge \frac{K\varepsilon_n}{2}\right\} \le \frac{\sqrt{2} c_2}{\sqrt{K}} \frac{\sqrt{U_n} n^{(1/4) - c_3(K^2/4)}}{\log^{1/4}(n)}.$$

Since the same statement holds for $|\operatorname{Im}(\Phi_n(u) - \Phi(u))|/|\Phi(u)|$, we arrive at the desired result. □

Appendix E: Proofs for Example 3

Lemma 4. *Let $\mathcal{K}(x) = (\sum_{r=0}^{R} b_r |x|^{k_r}) e^{-\rho|x|}$ with $\rho > 0$, $b_r \ge 0$ for all $r = 0, \dots, R$, and $0 \le k_0 < \cdots < k_R$. Then*

$$\frac{(\mathcal{K} \star \mathcal{K})(\Delta(k - j))}{(\mathcal{K} \star \mathcal{K})(0)} \le \kappa_0 (j - k)^{\kappa_1} e^{-\kappa_2 (j - k)} \tag{33}$$

for all $j > k$ with some positive constants κ_0, κ_1 and κ_2. Moreover, all eigenvalues of the matrix $\mathcal{M}$ are bounded from below and above by two finite positive numbers, provided κ_2 (equivalently ρ) is large enough.

Proof. Let us focus for simplicity on the case $\mathcal{K}(x) = |x|^r e^{-\rho|x|}$ with some $r > 0$, $\rho > 0$; proof for the general case follows the same lines. We have

$$(\mathcal{K} \star \mathcal{K})(0) = 2 \int_0^\infty x^{2r} e^{-2\rho x} \, dx = 2(2\rho)^{-2r-1} \Gamma(2r + 1)$$

and

$$\int_{\mathbb{R}} \mathcal{K}_{\Delta j}(v) \mathcal{K}_{\Delta k}(v) \, dv = \left(\int_{-\infty}^{\Delta k} + \int_{\Delta k}^{\Delta j} + \int_{\Delta j}^{\infty}\right) \mathcal{K}_{\Delta j}(v) \mathcal{K}_{\Delta k}(v) \, dv =: I_1 + I_2 + I_3,$$

where $\mathcal{K}_t(s) := \mathcal{K}(s - t), \forall s, t \in \mathbb{R}_+$. In the sequel we separately consider integrals I_1, I_2, I_3. We have

$$\begin{aligned} I_1 &= \int_{\Delta j}^{\infty} (v - \Delta j)^r (v - \Delta k)^r e^{-2\rho v + \Delta\rho(j+k)} \, dv \\ &= \int_{\mathbb{R}_+} u^r \big(u + \Delta(j - k)\big)^r e^{-2\rho u - \rho\Delta(j-k)} \, du \\ &= 2^r e^{-\rho\Delta(j-k)} \big(\Gamma(2r + 1) + \big(\Delta(j - k)\big)^r \Gamma(r + 1)\big) \end{aligned}$$

and

$$I_2 = \int_{\Delta k}^{\Delta j} \left[-(v-\Delta j)(v-\Delta k)\right]^r e^{-\rho\Delta(j-k)}\,dv$$
$$\leq \frac{\Delta^{2r+1}}{2^{2r}}(j-k)^{2r+1} e^{-\rho\Delta(j-k)},$$

because maximum of the quadratic function $f(v) := -(v-\Delta j)(v-\Delta k)$ is attained at the point $v = \Delta(k+j)/2$ and is equal to $(\Delta^2/4)(j-k)^2$. Furthermore,

$$I_3 = \int_{-\infty}^{\Delta k} (\Delta j - v)^r (\Delta k - v)^r e^{2\rho v - \rho\Delta(j+k)}\,dv$$
$$= \int_{\mathbb{R}_+} \bigl(u + \Delta(j-k)\bigr)^r u^r e^{-2\rho u - \rho\Delta(j-k)}\,du = I_1.$$

Next, the well-known Gershgorin circle theorem implies that the minimal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from below by

$$(\mathcal{K}\star\mathcal{K})(0) - 2\sum_{l>0}(\mathcal{K}\star\mathcal{K})(l)$$
$$= (\mathcal{K}\star\mathcal{K})(0)\left[1 - 2\kappa_0 \sum_{l>0} l^{\kappa_1} e^{-\kappa_2 l}\right].$$

