

A goodness-of-fit test for bivariate extreme-value copulas

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It is often reasonable to assume that the dependence structure of a bivariate continuous distribution belongs to the class of extreme-value copulas. The latter are characterized by their Pickands dependence function. In this paper, a procedure is proposed for testing whether this function belongs to a given parametric family. The test is based on a Cramér–von Mises statistic measuring the distance between an estimate of the parametric Pickands dependence function and either one of two nonparametric estimators thereof studied by Genest and Segers [*Ann. Statist.* **37** (2009) 2990–3022]. As the limiting distribution of the test statistic depends on unknown parameters, it must be estimated via a parametric bootstrap procedure, the validity of which is established. Monte Carlo simulations are used to assess the power of the test and an extension to dependence structures that are left-tail decreasing in both variables is considered.

Keywords: extreme-value copula; goodness of fit; parametric bootstrap; Pickands dependence function; rank-based inference

1. Introduction

Let X and Y be continuous random variables with cumulative distribution functions F and G , respectively. Following Sklar [36], the joint behavior of the pair (X, Y) can be characterized at every $(x, y) \in \mathbb{R}^2$ by the relation

$$H(x, y) = \Pr(X \leq x, Y \leq y) = C\{F(x), G(y)\} \quad (1)$$

through a unique copula C that captures the dependence between X and Y .

When H is known, its marginal distributions can easily be retrieved from it. The copula can also be readily identified as it is simply the joint distribution of the pair $(U, V) = (F(X), G(Y))$. In practice, however, H is often unknown, and the relation between X and Y must be modeled from data.

A copula model for H assumes that equation (1) holds for some F , G and C from specific parametric classes. This approach was used, for example, by Frees and Valdez [11] and Klugman and Parsa [25] to analyze data from the Insurance Services Office, Inc. on the indemnity

payment (X) and allocated loss adjustment expense (Y) for 1500 general liability claims randomly chosen from late settlement lags. Based on their work and subsequent analysis by other authors, it is reasonable to assume that for these data, F is inverse paralogistic, G is Pareto and C is a Gumbel–Hougaard extreme-value copula.

Extreme-value copulas characterize the limiting dependence structure of suitably normalized componentwise maxima. They are of special interest in insurance [7], finance [6,29] and hydrology [34], where the occurrence of joint extremes is a risk management concern.

Pickands [31] showed that if C is a bivariate extreme-value copula, then

$$C(u, v) = \exp \left[\log(uv) A \left\{ \frac{\log(v)}{\log(uv)} \right\} \right] \tag{2}$$

for all $u, v \in (0, 1)$ and a mapping $A : [0, 1] \rightarrow [1/2, 1]$, referred to as the Pickands dependence function, which is convex and such that $\max(t, 1 - t) \leq A(t) \leq 1$ for all $t \in [0, 1]$. For instance, an extreme-value copula is said to belong to the Gumbel–Hougaard family if there exists $\theta \in [1, \infty)$ such that for all $t \in [0, 1]$, we have

$$A(t) = \{t^\theta + (1 - t)^\theta\}^{1/\theta}. \tag{3}$$

A test that a copula C is of the form (2) was developed by Ghoudi *et al.* [20]; it was recently refined by Ben Ghorbal *et al.* [1]. Under the assumption that C is an extreme-value copula, it may be of interest to check whether its Pickands dependence function A belongs to a specific parametric class, say $\mathcal{A} = \{A_\theta : \theta \in \mathcal{O}\}$, where \mathcal{O} is an open subset of \mathbb{R}^p for some integer p .

The purpose of this paper is to examine how the hypothesis $H_0 : A \in \mathcal{A}$ can be tested with a random sample $(X_1, Y_1), \dots, (X_n, Y_n)$ from H . As for all goodness-of-fit tests reviewed by Berg [2] and Genest *et al.* [18], the proposed procedure is based on pseudo-observations $(U_1, V_1), \dots, (U_n, V_n)$ from copula C , defined, for $i \in \{1, \dots, n\}$, by

$$U_i = F_n(X_i), \quad V_i = G_n(Y_i), \tag{4}$$

where F_n and G_n are rescaled empirical counterparts of F and G , respectively, given by

$$F_n(x) = \frac{1}{n+1} \sum_{i=1}^n \mathbf{1}(X_i \leq x), \quad G_n(y) = \frac{1}{n+1} \sum_{i=1}^n \mathbf{1}(Y_i \leq y)$$

for all $x, y \in \mathbb{R}$. This approach is justified because, as copulas themselves, the pairs $(U_1, V_1), \dots, (U_n, V_n)$ of normalized ranks are invariant under strictly increasing transformations of X and Y . As shown by Kim *et al.* [24], it also leads to efficient and robust estimators.

