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Minima and maxima of elliptical arrays and spherical processes

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In this paper, we investigate first the asymptotics of the minima of elliptical triangular arrays. Motivated by the findings of Kabluchko (*Extremes* **14** (2011) 285–310), we discuss further the asymptotic behaviour of the maxima of elliptical triangular arrays with marginal distribution functions in the Gumbel or Weibull max-domain of attraction. We present an application concerning the asymptotics of the maximum and the minimum of independent spherical processes.

Keywords: asymptotics of sample maxima; Brown–Resnick copula; Brown–Resnick process; Davis–Resnick tail property; Gaussian process; Penrose–Kabluchko process; spherical process

1. Introduction

It is well known that the maxima of Gaussian random vectors have asymptotically independent components, a result going back to Sibuya [27]. Recently, Kabluchko [22] shows that the minima of the absolute values of Gaussian random vectors have also asymptotically independent components. The Gaussian framework is appealing from both theoretical and applied point of view. In order to still consider Gaussian random vectors for modelling asymptotically dependent risks, triangular arrays of Gaussian random vectors with increasing dependence should be considered – this approach is suggested in Hüsler and Reiss [20]. As shown in the aforementioned paper, the maxima of Gaussian triangular arrays can be attracted by some max-stable distribution function with dependent components which is referred to as the Hüsler–Reiss distribution function. In fact, the Hüsler–Reiss copula is a particular case of the Brown–Resnick copula; a canonical example of a max-stable Brown–Resnick process is first presented in Brown and Resnick [4] in the context of the asymptotics of the maximum of Brownian motions. See Kabluchko *et al.* [23] for the main properties of Brown–Resnick processes. Kabluchko [22] discusses a more general asymptotic framework analysing the maximum of independent Gaussian processes showing that the Brown–Resnick process appears as the limit process if the underlying covariance functions satisfy a certain asymptotic condition. Additionally, the aforementioned paper investigates the asymptotics of the minimum of the absolute value of independent Gaussian processes extending some previous results of Penrose [26].

Indeed, Gaussian random vectors are a canonical example of elliptically symmetric (for short elliptical) random vectors. Therefore, it is natural to consider Kabluchko's findings in the framework of elliptical random vectors and spherical processes. Belonging to the class of conditional Gaussian processes, spherical processes appear naturally in diverse applications, see, for example, Falk *et al.* [10], or Hüsler *et al.* [18,19].

As shown in Hashorva [11,16] the maxima and the minima (of absolute values) of elliptical random vectors have asymptotically independent components. Elliptical random vectors are defined by the marginal distribution functions and some nonnegative definite matrix Σ, see (2.1) below. If $\Sigma_n, n \ge 1$ are $k \times k$ correlation matrices pertaining to an elliptical triangular array, the crucial condition for the asymptotic behaviour of both maxima and minima is

$$\lim_{n\to\infty} c_n\big(\mathbf{1}\mathbf{1}^\top - \Sigma_n\big) = \Gamma =: (\gamma_{ij})_{i,j\le k} \qquad \text{with } \gamma_{ij} \in (0,\infty), i \ne j, i, j \le k, \tag{1.1}$$

where $c_n, n \ge 1$ is a sequence of positive constants determined by a marginal distribution function of the elliptical random vectors, and $\mathbf{1} = (1,\dots,1)^\top \in \mathbb{R}^k$ (here $^\top$ stands for the transpose sign).

In Theorem 3.1, we specify the constants c_n such that the minima of absolute values of triangular arrays are attracted by some min-infinitely divisible distribution function in $\mathbb{R}^k$; the dependence function of the limiting distribution function is indirectly determined by the marginal distribution functions of the triangular array. Utilising Kabluchko's approach, we reconsider the aforementioned results for the maxima deriving some new representations for the limiting distributions under the assumptions that the marginals of the elliptical random vectors have distribution function in the Gumbel or Weibull max-domain of attraction (MDA).

A direct application of our result concerns the asymptotics of maximum and minimum (of absolute values) of independent spherical processes. It turns out that the limiting process of the normalised maximum of spherical processes is the same as that of Gaussian processes discussed in Kabluchko [22], namely the max-stable Brown–Resnick process. However, the norming constants are necessarily different. One important consequence of our findings is that the Brown–Resnick process is shown to be also the limit of the maximum of non-Gaussian processes. When instead of maximum the minimum of absolute values of Gaussian processes is considered, from the aforementioned reference, we know that the limiting process is min-stable; we refer to that process as Penrose–Kabluchko process. As demonstrated in our application, Penrose–Kabluchko processes can be retrieved in the limit in the more general framework of spherical processes.

The paper is organised as follows: Section 2 introduces our notation and presents some preliminary results. In Section 3, we deal with the asymptotics of minima of absolute values of elliptical triangular arrays. Section 4 investigates the maxima of triangular arrays with marginal distribution functions in the MDA of the Gumbel or the Weibull distribution. The applications mentioned above are presented in Section 5. Proofs of all the results are relegated to Section 6.

2. Preliminaries

Let in the following I, J be two non-empty disjoint index sets such that $I \cup J = \{1,\dots,k\}, k \ge 2$, and define for $\mathbf{x} = (x_1,\dots,x_k)^\top \in \mathbb{R}^k$ the subvector of $\mathbf{x}$ with respect to I by $\mathbf{x}_I = (x_i, i \in I)^\top$. If $\Sigma \in \mathbb{R}^{k\times k}$ is a square matrix, then the matrix Σ_{IJ} is obtained by retaining both the rows and the columns of Σ with indices in I and in J, respectively; similarly we define $\Sigma_{JI}, \Sigma_{JJ}, \Sigma_{II}$. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ write

$$\mathbf{x} > \mathbf{y} \qquad \text{if } x_i > y_i, \forall i = 1,\dots,k,$$

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1,\dots,x_k + y_k)^\top, \qquad c\mathbf{x} = (cx_1,\dots,cx_k)^\top, c \in \mathbb{R}.$$

The notation $\mathcal{B}_{a,b}, a, b > 0$ stands for a beta random variable with probability density function

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \qquad x \in (0,1),$$

where $\Gamma(\cdot)$ is the Euler Gamma function; $\mathbf{Y} \sim F$ means that the random vector $\mathbf{Y}$ has distribution function F.

Throughout this paper, $\mathbf{U}$ is a k-dimensional random vector uniformly distributed on the unit sphere (with respect to the L_2-norm) $\mathcal{S}_k$ of $\mathbb{R}^k$ being further independent of $R_k > 0$ and $A, A_n, n \geq 1$ are k-dimensional square matrices such that $\Sigma = AA^\top$ and $\Sigma_n = A_n A_n^\top$ are positive definite correlation matrices (all entries in the main diagonal are equal to 1). We write $\mathbf{U}_m$ if $m < k$ to mean again that $\mathbf{U}_m$ has the uniform distribution on $\mathcal{S}_m$. The distribution function of $R_k, k \geq 1$ will be denoted by H_k, whereas the distribution function of $R_k U_1$ will be denoted by G; $\omega \in (0, \infty]$ is their common upper endpoint.

Let $\mathbf{X} = (X_1, \dots, X_k)^\top, k \geq 2$ be an elliptically symmetric random vector with stochastic representation

$$\mathbf{X} \stackrel{d}{=} R_k A \mathbf{U}, \tag{2.1}$$

where $\stackrel{d}{=}$ stands for equality of the distribution functions. As shown in Cambanis *et al.* [5] $\mathbf{S} \stackrel{d}{=} R_k \mathbf{U}$ is a spherically symmetric random vector with tractable distributional properties. For instance $(S_1, \dots, S_m)^\top \stackrel{d}{=} R_m \mathbf{U}_m, m < k$ with positive random radius R_m such that

$$R_m^2 \stackrel{d}{=} R_k^2 \mathcal{B}_{m/2,(k-m)/2}, \tag{2.2}$$

with $\mathcal{B}_{m/2,(k-m)/2}$ independent of R_k. Equation (2.2) can be written iteratively as

$$R_m^2 \stackrel{d}{=} R_{m+1}^2 \mathcal{B}_{m/2,1/2}, \qquad m = 1, \dots, k-1, \tag{2.3}$$

where R_{m+1}^2 and $\mathcal{B}_{m/2,1/2}$ are independent. Note that if R_k^2 is chi-square distributed with k degrees of freedom (abbreviate this by $R_k^2 \sim \chi_k^2$), then (2.3) holds for any $m \in \mathbb{N}$ with $R_m^2 \sim \chi_m^2$.