Note that for any natural number $\kappa_1 > 0$

$$\sum_{l\geq 1} l^{\kappa_1} e^{-\kappa_2 l} = (-1)^{\kappa_1} \frac{d^{\kappa_1}}{dx^{\kappa_1}}\left(\frac{e^{-x}}{1-e^{-x}}\right)\bigg|_{x=\kappa_2}.$$

Hence the minimal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from below by a positive number, if κ_2 is large enough. Analogously the maximal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from above by

$$(\mathcal{K}\star\mathcal{K})(0) + 2\sum_{l>0}(\mathcal{K}\star\mathcal{K})(l) = (\mathcal{K}\star\mathcal{K})(0)\left[1 + 2\kappa_0 \sum_{l>0} l^{\kappa_1} e^{-\kappa_2 l}\right]$$

which is finite. □

For the sake of simplicity in what follows we consider the case $\mathcal{K}(x) = |x|^r e^{-|x|}$, and assume that either the kernel $\mathcal{K}$ is symmetric or is supported on $\mathbb{R}_+$, so that it suffices to study the integral over $\mathbb{R}_+$.

1. *Upper bound for* $\Lambda_n := \int_{\mathbb{R}_+} [\mathcal{K}(x)]^{c+1} [1 + U_n \mathcal{K}(x)]^{-\alpha}\,dx$ In the sequel, we consider only the case $r > 0$, since for $r = 0$ the statement becomes obvious. Note that the function $\mathcal{K}(x) =$

$x^r e^{-x}$ has two intervals of monotonicity on $\mathbb{R}_+$: $[0, r]$ and $[r, \infty)$. Denote the corresponding inverse functions by $g_1 : [0, r^r e^{-r}] \to [0, r]$ and $g_2 : [0, r^r e^{-r}] \to [r, \infty)$. Then

$$\begin{aligned}
\Lambda_n &= \left(\int_0^r + \int_r^\infty\right)\left[\mathcal{K}(x)\right]^{c+1}\left[1 + U_n \mathcal{K}(x)\right]^{-\alpha} dx \\
&= \int_0^{r^r e^{-r}} w^{c+1}(1 + U_n w)^{-\alpha} g_1'(w)\, dw \\
&\quad + \int_{r^r e^{-r}}^0 w^{c+1}(1 + U_n w)^{-\alpha} g_2'(w)\, dw \\
&= \int_0^{r^r e^{-r}} w^{c+1}(1 + U_n w)^{-\alpha} G(w)\, dw \\
&= U_n^{-c-2}\left(\int_0^1 + \int_1^{r^r e^{-r} U_n}\right) y^{c+1}(1 + y)^{-\alpha} \cdot G(y/U_n)\, dy \\
&=: J_1 + J_2,
\end{aligned}$$

where $G(\cdot) = g_1'(\cdot) - g_2'(\cdot)$. In what follows, we separately analyze the summands J_1 and J_2.

1a. *Upper bound for* J_1 Clearly, the behavior of the function $G(\cdot)$ at zero is crucial for the analysis of J_1. Since $\mathcal{K}(g_1(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we get $g_1(0) = 0$ and moreover as $y \to 0$,

$$g_1'(y) = \frac{1}{\mathcal{K}'(g_1(y))} = \frac{1}{[g_1(y)]^{r-1} e^{-g_1(y)}(r - g_1(y))} \asymp \frac{1}{r[g_1(y)]^{r-1}}.$$

Analogously, due to $\mathcal{K}(g_2(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we conclude that $\lim_{y\to 0} g_2(y) = +\infty$, and as $y \to 0$

$$g_2'(y) = \frac{1}{[g_2(y)]^{r-1} e^{-g_2(y)}(r - g_2(y))} \asymp \frac{-1}{[g_2(y)]^r e^{-g_2(y)}} = \frac{-1}{\mathcal{K}(g_2(y))} = \frac{-1}{y}.$$

For further analysis of the asymptotic behaviour of $g_1(\cdot)$, we apply the asymptotic iteration method. We are interested in the behaviour of the solution $g_1(y)$ of the equation

$$f(x) := x^r e^{-x} - y = 0$$

as $y \to 0$. Note that the distinction between the solutions is in the asymptotic behaviour as $y \to 0$: $g_1(y) \to 0$, $g_2(y) \to \infty$. Let us iteratively apply the recursion

$$\varphi_{n+1} = \varphi_n - \frac{f(\varphi_n)}{f'(\varphi_n)} = \varphi_n - \frac{\varphi_n^r e^{-\varphi_n} - y}{\varphi_n^{r-1} e^{-\varphi_n}(r - \varphi_n)}, \qquad n = 1, 2, \ldots.$$