The proposed test is described in Section 2 and its asymptotic null distribution is given in Section 3, where a parametric bootstrap is proposed for the calculation of P -values. In Section 4, the distributional result is extended to alternatives that are left-tail decreasing in both variables. This is instrumental in studying the consistency and power of the test, which are considered in Sections 5 and 6, respectively. The paper concludes with an illustrative example. Technical proofs are grouped in a series of appendices.

All procedures discussed herein are implemented in the R package `copula` [38] available via the Comprehensive R Archive Network at <http://cran.r-project.org>.

2. Proposed goodness-of-fit test

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a random sample from some unknown continuous bivariate distribution H whose underlying copula is of the form (2) with Pickands dependence function A . In order to test the hypothesis

$$H_0 : A \in \mathcal{A} = \{A_\theta : \theta \in \mathcal{O}\},$$

a natural way to proceed is to compare a nonparametric estimator A_n of A to a parametric estimator A_{θ_n} . Several measures of distance can be used for this purpose, but the Cramér–von Mises statistic

$$S_n = \int_0^1 n |A_n(t) - A_{\theta_n}(t)|^2 dt \tag{5}$$

generally leads to more powerful tests than, say, the Kolmogorov–Smirnov statistic [18]. The choices of A_{θ_n} and A_n are discussed next.

2.1. Parametric estimation of A

Under H_0 , A_θ may be estimated by A_{θ_n} using a consistent estimate θ_n of θ . Such an estimate can be derived from the pairs $(U_1, V_1), \dots, (U_n, V_n)$ via the maximum pseudo-likelihood method considered by Genest *et al.* [14] and Shih and Louis [35].

To illustrate this approach in a concrete case, let A_θ be the generator of the Gumbel–Hougaard copula defined in (3). For all $u, v \in (0, 1)$, write

$$\begin{aligned} C_\theta(u, v) &= \exp \left[\log(uv) A_\theta \left\{ \frac{\log(v)}{\log(uv)} \right\} \right] \\ &= \exp[-\{|\log(u)|^\theta + |\log(v)|^\theta\}^{1/\theta}]. \end{aligned}$$

As A_θ is twice differentiable on $(0, 1)$, the copula C_θ has a density given by $c_\theta(u, v) = \partial^2 C_\theta(u, v) / \partial u \partial v$ everywhere on $(0, 1)^2$. The maximum pseudo-likelihood estimator θ_n is then the value $\theta \in \mathcal{O} = (1, \infty)$ at which the function

$$\ell(\theta) = \sum_{i=1}^n \log\{c_\theta(U_i, V_i)\}$$

reaches its global maximum. An advantage of this method is that it can be used even when the parameter space \mathcal{O} is multidimensional.

When θ is real-valued, a simpler technique which also yields a consistent estimator is based on the inversion of Kendall’s tau. As shown by Ghoudi *et al.* [20], the relation

$$\tau(C) = -1 + 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t)$$

is valid for any extreme-value copula C . When $A \in \mathcal{A}$, τ is a function of θ and a rank-based moment estimate of the latter is obtained by solving the equation $\tau_n = \tau(\theta)$ for θ , where τ_n is the sample value of Kendall's tau. In the Gumbel–Hougaard model, for instance, we find $\tau(\theta) = 1 - 1/\theta$ and hence $\theta_n = \max\{1, 1/(1 - \tau_n)\}$.