Another interesting result of Cambanis *et al.* [5] is that $\boldsymbol{\mu}^\top \mathbf{S} \stackrel{d}{=} \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\mu}} S_1$ for any $\boldsymbol{\mu} \in \mathbb{R}^k$. Consequently, the assumption that Σ is a correlation matrix yields

$$X_i \stackrel{d}{=} X_1 \stackrel{d}{=} R_k U_1, \qquad 1 \leq i \leq k.$$

We call a positive random variable $Z \sim F$ regularly varying at 0 with index $\gamma \in [0, \infty]$ if

$$\lim_{s \downarrow 0} \frac{F(st)}{F(s)} = t^\gamma \qquad \forall t > 0, \tag{2.4}$$

which is abbreviated as $Z \in RV_\gamma$ or $F \in RV_\gamma$. Condition (2.4) is equivalent with $1/Z$ (or its survival function) being regularly varying at infinity with index $-\gamma$. When $\gamma = -\infty$, then the survival function of $1/Z$ is called rapidly varying at infinity. See Jessen and Mikosch [21] or Omey and Segers [25] for details on regular variation.

Central for our results is an interesting fact discovered by Kabluchko [22] pointing out the importance of the incremental variance matrix (function) for the properties of the Brown–Resnick process. Given a k-dimensional Gaussian random vector $\mathbf{X}$ this $k \times k$ matrix is denoted by $\Gamma = (\gamma_{ij})_{i,j\le k}$, where $\gamma_{ij} = \mathbf{Var}\{X_i - X_j\}$. The covariance matrix Σ of $\mathbf{X}$ is related to Γ by

$$\Sigma = AA^\top = \big(\boldsymbol{\theta}\mathbf{1}^\top + \mathbf{1}\boldsymbol{\theta}^\top - \Gamma\big)/2, \qquad \boldsymbol{\theta} = \big(\mathbf{Var}\{X_1\}, \dots, \mathbf{Var}\{X_k\}\big)^\top. \tag{2.5}$$

If $\{Z(t), t \in T\}$ is a mean-zero Gaussian process with variance function $\sigma^2(\cdot)$, we define similarly to the discrete case the incremental variance function Γ by

$$\Gamma(t_1, t_2) = \mathbf{Var}\big\{Z(t_2) - Z(t_1)\big\}, \qquad t_1, t_2 \in T.$$

By Theorem 4.1 of Kabluchko [22], the stochastic process

$$\eta_\Gamma(t) = \min_{i\ge 1}\big|\Upsilon_i + Z_i(t)\big|, \qquad t \in \mathbb{R} \tag{2.6}$$

is the limit of the minima of absolute values of independent Gaussian processes, if additionally $\Xi_L = \sum_{i=1}^\infty \varepsilon_{\Upsilon_i}$ is a Poisson point process on $\mathbb{R}$ with points $\Upsilon_1, \Upsilon_2, \dots$ and intensity measure given by the Lebesgue measure being further independent of the Gaussian processes $\{Z_i(t), t \in \mathbb{R}\}, i \ge 1$. Here ε_x denotes the Dirac measure at x; $\varepsilon_x(B) = 1$ if $x \in B \subset \mathbb{R}$, and $\varepsilon_x(B) = 0$ when $x \notin B$.

In the sequel, for given $\boldsymbol{\theta} \in (0,\infty)^k, k \ge 2$ and A, Σ, Γ satisfying (2.5) we write $\mathbf{X} \approx \mathfrak{E}[\boldsymbol{\theta}, \Gamma; H_k]$ if $\mathbf{X} \stackrel{d}{=} R_k A\mathbf{U}$, $R_k \sim H_k$. We write simply $\mathbf{X} \approx \mathfrak{E}[\Gamma; H_k]$ if the specification of $\boldsymbol{\theta}$ is not necessary for the stated result, meaning that the result holds for any $\boldsymbol{\theta} \in (0,\infty)^k$. Further, if $R_k^2 \sim \chi_k^2$ we write $\mathbf{X} \approx \mathit{Gauss}[\Gamma]$, with $\mathbf{X}$ a mean-zero Gaussian random vector with incremental variance matrix Γ.

3. Minima of elliptical triangular arrays

Let $\mathbf{X}_n^{(i)} \stackrel{d}{=} R_k A_n \mathbf{U}$, $1 \le i \le n, n \ge 1$ be k-dimensional independent elliptical random vectors, where the square matrix A_n is such that $\Sigma_n = A_n A_n^\top, n \ge 1$ is a correlation matrix. Next, we discuss the asymptotic behaviour of $\mathbf{L}_n = (L_{n1}, \dots, L_{nk})^\top, n \ge 1$ defined by

$$L_{nj} = \min_{1\le i\le n} \big|X_{nj}^{(i)}\big|, \qquad j = 1, \dots, k, n \ge 1.$$

We have

$$X_{nj}^{(i)} \stackrel{d}{=} X_{11}^{(1)} =: X_{11}, \qquad L_{nj} \stackrel{d}{=} L_{n1}, \qquad j = 1, \dots, k, 2 \le i \le n$$

and $|X_{11}|^2 \stackrel{d}{=} R_k^2 \mathcal{B}_{1/2,(k-1)/2}$.

Next, we assume that $R_k \in RV_\gamma$ with index $\gamma \in (0, 1]$, which in view of Lemma 6.1 implies $|X_{11}| \in RV_\gamma$; note that the converse holds if $\gamma \in (0, 1)$. Define a sequence of constants $a_n, n \geq 1$ by

$$\mathbf{P}\big\{a_n^{-1} \geq X_{11} > 0\big\} = 1/n. \tag{3.1}$$

For such constants, we have the convergence in distribution ($n \to \infty$)

$$a_n L_{nj} \xrightarrow{d} \mathcal{L}_j \sim \mathcal{G}_\gamma, \qquad j = 1, \ldots, k,$$

with distribution function $\mathcal{G}_\gamma$ given by

$$\mathcal{G}_\gamma(x) = 1 - \exp\big(-2x^\gamma\big), \qquad x > 0. \tag{3.2}$$

In view of Hashorva [16], if Σ_n has all off-diagonal elements bounded by some constant $c \in (0, 1)$, then

$$a_n \mathbf{L}_n \xrightarrow{d} \boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \ldots, \mathcal{L}_k)^\top, \qquad n \to \infty \tag{3.3}$$

holds with $\mathcal{L}_1, \ldots, \mathcal{L}_k$ being mutually independent. By allowing the off-diagonal elements of Σ_n to converge to 1 as $n \to \infty$ with a certain speed, it is possible that the random vector $\boldsymbol{\mathcal{L}}$ has dependent components. If $H_i, i \leq k$ is the distribution function of R_i in (2.3) it turns out that $\mathcal{R}_m, m \leq k-1$ with distribution function

$$\mathcal{H}_m(z) = \int_0^z \frac{1}{r\mathbf{E}\{1/R_{m+1}\}} \, \mathrm{d}H_{m+1}(r), \qquad z > 0 \tag{3.4}$$

determine the distribution function of $\boldsymbol{\mathcal{L}}$ (assuming $\mathbf{E}\{1/R_k\} < \infty$). For the derivation of this result, we shall define an elliptical random vector $\mathbf{Z}^{K;j} \stackrel{d}{=} \mathcal{R}_{m-1}\Gamma_{m,K}\mathbf{U}_m$ with

$$\Gamma_{m,K}(\Gamma_{m,K})^\top = \big(\mathbf{1}\Gamma_{K_j,J}^\top + \Gamma_{K_j,J}\mathbf{1}^\top - \Gamma_{K_j,K_j}\big)/2,$$
$$\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{m-1}, K_j = K \setminus J, J = \{j\},$$

where $K \subset \{1, \ldots, k\}$ has $m \geq 2$ elements, and Γ is the matrix in (1.1).