Motivated by the power series expansion of the function e^{-x} at zero,

$$x^r e^{-x} = x^r - x^{r+1} + \frac{1}{2}x^{r+2} + o\left(x^{r+2}\right),$$

we take for the initial approximation of $g_1(y)$, the function $\varphi_0 = y^{1/r}$. Then

$$\begin{aligned}\varphi_1(y) &= y^{1/r} - \frac{ye^{-y^{1/r}} - y}{y^{(r-1)/r}e^{-y^{1/r}}(r - y^{1/r})} \\ &= y^{1/r}\left(1 - \frac{e^{-y^{1/r}} - 1}{e^{-y^{1/r}}(r - y^{1/r})}\right) \\ &= y^{1/r} + O\big(y^{2/r}\big).\end{aligned}$$

Finally, we conclude that as $y \to 0$,

$$G(y) = \frac{1}{ry^{(r-1)/r}}\big(1 + o(1)\big) + \frac{1}{y}\big(1 + o(1)\big) = \frac{1}{y}\big(1 + o(1)\big).$$

Therefore, J_1 can be upper bounded as follows:

$$J_1 \le C_3 U_n^{-c-1} \int_0^1 y^c (1 + y)^{-\alpha}\big(1 + o(1)\big)\, dy.$$

The integral in the right-hand side converges iff $\int_0^1 y^c\, dy < \infty$. Since $c \in (0, 1)$, we get $J_1 \lesssim U_n^{-c-1}$.

1b. *Asymptotic behaviour of* J_2 Analogously, the asymptotic behavior of J_2 crucially depends on the behavior of $G(y)$ at the point $y = r^r e^{-r}$. Note that as $y \to r^r e^{-r}$,

$$g_k'(y) = \frac{1}{\mathcal{K}'(g_k(y))} = \frac{1}{[g_k(y)]^{r-1}e^{-g_k(y)}(r - g_k(y))} \asymp \frac{C}{r - g_k(y)}$$

for $k = 1, 2$. Taking logarithms of both parts of the equation $x^r e^{-x} = y$ and changing the variables $u = x - r$ and $\delta = r^r e^{-r} - y$, we arrive at the equality

$$u = r \log\left(1 + \frac{u}{r}\right) - \log\left(1 - \frac{\delta}{r^r e^{-r}}\right).$$

Consider this equality as $u \to 0$ and $\delta \to 0+$, we get

$$u = r\left(\frac{u}{r} - \frac{1}{2}\frac{u^2}{r^2}\right) + \frac{\delta}{r^r e^{-r}} + O\big(\delta^2\big) + O\big(u^3\big),$$

and therefore

$$u = \pm\sqrt{2r^{1-r}e^r} \cdot \sqrt{\delta} + O(\delta) + O\big(u^{3/2}\big)$$

corresponding to the functions g_1 and g_2. Finally, we conclude

$$\big|G(y)\big| \asymp \frac{C\sqrt{2}}{\sqrt{r^{1-r}e^r}} \frac{1}{\sqrt{r^r e^{-r} - y}}, \qquad y \to r^r e^{-r},$$

and therefore

$$J_2 \sim U_n^{-c-3/2} \int_1^{r^r e^{-r} U_n} y^{c+1}(1+y)^{-\alpha} \cdot \frac{1}{\sqrt{r^r e^{-r} U_n - y}}\, dy.$$

We change the variable in the last integral:

$$z = \sqrt{\frac{r^r e^{-r} U_n - 1}{r^r e^{-r} U_n - y}}, \qquad y = r^r e^{-r} U_n + \frac{1 - r^r e^{-r} U_n}{z^2},$$

and get with $\widetilde{U}_n = r^r e^{-r} U_n$

$$J_2 \asymp U_n^{-c-3/2} \int_1^{\infty} \left(\widetilde{U}_n + \frac{1-\widetilde{U}_n}{z^2}\right)^{c+1} \cdot \left(1 + \widetilde{U}_n + \frac{1-\widetilde{U}_n}{z^2}\right)^{-\alpha} \cdot \frac{z}{\sqrt{\widetilde{U}_n - 1}} \frac{2(\widetilde{U}_n - 1)}{z^3}\, dz.$$

Therefore,

$$J_2 \asymp C_4 U_n^{-c-3/2} \widetilde{U}_n^{c+1} (\widetilde{U}_n + 1)^{-\alpha} \sqrt{\widetilde{U}_n - 1}, \qquad n \to \infty,$$

with some constant $C_4 > 0$ and we conclude that $J_2 \asymp C_5 U_n^{-\alpha}$ as $n \to \infty$. To sum up, $\Lambda_n \lesssim U_n^{-\min(\alpha, c+1)} = U_n^{-\alpha}$ as $n \to \infty$.