When $\mathcal{O} \subset \mathbb{R}$, we can also obtain consistent, rank-based estimates of θ by exploiting its one-to-one relationship with other nonparametric measures of dependence such as Spearman's rho, that is,

$$\rho(C) = -3 + 12 \iint_0^1 uv \, dC(u, v) = -1 + \int_0^1 \frac{1}{\{A(t)\}^2} \, dt.$$

2.2. Nonparametric estimation of A

Nonparametric estimators of A are proposed by Genest and Segers [19]. For $i \in \{1, \dots, n\}$, set $\xi_i(0) = -\log(U_i)$, $\xi_i(1) = -\log(V_i)$ and

$$\xi_i(t) = \min \left\{ \frac{-\log(U_i)}{1-t}, \frac{-\log(V_i)}{t} \right\}$$

for all $t \in (0, 1)$, where U_i and V_i are as in equation (4). Also, let

$$A_n^P(t) = 1 / \left\{ \frac{1}{n} \sum_{i=1}^n \xi_i(t) \right\}, \quad A_n^{CFG}(t) = \exp \left[-\gamma - \frac{1}{n} \sum_{i=1}^n \log\{\xi_i(t)\} \right],$$

where $\gamma = -\int_0^\infty \log(x)e^{-x} \, dx \approx 0.577$ is Euler's constant.

The functions A_n^P and A_n^{CFG} are rank-based versions of the estimators of A introduced by Pickands [31] and Capéraà *et al.* [4], respectively. As noted by Genest and Segers [19], these estimators can be altered to meet the end-point conditions $A_n^P(0) = A_n^{CFG}(0) = 1$ and $A_n^P(1) = A_n^{CFG}(1) = 1$. However, this makes no difference asymptotically.

Both A_n^P and A_n^{CFG} can be expressed as functionals of the empirical copula, which may be defined for all $u, v \in [0, 1]$ by

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(U_i \leq u, V_i \leq v).$$

To be specific, the following relations hold for all $t \in [0, 1]$:

$$A_n^P(t) = 1 / \int_0^1 C_n(x^{1-t}, x^t) \frac{dx}{x},$$

$$A_n^{CFG}(t) = \exp \left\{ -\gamma + \int_0^1 \{C_n(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1})\} \frac{dx}{x \log(x)} \right\}.$$

It was shown by Rüschenendorf [33] that under weak regularity conditions, the process $\sqrt{n}(C_n - C)$ converges in law to a Gaussian limit \mathbb{C} , that is, $\sqrt{n}(C_n - C) \rightsquigarrow \mathbb{C}$ as $n \rightarrow \infty$. We may thus

expect A_n^P and A_n^{CFG} to be consistent and asymptotically Gaussian. This is shown by Genest and Segers [19], provided that A is twice continuously differentiable. Their Theorem 3.2 states that

$$\mathbb{A}_n^P = \sqrt{n}(A_n^P - A) \rightsquigarrow \mathbb{A}^P, \quad \mathbb{A}_n^{\text{CFG}} = \sqrt{n}(A_n^{\text{CFG}} - A) \rightsquigarrow \mathbb{A}^{\text{CFG}}$$

as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$, where, for all $t \in [0, 1]$,

$$\begin{aligned} \mathbb{A}^P(t) &= -A^2(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t) \frac{dx}{x}, \\ \mathbb{A}^{\text{CFG}}(t) &= A(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t) \frac{dx}{x \log(x)}. \end{aligned}$$

Remark. Observe that, in principle, the statistics S_n^P and S_n^{CFG} could be extended to arbitrary dimension $d \geq 3$ because d -variate extreme-value copulas are characterized by $(d - 1)$ -place Pickands dependence functions [10]. At present, however, multivariate analogs of the rank-based estimators A_n^P and A_n^{CFG} are unavailable. To see how the estimation can proceed in the d -variate case when the marginal distributions are known, refer to [39] or [21].

3. Asymptotic null distribution of the test statistic

The asymptotic distribution of the goodness-of-fit statistic S_n depends on the joint behavior of $\Theta_n = \sqrt{n}(\theta_n - \theta)$ and either \mathbb{A}_n^P or $\mathbb{A}_n^{\text{CFG}}$ under H_0 . Suppose that the class $\mathcal{A} = \{A_\theta : \theta \in \mathcal{O}\}$ satisfies the following conditions:

- (A) the parameter space \mathcal{O} is an open subset of \mathbb{R}^p ;
- (B) for every $\theta \in \mathcal{O}$, A_θ is twice continuously differentiable on $(0, 1)$;
- (C) the gradient $\dot{A}_\theta(t)$ of $A_\theta(t)$ with respect to θ satisfies

$$\lim_{\epsilon \downarrow 0} \sup_{\|\theta^* - \theta\| < \epsilon} \sup_{t \in [0, 1]} \|\dot{A}_{\theta^*}(t) - \dot{A}_\theta(t)\| \rightarrow 0, \tag{6}$$

where $\|\cdot\|$ denotes the ℓ_2 -norm.