Theorem 3.1. *Let $\mathbf{X}_n^{(i)}, 1 \leq i \leq n, n \geq 1$ be a triangular array of k-dimensional elliptical random vectors with correlation matrices $\Sigma_n, n \geq 1$ as above, and $R_k \sim H_k$. Suppose that $|X_{11}^{(1)}| \in RV_\gamma, \gamma \in (0, 1]$ and $\mathbf{E}\{1/R_k\} < \infty$.*

If condition (1.1) is satisfied for $c_n = 2a_n^2$ with a_n determined by (3.1), then (3.3) holds and for all $\mathbf{x} \in (0, \infty)^k$

$$\mathbf{P}\{\boldsymbol{\mathcal{L}} > \mathbf{x}\} = \exp\Bigg(\sum_{m=1}^{k}(-1)^m \sum_{|K|=m} \int_{-x_j^\gamma}^{x_j^\gamma} \mathbf{P}\big\{\big|\mathrm{sign}(y)|y|^{1/\gamma} + Z_i^{K;j}\big| \leq x_i, i \in K \setminus \{j\}, j \in K\big\}\,\mathrm{d}y\Bigg), \tag{3.5}$$

where the summation above runs over all non-empty index sets K with $|K| = m$ elements and j is some index in K. Set the integral in (3.5) equal to $2x_j^{\gamma}$ if $K = \{j\}$.

Remarks.

(a) The result of Theorem 3.1 can be extended for Γ with off-diagonal elements equal to 0. For instance when $\Gamma = \mathbf{0}\mathbf{0}^{\top}$ with $\mathbf{0} = (0, \dots, 0)^{\top}$, then it follows that

$$\mathbf{P}\{\boldsymbol{\mathcal{L}} > \mathbf{x}\} = 1 - \mathcal{G}_{\gamma}\Big(\min_{1\le i\le k} x_i\Big), \qquad \mathbf{x} \in (0,\infty)^k.$$

(b) In view of (3.5) the random vector $(\mathcal{L}_d, \mathcal{L}_l), d \neq l$ has joint distribution function depending on the element γ_{dl} of Γ.

Example 1. Let $\mathbf{X}_n^{(i)}, 1 \le i \le n, n \ge 1$ be a triangular array of k-dimensional mean-zero Gaussian random vectors with covariance matrix $\Sigma_n, n \ge 1$. Since $\mathcal{R}_m^2 \sim \chi_m^2, m \le k$, then a_n defined by (3.1) satisfies

$$a_n = \big(1 + \mathrm{o}(1)\big)\frac{n}{\sqrt{2\pi}}, \qquad n \to \infty.$$

Hence, when (1.1) is valid with $c_n = 2a_n^2$, then (3.5) holds with $\mathbf{Z}^{K;j}$ a mean-zero Gaussian random vector with covariance matrix $\Gamma_{m,K}(\Gamma_{m,K})^{\top}$.

Next, we extend Theorem 3.1 imposing a smoothness assumption on R_k, namely that (2.3) holds also for $m = k$.

Theorem 3.2. *Under the assumptions and notation of Theorem 3.1, if further (2.3) holds for $m = k$ with $R_{k+1} \sim H_{k+1}$, then*

$$\mathbf{P}\{\boldsymbol{\mathcal{L}} > \mathbf{x}\} = \exp\bigg(-\int_{\mathbb{R}} \mathbf{P}\big\{\exists i \le k\colon \big|\mathrm{sign}(y)|y|^{1/\gamma} + Z_i\big| \le x_i\big\}\,\mathrm{d}y\bigg), \qquad \mathbf{x} \in (0,\infty)^k, \tag{3.6}$$

with $\mathbf{Z} \approx \mathfrak{E}[\Gamma; \mathcal{H}_k]$ and $\mathcal{H}_k$ defined by (3.4).

Remark. The assumption (2.3) is satisfied for $m = k$, provided that $\mathbf{X}_n^{(i)}, i \le n$ is a subvector of an elliptical random vector, see Cambanis *et al.* [5]. In particular, it holds if $R_k \stackrel{d}{=} S\tilde{R}_k$ with S a positive random variable independent of $\tilde{R}_k^2 \sim \chi_k^2$.

Example 2. Let $\mathbf{X}_n^{(i)}, 1 \le i \le n, n \ge 1$ be as in Example 1. Next, define

$$\mathbf{Y}_n^{(i)} = S_n \mathbf{X}_n^{(i)}, \qquad 1 \le i \le n, n \ge 1,$$

with $S, S_n, n \ge 1$ independent positive random variables with distribution function F being further independent of $\mathbf{X}_n^{(i)}, 1 \le i \le n$. If $F \in RV_{\gamma}, \gamma \in (0,1]$, then by Lemma 6.1 $|Y_{n1}^{(1)}| \in RV_{\gamma}$. Define constants $a_n, n \ge 1$ such that $\mathbf{P}\{0 < SX_{11}(1) \le 1/a_n\} = 1/n$ holds for all large n. If further (1.1) is satisfied with $c_n = 2a_n^2$, then (3.6) holds. Note in passing that $\mathcal{H}_k$ satisfies (3.4) with $R_{k+1}^2 \sim \chi_{k+1}^2, R_{k+1} > 0$.

4. Maxima of elliptical triangular arrays

With the same notation as above we consider again the triangular array $\mathbf{X}_n^{(i)}, 1 \le i \le n, n \ge 1$ of k-dimensional independent elliptical random vectors with stochastic representation (2.1) and $\Sigma_n = A_n A_n^\top, n \ge 1$ given correlation matrices. Define the componentwise maxima $\mathbf{M}_n = (M_{n1}, \dots, M_{nk})^\top$ by

$$M_{nj} = \max_{1 \le i \le n} X_{nj}^{(i)}, \qquad j = 1, \dots, k, n \ge 1.$$

The asymptotic behaviour of the maxima of elliptical triangular arrays is discussed in Hashorva [12] assuming that the random radius R_k has distribution function H_k in the Gumbel MDA. A canonical example of such triangular arrays is that of the Gaussian arrays for which the limit distribution of the maxima is the Hüsler–Reiss copula which is a particular case of the Brown–Resnick copula. When H_k is in the Weibull MDA the limit distribution of the maxima is a max-infinitely divisible distribution function, see Hashorva [11].

We reconsider the findings of the aforementioned papers showing novel representations of the limit distributions given in terms of the distribution of the maxima of some point processes shifted by elliptical random vectors. For the derivation of the next results, we impose asymptotic assumptions on either the marginal distribution functions or on the associated random radius R_k, which is of some interest for statistical applications where some data might be missing, or some component of the random vector might be unobservable, and therefore the random radius itself cannot be estimated.

4.1. Gumbel max-domain of attraction

The main assumption in this section is that the marginal distribution functions of the elliptical triangular array are in the Gumbel MDA. A univariate distribution function G is in the Gumbel MDA (abbreviated $G \in \mathrm{GMDA}(w)$) if for any $x \in \mathbb{R}$

$$\lim_{t \uparrow \omega} \frac{1 - G(t + x/w(t))}{1 - G(t)} = \exp(-x), \qquad \omega = \sup\{t\colon\ G(t) < 1\}, \tag{4.1}$$

with $w(\cdot)$ some positive scaling function. If $\omega = \infty$, an important property for the distribution function G satisfying (4.1) is a key finding of Davis and Resnick [6], namely by Proposition 1.1 therein (see also Embrechts *et al.* [9], page 586) for any $\mu \in \mathbb{R}, \tau > 1$ we have

$$\lim_{x \to \infty} \big(x w(x)\big)^\mu \frac{1 - G(\tau x)}{1 - G(x)} = 0. \tag{4.2}$$

Indeed (4.2), which we refer to as the *Davis–Resnick tail property* is crucial for several asymptotic approximations.

Theorem 4.1. *Let* $R \sim H_k, \mathbf{X}_n^{(i)}, 1 \le i \le n, \Sigma_n, n \ge 1$ *be as in Theorem* 3.1. *If either* $G \in \mathrm{GMDA}(w)$ *or* $H_k \in \mathrm{GMDA}(w)$ *and condition* (1.1) *is satisfied with*

$$c_n = 2\frac{b_n}{a_n}, \qquad b_n = G^{-1}(1-1/n), a_n = 1/w(b_n), n > 1, \tag{4.3}$$

then for any $\mathbf{x} \in \mathbb{R}^k$ *and* $\mathbf{Z} \approx$ *Gauss*$[\Gamma]$ *we have*

$$\begin{aligned}&\lim_{n\to\infty} \mathbf{P}\big\{(\mathbf{M}_n - b_n\mathbf{1})/a_n \le \mathbf{x}\big\}\\ &\quad = Q_\Gamma(\mathbf{x}) = \exp\left(-\int_{\mathbb{R}} \mathbf{P}\{\exists i \le k:\ Z_i > x_i - y + \theta_i/2\} \exp(-y)\,\mathrm{d}y\right),\end{aligned} \tag{4.4}$$

where $\theta_i = \mathbf{Var}\{Z_i\}, i \le k$.