2. *Upper bound for* $H_n := \int_{\{|v| \le V_n\}} |Q(1-c-\mathrm{i}v)|^{-1}\, dv$ Recall that

$$H_n = \int_{\{|v| \le V_n\}} \frac{e^{-\pi v/2}}{|\Gamma(1-c-\mathrm{i}v)| \cdot |\int_{\mathbb{R}} (\mathcal{K}(x))^{c+1+\mathrm{i}v}\, dx|}\, dv.$$

Note that for our choice of the function $\mathcal{K}(\cdot)$, it holds for any $z \in \mathbb{C}$

$$\int_{\mathbb{R}} \big(K(x)\big)^z\, dx = 2 \int_{\mathbb{R}_+} \big(x^r e^{-x}\big)^z\, dx = 2\left[\lim_{R \to +\infty} \int_{\gamma_R(z)} u^{rz} e^{-u}\, du\right] \cdot z^{-(rz+1)},$$

where $\gamma_R(z)$ is the part of the complex line $\{(x \operatorname{Re}(z), x \operatorname{Im}(z)), x \in [0, R]\}$. Note that due to the Cauchy theorem, for any z with positive real part

$$\int_{\mathbb{R}_+} u^{rz} e^{-\rho u}\, du = \lim_{R \to +\infty} \int_{\gamma_R(z)} u^{rz} e^{-u}\, du + \lim_{R \to +\infty} \int_{c_R} u^{rz} e^{-u}\, du \tag{34}$$

with $c_R := \{(R\cos(\theta), R\sin(\theta)), \theta \in (0, \arctan(\operatorname{Im}(z)/\operatorname{Re}(z)))\}$. Since the last limit in (34) is equal to 0, we conclude that

$$\int_{\mathbb{R}} \big(K(x)\big)^{c+1+\mathrm{i}v}\, dx = 2\Gamma\big(r(c+1) + 1 + \mathrm{i}vr\big) \cdot e^{-(r(c+1)+1+\mathrm{i}vr)\cdot \log(c+1+\mathrm{i}v)}.$$

Next, using the fact that there exists a constant $\bar{C} > 0$ such that $|\Gamma(\alpha + \mathrm{i}\beta)| \geq \bar{C}|\beta|^{\alpha-1/2}e^{-|\beta|\pi/2}$ for any $\alpha \geq -2$, $|\beta| \geq 2$ (see Corollary 7.3 from [8]), we get that

$$\frac{e^{-\pi v/2}}{|\Gamma(1-c-\mathrm{i}v)|} \leq v^{c-1/2},$$

and moreover

$$\left|\int_{\mathbb{R}} \big(K(x)\big)^{c+1+\mathrm{i}v}\,dx\right| = 2\frac{|\Gamma(r(c+1)+1+\mathrm{i}vr)|}{((c+1)^2+v^2)^{(r(c+1)+1)/2}e^{-vr\arctan(v/(c+1))}}.$$

The asymptotic behavior of the last expression depends on the value r. More precisely,

$$\left|\int_{\mathbb{R}} \big(K(x)\big)^{c+1+\mathrm{i}v}\,dx\right| \sim \begin{cases} 2\dfrac{c(vr)^{r(c+1)+1/2}e^{-vr\pi/2}}{((c+1)^2+v^2)^{(r(c+1)+1)/2}e^{-vr\arctan(v/(c+1))}} \sim v^{-1/2}, \\ \qquad \text{if } r = 1, 2, \ldots, \\ v^{-1}, \qquad \text{if } r = 0 \end{cases}$$

as $v \to +\infty$. Finally, we conclude that $H_n \lesssim V_n^{c+1}$, if $r = 1, 2, \ldots$, and $H_n \lesssim V_n^{c+3/2}$ if $r = 0$.

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