As is proved in Appendix A, the process $\mathbb{A}_{n, \theta_n} = \sqrt{n}(A_n - A_{\theta_n})$ is then asymptotically Gaussian, both when $A_n = A_n^P$ and $A_n = A_n^{\text{CFG}}$.

Proposition 1. *Assume H_0 holds, that is, C is an extreme-value copula with Pickands dependence function $A = A_{\theta_0}$ for some $\theta_0 \in \mathcal{O}$. Further, assume that $\mathcal{A} = \{A_\theta : \theta \in \mathcal{O}\}$ meets conditions (A)–(C).*

- (a) *If $(\mathbb{A}_n^P, \Theta_n)$ converges to a Gaussian limit (\mathbb{A}^P, Θ) , then $\mathbb{A}_{n, \theta_n} \rightsquigarrow \mathbb{A}^P - \dot{A}_{\theta_0}^\top \Theta$ as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$.*
- (b) *If $(\mathbb{A}_n^{\text{CFG}}, \Theta_n)$ converges to a Gaussian limit $(\mathbb{A}^{\text{CFG}}, \Theta)$, then $\mathbb{A}_{n, \theta_n} \rightsquigarrow \mathbb{A}^{\text{CFG}} - \dot{A}_{\theta_0}^\top \Theta$ as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$.*

The weak convergence of the statistic S_n defined in (5) follows immediately from Proposition 1 and the continuous mapping theorem (see, e.g., [37], Theorem 1.3.6). As the limit depends on the unknown parameter value θ_0 , we must resort to resampling techniques to carry out the test. The following parametric bootstrap procedure can be used to this end. Its validity depends on regularity conditions adapted from [17]. These conditions, listed in Appendix B, can be verified for many families of extreme-value copulas.

Parametric bootstrap procedure

- (1) Compute A_n from the pairs $(U_1, V_1), \dots, (U_n, V_n)$ of normalized ranks and estimate θ using a rank-based estimator, as discussed in Section 2.
- (2) Compute the test statistic S_n defined in (5).
- (3) For some large integer N , repeat the following steps for every $k \in \{1, \dots, N\}$:
 - (3.1) generate a random sample $(X_{1k}, Y_{1k}), \dots, (X_{nk}, Y_{nk})$ from copula C_{θ_n} and deduce the associated pairs $(U_{1k}, V_{1k}), \dots, (U_{nk}, V_{nk})$ of normalized ranks;
 - (3.2) let A_{nk} and θ_{nk} stand for the versions of A_n and θ_n derived from the pairs $(U_{1k}, V_{1k}), \dots, (U_{nk}, V_{nk})$;
 - (3.3) compute

$$S_{nk} = \int_0^1 n |A_{nk}(t) - A_{\theta_{nk}}(t)|^2 dt.$$

- (4) An approximate P -value for the test is given by $N^{-1} \sum_{k=1}^N \mathbf{1}(S_{nk} \geq S_n)$.

4. Extension to left-tail decreasing copulas

The statistic S_n can be used to build goodness-of-fit tests for the more general hypothesis

$$H_0^* : C \in \mathcal{C} = \{C_\theta : \theta \in \mathcal{O}\},$$

where \mathcal{C} is a parametric family of copulas that are left-tail decreasing (LTD) in both arguments. From [30], Exercise 5.35, a copula C is LTD in this sense if and only if, for all $0 < u \leq u' \leq 1$ and $0 < v \leq v' \leq 1$,

$$\frac{C(u, v)}{uv} \geq \frac{C(u', v')}{u'v'}. \tag{7}$$

This condition is satisfied for extreme-value copulas, which Garralda-Guillem [13] showed to be stochastically increasing in both variables.