Since the above result holds for Gaussian triangular arrays with scaling function $w(x) = x$, the distribution function Q_Γ is the multivariate max-stable Hüsler–Reiss distribution function. For a particular choice of a Gaussian process $\{Z(t), t \in \mathbb{R}\}$ this distribution has the Brown–Resncik copula; in fact it can be directly defined by Brown–Resnick processes $\beta_{\mathcal{R};\Gamma}$ with independent Gaussian points $\xi_i(t) := Z_i(t) - \sigma^2(t)/2, i \ge 1$ given as

$$\beta_{\mathcal{R};\Gamma}(t) = \max_{i \ge 1}\big[\Upsilon_i + \xi_i(t)\big], \qquad t \in \mathbb{R}. \tag{4.5}$$

Here $\Xi = \sum_{i=1}^{\infty} \varepsilon_{\Upsilon_i}$ is a Poisson point process with intensity measure $\exp(-x)\,\mathrm{d}x$ being independent of $\{Z_i(t), t \in \mathbb{R}\}, i \ge 1$. In view of our result, the Brown–Resnick process with Gaussian points does not depend on the variance function, which is already established in Theorem 2.1 of Kabluchko *et al.* [23].

4.2. Weibull max-domain of attraction

The unit Weibull distribution with index $\alpha \in (0,\infty)$ is $\Psi_\alpha(x) = \exp(-|x|^\alpha), x < 0$. In view of Hashorva and Pakes [17] the distribution function G is in the Weibull MDA if H_k is in the Weibull MDA. We assume for simplicity that H_k has upper endpoint equal to 1. By definition, H_k is in the MDA of Ψ_α (for short $H_k \in \mathrm{WMDA}(\alpha)$) if for any $x \in (0,\infty)$

$$\lim_{n\to\infty} H_k^n\big(1 - a(n)x\big) = \Psi_\alpha(x), \qquad a_n = 1 - H_k^{-1}(1-1/n). \tag{4.6}$$

If $H_k \in \mathrm{WMDA}(\alpha)$, with some index $\alpha \in (0,\infty)$ and H_k has upper endpoint equal to 1, then by Theorem 2.1 in Hashorva [13]

$$\lim_{n\to\infty} \mathbf{P}\big\{(\mathbf{M}_n - \mathbf{1})/a_n \le \mathbf{x}\big\} = \widetilde{Q_{\Gamma,\alpha}}(\mathbf{x}) \qquad \forall \mathbf{x} \in (-\infty, 0)^k, \tag{4.7}$$

with $\widetilde{Q_{\Gamma,\alpha}}$ a max-infinitely divisible distribution function, provided that (1.1) holds with $c_n = 2/a_n, a_n = 1 - G^{-1}(1-1/n), n > 1$.

In the next theorem, we show that (4.7) holds if either G or H_k is in the Weibull MDA. Furthermore, we give a new representation for the limit distribution function $\widetilde{\mathcal{Q}_{\Gamma,\alpha}}$.

Theorem 4.2. *Let* $R \sim H_k, \mathbf{X}_n^{(i)}, 1 \le i \le n, \Sigma_n, n \ge 1$ *be as in Theorem* 3.1, *and assume that* G *has upper endpoint* 1. *If either* $G \in \mathrm{WMDA}(\alpha + (k-1)/2)$, *or* $H_k \in \mathrm{WMDA}(\alpha)$, *with* $\alpha \in (0, \infty)$, *then* (4.7) *holds where*

$$\widetilde{\mathcal{Q}_{\Gamma,\alpha}}(\mathbf{x}) = \exp\biggl(-\int_0^\infty \mathbf{P}\{\exists i \le k\colon \sqrt{2y} Z_i > x_i + y + \theta_i/2\} \, \mathrm{d}y^{\alpha+(k-1)/2}\biggr), \tag{4.8}$$

with $\mathbf{Z} \approx \mathfrak{E}[\Gamma; H_k], \boldsymbol{\theta} \in (0,\infty)^k$ *and* $\widetilde{\mathcal{H}}_\alpha$ *the distribution function of* $\widetilde{\mathcal{R}}_\alpha > 0$ *which satisfies* $\widetilde{\mathcal{R}_\alpha}^2 \stackrel{d}{=} \mathcal{B}_{k/2,\alpha}$.

We remark that $\widetilde{\mathcal{Q}_{\Gamma,\alpha}}$ has Weibull marginal distributions $\Psi_{\alpha+(k-1)/2}$. It follows from our result that $\widetilde{\mathcal{Q}_{\Gamma,\alpha}}$ is determined by Γ and α but not by the vector $\boldsymbol{\theta}$, and further $\widetilde{\mathcal{Q}_{\Gamma,\alpha}}$ is not a max-stable distribution function; clearly, it is a max-infinitely divisible distribution function.

5. Results for spherical processes

It is well-known that spherical random sequences are mixtures of Gaussian random sequences. Specifically, if the random variables $X_i, i \ge 1$ with some common non-degenerate distribution function G are such that $(X_1, \dots, X_k)$ is centered and spherically distributed for any $k \ge 1$, then $X_i \stackrel{d}{=} SX_i^*, i \ge 1$ with $X_i^*, i \ge 1$ is a sequence of independent standard Gaussian random variables being further independent of $S > 0$. Consequently, a spherical random process $\{X(t), t \in \mathbb{R}\}$ such that $X(t)$ has distribution function G for any $t \in \mathbb{R}$ can be expressed as $\{X(t) = SY(t), t \in \mathbb{R}\}$ with $Y(t)$ a mean-zero Gaussian process and S a positive random variable independent of $\{Y(t), t \in \mathbb{R}\}$; see Theorem 7.4.4 in Bogachev [2] for a general result on spherically symmetric measures. We note in passing that $\{X(t), t \in T\}$ is a particular instance of Gaussian processes with random variance, see Hüsler *et al.* [19] for recent results on extremes of those processes.

We shall discuss first the asymptotic behaviour of the maximum of independent spherical processes. Then we shall briefly investigate the asymptotics of the minima of absolute values of those processes.

Model A: Assume that S has an infinite upper endpoint such that for given constants $\alpha_1 \in \mathbb{R}$ and $C_1, L_1, p_1 \in (0, \infty)$

$$\mathbf{P}\{S > x\} = \bigl(1 + \mathrm{o}(1)\bigr) C_1 x^{\alpha_1} \exp\bigl(-L_1 x^{p_1}\bigr), \qquad x \to \infty \tag{5.1}$$

is valid. We abbreviate (5.1) as $S \in \mathcal{W}(C_1, \alpha_1, L_1, p_1)$.

Model B: Consider S with upper endpoint equal to 1 such that

$$\lim_{u\to\infty} \frac{\mathbf{P}\{S > 1 - x/u\}}{\mathbf{P}\{S > 1 - 1/u\}} = x^\gamma, \qquad x \in (0, \infty), \tag{5.2}$$

with $\gamma \in [0, \infty)$ some constant.

Since for $S = 1$ almost surely, the spherical process is simply a Gaussian one (which is covered by Model B for $\gamma = 0$) intuitively, we expect that under the Model B the maximum of independent elliptical processes will behave asymptotically as the maximum of independent Gaussian processes. This intuition is confirmed by Theorem 5.1 below. In fact, it turns out that the limit process of the maximum of independent spherical processes is in both models the Brown–Resnick process. Next, if $\Gamma(\cdot,\cdot)$ is a negative definite kernel in $\mathbb{R}^2$ we define as previously the Brown–Resnick stochastic process with Gaussian points as

$$\beta_{\mathcal{R};\Gamma}(t) = \max_{i\geq 1}\bigl(\Upsilon_i + Z_i(t) - \sigma^2(t)/2\bigr), \qquad t \in T \subset \mathbb{R}, \tag{5.3}$$

with $\{Z_i(t), t \in T\}$ independent Gaussian processes with incremental variance function Γ, variance function $\sigma^2(\cdot)$ being further independent of the point process Ξ with points $\Upsilon_i, i \geq 1$ appearing in (4.5). For simplicity, we deal below with the case $T = \mathbb{R}$ establishing weak convergence of finite-dimensional distributions (denoted below as $\Longrightarrow$).