The following result, proved in Appendix C, implies that when C is an LTD copula, A_n^P and A_n^{CFG} are consistent, asymptotically Gaussian estimators of A_C^P and A_C^{CFG} , respectively, where, for all $t \in [0, 1]$,

$$A_C^P(t) = 1 / \int_0^1 C(x^{1-t}, x^t) \frac{dx}{x}$$

and

$$A_C^{CFG}(t) = \exp \left[-\gamma + \int_0^1 \{C(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1})\} \frac{dx}{x \log(x)} \right].$$

Proposition 2. Suppose that the copula C has a continuous density and satisfies condition (7). Then $\sqrt{n}(A_n^P - A_C^P) \rightsquigarrow \mathbb{A}_C^P$ and $\sqrt{n}(A_n^{CFG} - A_C^{CFG}) \rightsquigarrow \mathbb{A}_C^{CFG}$ as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$, where, for all $t \in [0, 1]$,

$$\mathbb{A}_C^P(t) = -\{A_C^P(t)\}^2 \int_0^1 \mathbb{C}(x^{1-t}, x^t) \frac{dx}{x},$$

$$\mathbb{A}_C^{CFG}(t) = A_C^{CFG}(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t) \frac{dx}{x \log(x)}.$$

Incidentally, the mappings A_C^P and A_C^{CFG} are well defined for any copula C , whether or not it is LTD. They reduce to the Pickands dependence function A when C is of the form (2). Otherwise, they typically differ from one another, but retain some of the properties of Pickands dependence functions. These facts are summarized in the following proposition, the proof of which is left to the reader.

Proposition 3. Let C be a copula and let A_C denote either A_C^P or A_C^{CFG} . Also, let W and M denote the lower and upper Fréchet–Hoeffding bounds, respectively. The following statements then hold:

- (a) $A_W(t) \geq A_C(t) \geq A_M(t) = \max(t, 1 - t)$ for all $t \in [0, 1]$;
- (b) if $C(u, v) \geq uv$ for all $u, v \in [0, 1]$, then $A_C(t) \leq 1$ for all $t \in [0, 1]$;
- (c) if $C(u, v) = C(v, u)$ for all $u, v \in [0, 1]$, then $A_C(t) = A_C(1 - t)$ for all $t \in [0, 1]$;
- (d) if C is an extreme-value copula with Pickands dependence function A , then $A_C = A$.

The bounds A_W^P , A_W^{CFG} and $A_M^P = A_M^{CFG}$ are depicted in the left panel of Figure 1. As a further example, consider the Farlie–Gumbel–Morgenstern copula with parameter $\theta \in [-1, 1]$, defined

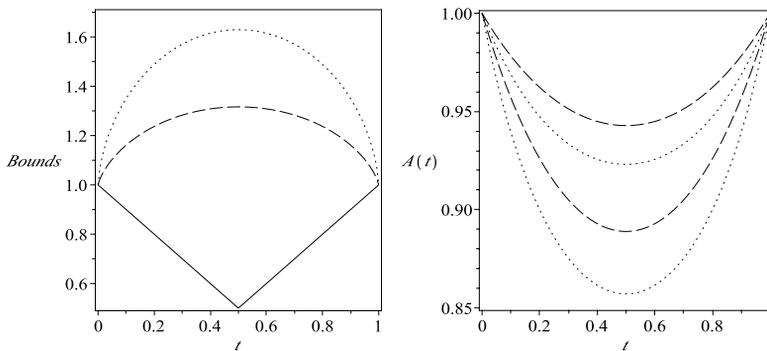


Figure 1. Left panel: graph of the bounds A_W^P (top curve), A_W^{CFG} (middle curve) and $A_M^P = A_M^{CFG}$ (bottom curve). Right panel: graph of A_C^P (dashed) and A_C^{CFG} (dotted) for the Farlie–Gumbel–Morgenstern copula with $\theta = 1/2$ (upper curves) and 1 (lower curves).

for all $u, v \in [0, 1]$ by $C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v)$. Condition (7) is met if $\theta \geq 0$ and it is easy to check that for all $t \in [0, 1]$,

$$A_\theta^P(t) = \frac{2t^2 - 2t - 4}{2t^2 - 2t - 4 + (3t^2 - 3t)\theta}, \quad A_\theta^{\text{CFG}}(t) = \left(\frac{2}{2 + t - t^2} \right)^\theta. \tag{8}$$

These functions are graphed in the right panel of Figure 1.

Invoking Proposition 2, we can proceed as in Appendix A to show the convergence of the goodness-of-fit process in the case of LTD copulas, whence the following result. The parametric bootstrap algorithm described in Section 3 also applies *mutatis mutandis* and remains valid under such H_0^* .

Proposition 4. Assume H_0^* holds, that is, C is an LTD copula such that $C = C_{\theta_0}$ for some $\theta_0 \in \mathcal{O}$. Let $\mathcal{A}^P = \{A_C^P : C \in \mathcal{C}\}$ and $\mathcal{A}^{\text{CFG}} = \{A_C^{\text{CFG}} : C \in \mathcal{C}\}$.

- (a) If \mathcal{A}^P meets conditions (A)–(C) and $(\mathbb{A}_n^P, \Theta_n)$ converges to a Gaussian limit (\mathbb{A}_C^P, Θ) , then $\mathbb{A}_{n, \theta_n} \rightsquigarrow \mathbb{A}_C^P - \dot{A}_{\theta_0}^\top \Theta$ as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$.
- (b) If \mathcal{A}^{CFG} meets conditions (A)–(C) and $(\mathbb{A}_n^{\text{CFG}}, \Theta_n)$ converges to a Gaussian limit $(\mathbb{A}_C^{\text{CFG}}, \Theta)$, then $\mathbb{A}_{n, \theta_n} \rightsquigarrow \mathbb{A}_C^{\text{CFG}} - \dot{A}_{\theta_0}^\top \Theta$ as $n \rightarrow \infty$ in $\mathcal{C}[0, 1]$.

5. Consistency of the test

Suppose that $C \notin \mathcal{C}$ is an LTD copula and that the hypothesis $H_0^* : C \in \mathcal{C}$ is being tested with the Cramér–von Mises statistic S_n . Let A_n denote either A_n^P or A_n^{CFG} and let A stand for A_C^P or A_C^{CFG} , as the case may be. Further, assume that θ_n is a consistent, rank-based estimator of some $\theta^* \in \mathcal{O}$. The test based on S_n is then consistent, provided that $A \neq A_{\theta^*}$.

To see this, decompose the process \mathbb{A}_{n, θ_n} as

$$\sqrt{n}(A_n - A_{\theta_n}) = \sqrt{n}(A_n - A) - \sqrt{n}(A_{\theta_n} - A_{\theta^*}) + \sqrt{n}(A - A_{\theta^*}). \tag{9}$$

Assume conditions (A)–(C) hold for $\mathcal{A} = \mathcal{A}^P$ or \mathcal{A}^{CFG} and that as $n \rightarrow \infty$, $(\sqrt{n}(A_n - A), \sqrt{n}(\theta_n - \theta^*)) \rightsquigarrow (\mathbb{A}, \Theta^*)$ to a Gaussian limit, where \mathbb{A} stands for either \mathbb{A}^P or \mathbb{A}^{CFG} . We can then proceed exactly as in Appendix A to see that as $n \rightarrow \infty$, $\sqrt{n}(A_n - A) - \sqrt{n}(A_{\theta_n} - A_{\theta^*}) \rightsquigarrow \mathbb{A} - \dot{A}_{\theta^*}^\top \Theta^*$. If $A \neq A_{\theta^*}$, then $\sup_{t \in [0, 1]} \sqrt{n}|A(t) - A_{\theta^*}(t)| \rightarrow \infty$ and hence, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \Pr(S_n > \epsilon) = 1.$$

In particular, the test based on S_n is consistent whenever C is an extreme-value copula and the hypothesized family \mathcal{C} also consists of extreme-value copulas. However, consistency may fail otherwise, for it may happen that $A = A_{\theta^*}$, even if H_0^* is false.

To illustrate this point, consider the functions A_θ^P and A_θ^{CFG} given in (8). As the latter are convex, they can be used to generate new families of extreme-value copulas, which may be called the FGM–P and FGM–CFG families.

Now, suppose that C is the Farlie–Gumbel–Morgenstern copula with parameter $\theta > 0$ and that the statistic S_n is used to test $H_0: A \in \mathcal{A}$ when:

- (a) \mathcal{A} is the Gumbel–Hougaard family of copulas;
- (b) \mathcal{A} is the FGM–CFG family of extreme-value copulas.