Theorem 5.1. *Let $\{Y_{ni}(t), t \in \mathbb{R}\}, 1 \leq i \leq n, n \geq 1$ be independent Gaussian processes with mean-zero, unit variance function and correlation function $\rho_n(s,t), s,t \in \mathbb{R}$. Let $S, S_{ni}, n \geq 1$ be independent and identically distributed positive random variables. Set $\{X_{ni}(t) = S_{ni}Y_{ni}(t), t \in \mathbb{R}\}, n \geq 1$, and let G be the distribution function of $X_{11}(1)$. Suppose that*

$$\lim_{u\to\infty} c_n\bigl(1 - \rho_n(t_1,t_2)\bigr) = \Gamma(t_1,t_2) \in (0,\infty), \qquad t_1 \neq t_2 \in \mathbb{R}, \tag{5.4}$$

where $c_n = 2b_n/a_n$ and $a_n = 1/w(b_n), b_n = G^{-1}(1 - 1/n)$ with G^{-1} the inverse of G.

(A) *If (5.1) holds, then as $n \to \infty$*

$$\frac{1}{a_n}\Bigl[\max_{1\leq i\leq n} X_{ni}(t) - b_n\Bigr] \Longrightarrow \beta_{\mathcal{R};\Gamma}(t), \qquad t \in \mathbb{R}, \tag{5.5}$$

where $\Longrightarrow$ means the weak convergence of the finite-dimensional distributions, and

$$\frac{b_n}{a_n} = \bigl(1 + o(1)\bigr)\frac{2p_1 \ln n}{2 + p_1}, \qquad b_n = \bigl(1 + o(1)\bigr)\left(\frac{\ln n}{L_1 A^{-p_1} + A^2/2}\right)^{(2+p_1)/(2p_1)},$$
$$A = (p_1 L_1)^{1/(2+p_1)}.$$

(B) *If (5.2) holds with $\gamma \in [0,\infty)$, then (5.5) is satisfied and $\lim_{n\to\infty} b_n/\sqrt{2\ln n} = \lim_{n\to\infty} a_n\sqrt{2\ln n} = 1$.*

Next, we discuss the asymptotic behaviour of the minimum of absolute values in the framework of independent spherical processes.

Theorem 5.2. *Let $\{Y_{ni}(t), Z_i(t), t \in \mathbb{R}\}, 1 \leq i \leq n, n \geq 1$ be as in Theorem 5.1, and let $\{S_{ni}(t), t \in \mathbb{R}\}, n \geq 1$ be independent copies of $\{S(t), t \in \mathbb{R}\}$, being further independent of the*

Gaussian processes. Define the spherical processes $\{X_{ni}(t) = S_{ni}(t)Y_{ni}(t), t \in \mathbb{R}\}, n \geq 1$, *and suppose that* $S(t) > \kappa, t \in \mathbb{R}$ *almost surely for some positive constant* κ. *If* $a_n = n/\sqrt{2\pi}$ *and* (5.4) *holds with* $c_n = 2a_n^2$, *then as* $n \to \infty$

$$\min_{1\leq i\leq n} a_n\big|X_{ni}(t)\big| \Longrightarrow \min_{i\geq 1} S_i(t)\big|\Upsilon_i + Z_i(t)\big| = \zeta_{\Gamma,S}(t), \qquad t \in \mathbb{R}, \tag{5.6}$$

where $\Upsilon_i, i \geq 1$ *are the points of* Ξ *defined in* (4.5) *being independent of both* $Z_i(t), S_i(t), t \in \mathbb{R}, i \geq 1$.

Remarks.

(a) In Theorem 5.2, we can relax the assumption that $S(t)$ is bounded from below by assuming instead $\mathbf{E}\{[S(t)]^{-1-\varepsilon}\} < \infty$ for some $\varepsilon > 0$.
(b) The process $\{\zeta_{\Gamma,S}(t), t \in \mathbb{R}\}$ is defined by Γ and $\{S(t), t \in \mathbb{R}\}$ but does not depend on the variance function $\sigma^2(\cdot)$. The processes $\zeta_{\Gamma,1}$ appears first in Penrose [26] and recently in Kabluchko [22]. We refer to $\{\eta_{\Gamma,S}(t), t \in \mathbb{R}\}$ as Penrose–Kabluchko process.

6. Further results and proofs

Lemma 6.1. *Let* $\mathbf{X} \stackrel{d}{=} RA\mathbf{U}$ *be an elliptical random vector in* $\mathbb{R}^k, k \geq 2$ *with* A *such that* $AA^\top$ *is a positive definite correlation matrix and* $R > 0$.

(a) *If for some* $\gamma \in [0, \infty]$ *we have* $R \in RV_\gamma$, *then* $|X_1| \in RV_{\gamma^*}$ *with* $\gamma^* = \min(\gamma, 1)$. *Conversely, if* $|X_1| \in RV_{\gamma^*}$ *with* $\gamma^* \in (0, 1)$, *then* $R \in RV_{\gamma^*}$.
(b) *If* $\mathbf{E}\{R^{-1-\varepsilon}\} < \infty$ *for some* $\varepsilon > 0$, *then* $|X_1| \in RV_1$.

Proof. (a) If $\gamma \in [0, \infty)$ the proof follows from Theorem 4.1 in Hashorva [16]. When $\gamma = \infty$, then $1/R$ is rapidly varying at infinity. Hence from Theorem 5.4.1 of de Haan and Ferreira [8] $\mathbf{E}\{R^{-p}\} < \infty$ for any $p \in (0, \infty)$, and thus the claim follows once the statement (b) is proved. Statement (b) can be directly established by applying Breiman's lemma (see Breiman [3], Davis and Mikosch [7]), and thus the proof is complete. □

Proof of Theorem 3.1. By the relation between the minima and maxima, in view of Lemma 4.1.3 in Falk *et al.* [10] the proof follows if

$$\lim_{n\to\infty} n\mathbf{P}\big\{a_n|X_{ni}| \leq x_i, i \in K\big\} = L_K(\mathbf{x}_K), \qquad \mathbf{x} \in (0, \infty)^k \tag{6.1}$$

holds for any non-empty index set $K \subset \{1, \ldots, k\}$ with $m \geq 2$ elements, and $L_K(\cdot)$ some right-continuous functions. In the sequel, we write simply $\mathbf{X}_n$ instead of $\mathbf{X}_n^{(1)}$; the subvector $(\mathbf{X}_n)_K$ is an elliptical random vector with associated random radius $R_m \sim H_m$ satisfying (2.3). By Lemma 6.1, $H_k \in RV_\gamma, \gamma \in (0, 1]$ implies $H_m \in RV_\gamma, 1 \leq m \leq k-1$. Consequently, it suffices to show (6.1) for the case $m = k$. Since the distribution function of $\mathbf{X}_n$ depends on Σ_n and not on A_n, and further Σ_n is positive definite, we can assume that A_n is a lower triangular matrix. Define $q_n(y) = y/a_n, y \in \mathbb{R}$ and recall that G denotes the distribution function of X_{11}. It follows that conditioning on $X_{nk} = q_n(y)$ with $y \neq 0$ such that $G(|y|/a_n) \in (0,1), n \geq 1$ we have the stochastic representation (set $I = \{1, \ldots, k-1\}$, $J = \{k\}$)

$$(\mathbf{X}_n)_I | X_{nk} = q_n(y) \stackrel{d}{=} R_{y,n,k-1} B_{nk} \mathbf{U}_{k-1} + (\Sigma_n)_{IJ} q_n(y), \qquad n \geq 1, \tag{6.2}$$

where B_{nk} is a lower triangular matrix satisfying $B_{nk} B_{nk}^\top = (\Sigma_n)_{II} - (\Sigma_n)_{IJ}(\Sigma_n)_{JI}$. In view of Cambanis *et al.* [5], $\mathbf{U}_{k-1}$ is independent of $R_{y,n,k-1}, n \geq 1$ which has survival function $\overline{Q}_{y,n,k-1}$ given by

$$\overline{Q}_{y,n,k-1}(z) = \frac{\int_{((y/a_n)^2+z^2)^{1/2}}^{\omega} (r^2 - (y/a_n)^2)^{(k-1)/2-1} r^{-k+2} \, \mathrm{d}H_k(r)}{\int_{y/a_n}^{\omega} (r^2 - (y/a_n)^2)^{(k-1)/2-1} r^{-k+2} \, \mathrm{d}H_k(r)}, \quad z \in \bigl(0, \sqrt{\omega^2 - y^2/a_n^2}\bigr). \tag{6.3}$$