In case (a), the tests based on A_n^P and A_n^{CFG} would be consistent because A_θ^P and A_θ^{CFG} both differ from the Pickands dependence function of the Gumbel–Hougaard given in (3). In case (b), the test based on A_n^P would also be consistent because $A_\theta^P \neq A_{\theta^*}^{CFG}$. The test based on A_n^{CFG} may fail to be consistent, however, given that A_θ^{CFG} coincides with the Pickands dependence function of the FGM–CFG family. Consistency of the test would then depend on the behavior of θ_n .

Suppose, for instance, that θ is estimated by inversion of Kendall’s tau. As $n \rightarrow \infty$, θ_n would approach $2\theta/9$, which is the population value of this dependence measure for the FGM copula. For the FGM–CFG family, however, Kendall’s tau is $7\theta/10 + \theta^2/30$, which coincides with $2\theta/9$ only when $\theta = 0$, that is, at independence where the difference between the two models is immaterial. Therefore, the test based on A_n^{CFG} would be consistent in this case, provided that θ is estimated by inversion of Kendall’s tau. A similar conclusion would be reached for inversion of Spearman’s rho and maximum pseudo-likelihood estimation.

6. Power study

Equation (9) and the accompanying discussion suggest that just as for consistency, the power of the test based on S_n depends on how different $A = A_C^P$ or A_C^{CFG} is from its parametric estimate A_{θ^*} under H_0 . This issue is investigated graphically in Section 6.1 and via simulations in Sections 6.2 and 6.3.

6.1. General considerations

Consider the following three sets of LTD copula families.

Group I: *Symmetric extreme-value copulas:* the Gumbel–Hougaard (GH), Galambos (GA), Hüsler–Reiss (HR) and Student extreme-value (t-EV) copula with four degrees of freedom.

Group II: *Symmetric non-extreme-value copulas:* the Clayton (C), Frank (F), Normal (N) and Plackett (P).

Group III: *Asymmetric extreme-value copulas:* asymmetric versions of the Gumbel–Hougaard (a-GH), Galambos (a-GA), Hüsler–Reiss (a-HR), and Student extreme-value (a-t-EV) copula with four degrees of freedom.

Figure 2 shows the Pickands dependence functions of the copulas in Group I when $\tau = 0.25, 0.50, 0.75$. Although the curves are not identical, they are very similar. When the statistic S_n is used to distinguish between these models, therefore, the test will be consistent, but can be expected to have little power, even in moderate sample sizes.

In Figure 3, the functions A_C^P and A_C^{CFG} are plotted for the copulas in Group II and the same values of tau. For comparison purposes, the curve corresponding to the Gumbel–Hougaard copula is added. Here, the differences between the curves are much more pronounced. Thus, the

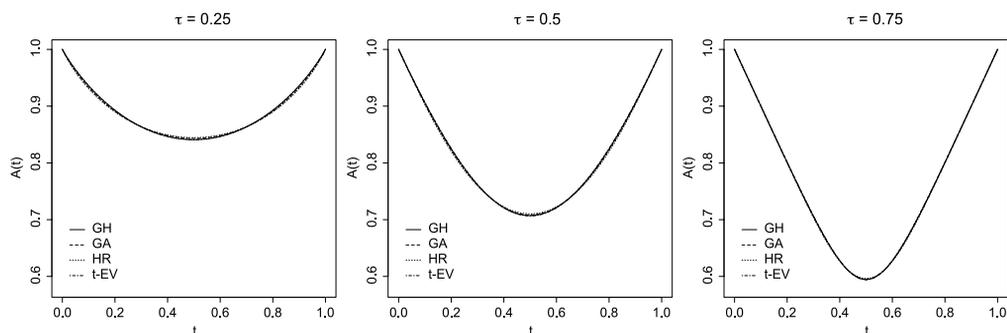


Figure 2. Pickands dependence functions of the Gumbel–Hougaard, Galambos, Hüsler–Reiss and t-EV copulas when $\tau = 0.25$, $\tau = 0.50$ and $\tau = 0.75$.

power of the test based on S_n may be expected to rise quickly (and be approximately the same) if the copula family under H_0 is from Group I.