Clearly, $\lim_{n\to\infty} a_n = \infty$ and the monotone convergence theorem implies the convergence in distribution

$$R_{y,n,k-1} \stackrel{d}{\to} \mathcal{R}_{k-1}, \qquad n \to \infty,$$

where $\mathcal{R}_{k-1} \sim \mathcal{H}_{k-1}$ with

$$\mathcal{H}_{k-1}(z) = 1 - \frac{\int_z^\omega r^{-1} \, \mathrm{d}H_k(r)}{\mathbf{E}\{1/R_k\}}, \qquad z \in (0, \omega). \tag{6.4}$$

In view of relation (2.2) and since for any integer $m \geq 2$ we have $\mathbf{E}\{1/\mathcal{B}_{m/2,(k-m)/2}\} < \infty$ the assumption $\mathbf{E}\{1/R_k\} < \infty$ implies $\mathbf{E}\{1/R_m\} < \infty$. Hence, the above convergence holds also for the omitted case $k = m$. Next, by (1.1) and the fact that $B_{nk} B_{nk}^\top$ (and not the matrix B_{nk}) defines the conditional distribution in (6.2) we can choose B_{nk} such that $\lim_{n\to\infty} a_n B_{nk} = B_k$ with

$$B_k B_k^\top = \bigl(\mathbf{1}\boldsymbol{\theta}^\top + \boldsymbol{\theta}\mathbf{1}^\top - \Gamma_{II}\bigr)/2, \qquad \boldsymbol{\theta} = \Gamma_{IJ}.$$

Hence, for any $\mathbf{x} \in (0,\infty)^k$ utilising further (6.2) and the fact that G is symmetric about 0 we obtain (set $G_n(y) = G(y/a_n), n \geq 1$ and $K = \{1, \ldots, k\}$)

$$\begin{aligned}
&\mathbf{P}\bigl\{a_n|X_{ni}| \leq x_i, \forall i = 1, \ldots, k\bigr\} \\
&\quad = \int_{\mathbb{R}} \mathbf{P}\bigl\{a_n|X_{ni}| \leq x_i, \forall i \in I | X_{nk} = y\bigr\} \, \mathrm{d}G(y) \\
&\quad = \int_{-x_k}^{x_k} \mathbf{P}\bigl\{a_n|X_{ni}| \leq x_i, \forall i \in I | X_{nk} = y/a_n\bigr\} \, \mathrm{d}G_n(y) \\
&\quad = \int_0^{x_k} \bigl[\mathbf{P}\bigl\{a_n|X_{ni}| \leq x_i, \forall i \in I | X_{nk} = y/a_n\bigr\} \\
&\qquad\qquad + \mathbf{P}\bigl\{a_n|X_{ni}| \leq x_i, \forall i \in I | X_{nk} = -y/a_n\bigr\}\bigr] \, \mathrm{d}G_n(y) \\
&\quad = \int_0^{x_k} \bigl[\mathbf{P}\bigl\{a_n|Z_{ni} + d_{ni} y/a_n| \leq x_i, i \in I\bigr\} + \mathbf{P}\bigl\{a_n|Z_{ni} - d_{ni} y/a_n| \leq x_i, \forall i \in I\bigr\}\bigr] \, \mathrm{d}G_n(y),
\end{aligned}$$

with $\mathbf{Z}_n = R_{y,n,k-1} B_{nk} \mathbf{U}_{k-1}$ and d_{ni} the ith component of $(\Sigma_n)_{IJ}$. By the construction we have the convergence in distribution ($n \to \infty$)

$$R_{y,n,k-1}(a_n B_{nk})\mathbf{U}_{k-1} \xrightarrow{d} \mathcal{R}_{k-1} B_k \mathbf{U}_{k-1} =: (Z_1, \dots, Z_{k-1})^\top.$$

Further, by the regular variation at 0 of the distribution function of $|X_{11}|$, the fact that X_{11} is symmetric about 0, and the choice of $a_n, n \geq 1$ we have

$$\lim_{n\to\infty} n\big[G_n(t) - G_n(s)\big] = t^\gamma - s^\gamma \qquad \forall s, t \in (0, \infty). \tag{6.5}$$

Consequently, since $\lim_{n\to\infty} d_{ni} = 1$

$$\begin{aligned}
&\lim_{n\to\infty} n\mathbf{P}\big\{a_n|X_{ni}| \leq x_i, \forall i = 1, \dots, k\big\} \\
&\quad = \int_0^{x_k} \big[\mathbf{P}\big\{|Z_i + y| \leq x_i, i \in I\big\}\,\mathrm{d}y^\gamma + \mathbf{P}\big\{|Z_i - y| \leq x_i, i \in I\big\}\big]\,\mathrm{d}y^\gamma \\
&\quad = \int_0^{x_k^\gamma} \big[\mathbf{P}\big\{\big|Z_i + y^{1/\gamma}\big| \leq x_i, i \in I\big\}\,\mathrm{d}y + \mathbf{P}\big\{\big|Z_i - y^{1/\gamma}\big| \leq x_i, i \in I\big\}\big]\,\mathrm{d}y \\
&\quad = \int_{-x_k^\gamma}^{x_k^\gamma} \mathbf{P}\big\{\big|Z_i + \mathrm{sign}(y)|y|^{1/\gamma}\big| \leq x_i, i \in I\big\}\,\mathrm{d}y,
\end{aligned}$$

hence the proof follows. □

Proof of Theorem 3.2. First, we show that $\mathbf{X}_n = \mathbf{X}_n^{(1)}, n \geq 1$ is the k-dimensional marginal of some $(k+1)$-dimensional elliptical random vector. Define therefore a new random vector $\mathbf{Y}_n, n \geq 1$ with stochastic representation

$$\mathbf{Y}_n \stackrel{d}{=} R_{k+1} A_n^* \mathbf{U}_{k+1},$$

where $\mathbf{U}_{k+1}$ is uniformly distributed on $\mathcal{S}_{k+1}$ independent of $R_{k+1} \sim H_{k+1}$, and A_n^* is a non-singular $(k+1)$-dimensional square matrix. Choose $A_n^*, n \geq 1$ such that $\Sigma_n^* = A_n^*(A_n^*)^\top$ is again a correlation matrix satisfying

$$\big(\Sigma_n^*\big)_{II} = \Sigma_n, \qquad I = \{1, \dots, k\}, J = \{k+1\},$$

and

$$\lim_{n\to\infty} a_n^2\big(\mathbf{1}\mathbf{1}^\top - \Sigma_n^*\big) = \Gamma^* \in (0, \infty)^{(k+1)\times(k+1)}, \qquad \big(\Gamma^*\big)_{II} = \Gamma, \mathbf{1} \in \mathbb{R}^{k+1}.$$

Since Σ_n, Σ_n^* are positive definite, by condition (1.1) this construction is possible. Note that Σ_n^* satisfies (1.1) with $c_n = 2b_n/a_n$ and limit matrix $\Gamma^* \in [0, \infty)^{(k+1)\times(k+1)}$. We write for notational simplicity $(\Gamma^*)_{IJ} = \boldsymbol{\theta}/2$ and assume that $\boldsymbol{\theta}$ has positive components. It is well-known (see Cambanis [5]) that

$$\mathbf{U}_{k+1} \stackrel{d}{=} \big(\mathbf{U}W, \sqrt{1 - W^2}\mathcal{J}\big),$$

with W a positive random variable such that $W^2 \stackrel{d}{=} \mathcal{B}_{k/2,1/2}$, and $\mathcal{J}$ a Bernoulli random variable taking values $-1, 1$ with equal to probability $1/2$. Furthermore $\mathcal{J}, \mathbf{U}$, and W are mutually independent.

By the assumption, $R_k^2 \stackrel{d}{=} (R_{k+1})^2 \mathcal{B}_{k/2,1/2}$ with $R_{k+1} \sim H_{k+1}$ independent of $\mathcal{B}_{k/2,1/2}$, implying $\mathbf{Y}_{n,I} \stackrel{d}{=} \mathbf{X}_n$. Since the distribution function of $\mathbf{X}_n$ depends on Σ_n and not on A_n, and further Σ_n is positive definite we can assume that A_n is a lower triangular matrix. We construct A_n^* to be also a non-singular lower triangular matrix. With the same notation as in the proof of Theorem 3.1 we have

$$(\mathbf{Y}_n)_I | Y_{n,k+1} = q_n(y) \stackrel{d}{=} R_{y,n,k} B_n \mathbf{U} + \left(\Sigma_n^*\right)_{IJ} q_n(y), \qquad n \geq 1, \tag{6.6}$$

where B_n is a lower triangular matrix satisfying $B_n B_n^\top = \Sigma_n - (\Sigma_n^*)_{IJ} (\Sigma_n^*)_{JI}$, and $R_{y,n,k}, n \geq 1$ (being independent of $\mathbf{U}$) has survival function $\overline{Q}_{y,n,k+1}$ given by (6.4). As in the proof of Theorem 3.1

$$R_{y,n,k} \stackrel{d}{\to} \mathcal{R}_k \sim \mathcal{H}_k, \qquad n \to \infty.$$

By (1.1) and the fact that $B_n B_n^\top$ (and not the matrix B_n) defines the conditional distribution we can choose B_n such that $\lim_{n\to\infty} a_n B_n = B$ with $BB^\top = (\mathbf{1}\boldsymbol{\theta}^\top + \boldsymbol{\theta}\mathbf{1}^\top - \Gamma)/2$. Hence, for any $\mathbf{x} \in (0, \infty)^k$ utilising further (6.6) we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\{a_n \mathbf{L}_n > \mathbf{x}\} \\
&\quad = \lim_{n\to\infty} \mathbf{P}\{\forall i \leq k\colon\ a_n L_{ni} > x_i\} \\
&\quad = \lim_{n\to\infty} \mathbf{P}\bigl\{\forall i \leq k\colon\ a_n |X_{ni}| > x_i\bigr\}^n \\
&\quad = \exp\Bigl(-\lim_{n\to\infty} n\mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |X_{ni}| \leq x_i\bigr\}\Bigr) \\
&\quad = \exp\biggl(-\lim_{n\to\infty} n\biggl[\int_0^\infty \mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Y_{ni}| \leq x_i | Y_{n,k+1} = y/a_n\bigr\}\,\mathrm{d}G_n(y) \\
&\qquad\qquad + \int_{-\infty}^0 \mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Y_{ni}| \leq x_i | Y_{n,k+1} = y/a_n\bigr\}\,\mathrm{d}G_n(y)\biggr]\biggr) \\
&\quad = \exp\biggl(-\lim_{n\to\infty} n \int_0^\infty \bigl[\mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Y_{ni}| \leq x_i | Y_{n,k+1} = y/a_n\bigr\} \\
&\qquad\qquad + \mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Y_{ni}| \leq x_i | Y_{n,k+1} = -y/a_n\bigr\}\bigr]\,\mathrm{d}G_n(y)\biggr) \\
&\quad = \exp\biggl(-\lim_{n\to\infty} n \int_0^\infty \bigl[\mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Z_{ni} + d_{ni} y/a_n| \leq x_i\bigr\} \\
&\qquad\qquad + \mathbf{P}\bigl\{\exists i \leq k\colon\ a_n |Z_{ni} - d_{ni} y/a_n| \leq x_i\bigr\}\bigr]\,\mathrm{d}G_n(y)\biggr)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigl(-\int_0^\infty \bigl[\mathbf{P}\bigl\{\exists i \le k\colon |Z_i + y| \le x_i\bigr\} + \mathbf{P}\bigl\{\exists i \le k\colon |Z_i - y| \le x_i\bigr\}\bigr]\,\mathrm{d}y^\gamma\Bigr)\\
&= \exp\Bigl(-\int_0^\infty \bigl[\mathbf{P}\bigl\{\exists i \le k\colon \bigl|Z_i + y^{1/\gamma}\bigr| \le x_i\bigr\} + \mathbf{P}\bigl\{\exists i \le k\colon \bigl|Z_i - y^{1/\gamma}\bigr| \le x_i\bigr\}\bigr]\,\mathrm{d}y\Bigr)\\
&= \exp\Bigl(-\int_{\mathbb{R}} \mathbf{P}\bigl\{\exists i \le k\colon \bigl|Z_i + \mathrm{sign}(y)|y|^{1/\gamma}\bigr| \le x_i\bigr\}\,\mathrm{d}y\Bigr),
\end{aligned}
$$

with $(Z_1, \ldots, Z_k)^\top = \mathcal{R}_k B\mathbf{U}$, and thus the claim follows. □

Proof of Theorem 4.1. By Theorem 4.1 in Hashorva and Pakes [17], $H \in \mathrm{GMDA}(w)$ is equivalent with $G \in \mathrm{GMDA}(w)$. Let $B_n, \mathbf{Y}_n, n \ge 1$ be as in the proof of Theorem 3.2 and adopt below the same notation as therein. Conditioning on $Y_{n,k+1} = q_n(y) = a_n y + b_n$, with $y \in \mathbb{R}$ such that $G(q_n(y)) \in (0,1), n \ge 1$ we have that (6.6) holds, with $R_{y,n,k}$ independent of $\mathbf{U}$ satisfying (see Hashorva [15])

$$
\frac{1}{\sqrt{a_n b_n}} R_{y,n,k} \overset{d}{\to} \mathcal{R}, \qquad n \to \infty,
$$

where $\mathcal{R}^2 \sim \chi^2_{k+1}$, and $\mathcal{R}_k > 0$. Next, $G \in \mathrm{GMDA}(w)$, (1.1) and the choice of B_n imply for any $\mathbf{x} \in \mathbb{R}^k$ (omitting some details)

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\{\mathbf{M}_n \le a_n \mathbf{x} + b_n \mathbf{1}\}\\
&\quad = \lim_{n\to\infty} \bigl[1 - \mathbf{P}\bigl\{\exists i \le k\colon X_{ni} > q_n(x_i)\bigr\}\bigr]^n\\
&\quad = \exp\Bigl(-\lim_{n\to\infty} n\mathbf{P}\bigl\{\exists i \le k\colon X_{ni} > q_n(x_i)\bigr\}\Bigr)\\
&\quad = \exp\Bigl(-\lim_{n\to\infty} n \int_{\mathbb{R}} \mathbf{P}\bigl\{\exists i \le k\colon Y_{ni} > q_n(x_i) | Y_{n,k+1} = q_n(y)\bigr\}\,\mathrm{d}G\bigl(q_n(y)\bigr)\Bigr)\\
&\quad = \exp\Bigl(-\lim_{n\to\infty} n \int_{\mathbb{R}} \mathbf{P}\Bigl\{\exists i \le k\colon \frac{1}{\sqrt{a_n b_n}} R_{y,n,k}\bigl([\sqrt{b_n/a_n} B_n]\mathbf{U}\bigr)_i\\
&\qquad\qquad > x_i - y d_{ni} + [1 - d_{ni}] b_n / a_n\Bigr\}\,\mathrm{d}G\bigl(q_n(y)\bigr)\Bigr)\\
&\quad = \exp\Bigl(-\int_{\mathbb{R}} \mathbf{P}\{\exists i \le k\colon Z_i > x_i - y + \theta_i/2\} \exp(-y)\,\mathrm{d}y\Bigr),
\end{aligned}
$$

with $\mathbf{Z} \eqsim \mathit{Gauss}[\Gamma]$. Recall $\mathcal{R}_k\mathbf{U}$ is a k-dimensional Gaussian random vector with independent components, and further note that the choice of θ_i above is arbitrary. The assumption that (2.3) holds also for $m = k$ needed to define $\mathbf{Y}_n$ can now be dropped since the limit distribution is independent of that assumption, and further the convergence in distribution holds without imposing that assumption, hence the proof is complete. □

Proof of Theorem 4.2. First note that Theorem 4.5 in Hashorva [17] states that $H \in \mathrm{WMDA}(\alpha)$, $\alpha > 0$ is equivalent with $G \in \mathrm{WMDA}(\alpha + (k-1)/2)$. We proceed as in the proof of Theorem 4.1 (keeping the same notation). Conditioning on the event $Y_{n,k+1} = q_n(y) = 1 - a_n y$, with y such that $G(q_n(y)) \in (0,1)$, $n \geq 1$ and constants a_n defined in (3.1) we have that again (6.6) holds. In view of Hashorva [15] for any $y > 0$

$$\frac{1}{\sqrt{a_n}} R_{y,n,k} \xrightarrow{d} \sqrt{2y}\widetilde{\mathcal{R}_\alpha}, \qquad n \to \infty,$$

with $\widetilde{\mathcal{R}_\alpha} \sim \widetilde{\mathcal{H}}_\alpha$ where $\widetilde{\mathcal{H}}_\alpha(0) = 0$ and $\widetilde{\mathcal{R}_\alpha}^2 \stackrel{d}{=} \mathcal{B}_{k/2,\alpha}$. Furthermore

$$\lim_{u \to \infty} \frac{1 - G(1 - x/u)}{1 - G(1 - 1/u)} = x^{\alpha + (k-1)/2} \qquad \forall x \in (0, \infty)$$

holds. Hence for any $\mathbf{x} \in (-\infty, 0)^k$, we obtain (set $G_n(y) = G(1 - a_n y)$)

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\{\mathbf{M}_n \leq \mathbf{1} + a_n \mathbf{x}\} \\
&\quad = \exp\biggl(-\lim_{n\to\infty} n \int_0^\infty \mathbf{P}\biggl\{\exists i \leq k\colon \frac{1}{\sqrt{a_n}} R_{y,n,k}\biggl(\frac{B_n}{\sqrt{a_n}}\mathbf{U}\biggr)_i \\
&\qquad\qquad > x_i + y d_{ni} + [1 - d_{ni}]/a_n\biggr\} \, \mathrm{d}G_n(y)\biggr) \\
&\quad = \exp\biggl(-\int_0^\infty \mathbf{P}\bigl\{\exists i \leq k\colon Z_i > [x_i + y + \theta_i/2]/\sqrt{2y}\bigr\} \, \mathrm{d}y^{\alpha + (k-1)/2}\biggr),
\end{aligned}$$

with $\mathbf{Z} \approx \mathfrak{E}[\Gamma; \widetilde{\mathcal{H}}_\alpha]$, and thus the proof is complete. □

Proof of Theorem 5.1.

(A) Let G denote the distribution function of $S_1 Y_{11}(1)$, and let Φ denote the standard Gaussian distribution function on $\mathbb{R}$. The Mills ratio asymptotics (see, e.g., Lu and Li [24]) implies $Y_{11}(1) \in \mathcal{W}(1/\sqrt{2\pi}, -1, 1/2, 2)$. Consequently, by Lemma 2.1 in Arendarczyk and Dębicki [1]

$$\begin{aligned}
1 - G(x) &= \bigl(1 + o(1)\bigr)\biggl(\frac{2\pi}{2 + p_1}\biggr)^{1/2} \frac{C_1}{\sqrt{2\pi}} A^{-\alpha_1} x^{(\alpha_1(p_1 - 1) + p_1)/(2 + p_1)} \\
&\quad \times \exp\bigl(-\bigl(L_1 A^{-p_1} + A^2/2\bigr) x^{2p_1/(2+p_1)}\bigr) \\
&= \bigl(1 + o(1)\bigr) \frac{C_1}{\sqrt{2 + p_1}} A^{-\alpha_1} x^{(\alpha_1(p_1 - 1) + p_1)/(2 + p_1)} \exp\bigl(B x^{2p_1/(2+p_1)}\bigr), \qquad x \to \infty,
\end{aligned}$$

with $A = (p_1 L_1)^{1/(2+p_1)}$, $B = L_1 A^{-p_1} + A^2/2 > 0$. Hence $G \in \mathrm{GMDA}(w)$ with

$$w(x) = B \frac{2p_1}{2 + p_1} x^{(p_1 - 2)/(2 + p_1)}, \qquad x > 0.$$

Set $b_n = G^{-1}(1-1/n), n > 1$ with G^{-1} the generalised inverse of G. Now, by (4.2)

$$\lim_{n\to\infty} \frac{b_n}{b_n^*} = 1, \tag{6.7}$$

where $b_n^* = \Psi^{-1}(1-1/n), n > 1$ and Ψ is some distribution function satisfying

$$1 - \Psi(x) = \big(1 + \mathrm{o}(1)\big) \exp\big(-Bx^{2p_1/(2+p_1)}\big), \qquad x \to \infty.$$

The above asymptotics implies

$$\lim_{n\to\infty} n\big(1 - G(a_n x + b_n)\big) = \exp(-x) \qquad \forall x \in \mathbb{R}, \tag{6.8}$$

with

$$b_n = \big(1 + \mathrm{o}(1)\big)\left(\frac{\ln n}{B}\right)^{(2+p_1)/(2p_1)}, \qquad a_n = \frac{1}{w(b_n)} = \frac{(2+p_1) b_n^{(2-p_1)/(2+p_1)}}{2p_1 B}, \qquad n \to \infty.$$

Consequently, as $n \to \infty$

$$\frac{b_n}{a_n} = \big(1 + \mathrm{o}(1)\big)\frac{2p_1}{2+p_1} \ln n,$$

hence (5.5) follows by Theorem 3.1 of Kabluchko [22] and Theorem 4.1.

(B) Since $\Phi \in \mathrm{GMDA}(w)$ with scaling function $w(x) = x, x > 0$ Theorem 3 in Hashorva [14] implies

$$1 - G(x) = \big(1 + \mathrm{o}(1)\big)\Gamma(\alpha + 1)\mathbf{P}\big\{S > 1 - 1/\big(xw(x)\big)\big\}\mathbf{P}\big\{Y_{11}(1) > x\big\}, \qquad x \to \infty$$

and thus $G \in \mathrm{GMDA}(w)$. If $a_n, b_n, n \geq 1$ are defined by (6.8), then Theorem 3.1 in Kabluchko [22] and Theorem 4.1 establishes (5.5). By the form of $w(\cdot)$ we have $\lim_{n\to\infty} a_n b_n = 1$, and further (6.7) holds with $b_n^* = \Phi^{-1}(1-1/n), n > 1$. Consequently, $b_n = (1 + \mathrm{o}(1))\sqrt{2\ln n}$ for all large n, and thus the result follows. □

Proof of Theorem 5.2. Let $\mathbf{S}_n^{(i)}$ and $\mathbf{X}_n^{(i)}, i \leq n, n \geq 1$ be such that $S_{nj}^{(i)} = S_{ni}(t_j), t_j \in \mathbb{R}, j \leq k$ and $\mathbf{X}_n^{(i)}, i \leq n$ are independent copies of the Gaussian random vector $X_{n1}(t_j), 1 \leq j \leq k$. By the assumptions of the theorem, the proof follows if we show that the limit of the minima of the absolute values for the triangular array $\mathbf{S}_n^{(i)}\mathbf{X}_n^{(i)}, i \leq n, n \geq 1$ converges to the random vector $\boldsymbol{\mathcal{L}}$ such that

$$\mathbf{P}\{\boldsymbol{\mathcal{L}} > \mathbf{x}\} = \exp\left(-\int_{\mathbb{R}} \mathbf{P}\big\{\exists i \leq k\colon\ S_i|y + Z_i| \leq x_i\big\}\,\mathrm{d}y\right), \qquad \mathbf{x} \in (0,\infty)^k,$$

where $\mathbf{S} := \mathbf{S}_1^{(1)}$ is independent centered Gaussian random vector $\mathbf{Z}$ with incremental variance matrix Γ which has components $\gamma_{ij} = \Gamma(t_i, t_j)$. The proof follows with similar arguments as that of Theorem 3.2 since $\mathbf{S}_n^{(i)}$ is, by the assumption, independent of $\mathbf{X}_n^{(i)}$. □

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