Figure 4 shows the Pickands dependence functions of the copulas in Group III. These copulas were derived using Khoudraji's device [15,23,28], which transforms any symmetric copula C_θ

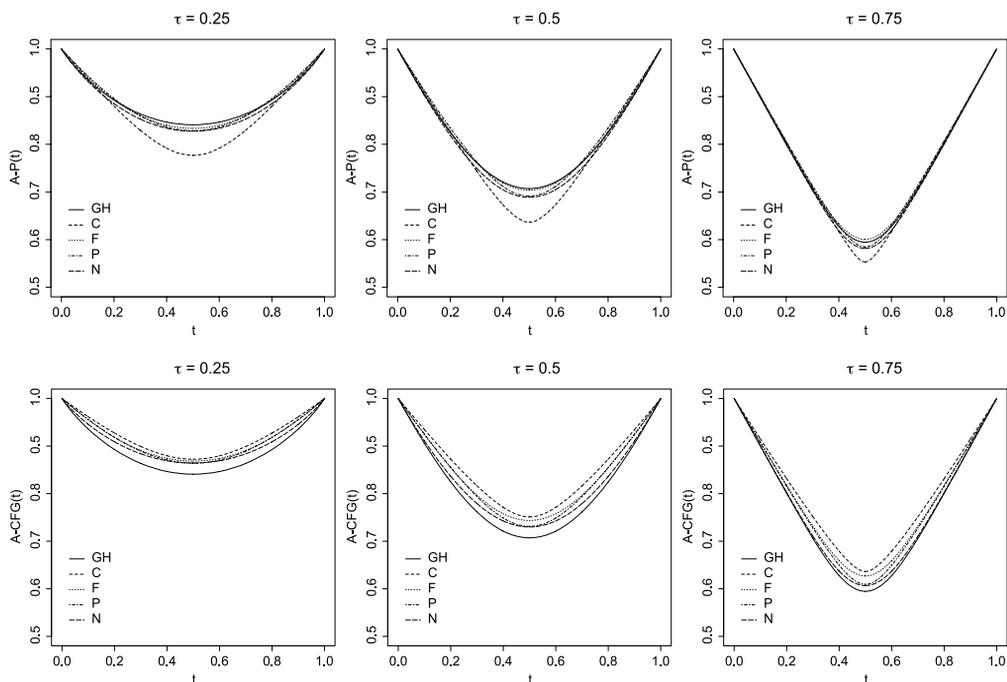


Figure 3. Plots of A_C^P (top) and A_C^{CFG} (bottom) when C is the Gumbel–Hougaard (GH), Clayton (C), Frank (F), Normal (N) and Plackett (P) copula with $\tau = 0.25$ (left), $\tau = 0.50$ (middle) and $\tau = 0.75$ (right).

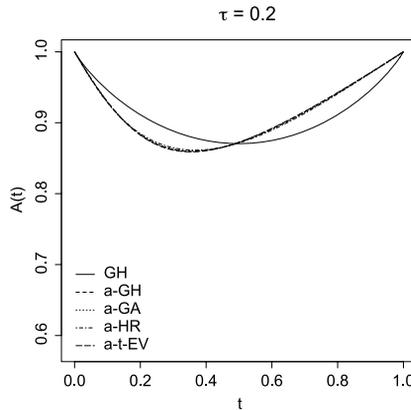


Figure 4. Pickands dependence functions for the Gumbel–Hougaard copula and four asymmetric extreme-value copulas with $\tau = 0.20$: the asymmetric Gumbel–Hougaard (a-GH), Galambos (a-GA), Hüsler–Reiss (a-HR) and t-EV (a-t-EV) with four degrees of freedom.

into a non-exchangeable model via the formula

$$C_{\lambda,\kappa,\theta}(u, v) = u^{1-\lambda}v^{1-\kappa}C_{\theta}(u^{\lambda}, v^{\kappa})$$

for all $u, v \in [0, 1]$ and arbitrary choices of $\lambda \neq \kappa \in (0, 1)$. Furthermore, if C_{θ} is an extreme-value copula with Pickands dependence function A_{θ} , then $C_{\lambda,\kappa,\theta}$ is also an extreme-value copula. Its Pickands dependence function is given, at all $t \in [0, 1]$, by

$$A_{\lambda,\kappa,\theta}(t) = (1 - \kappa)t + (1 - \lambda)(1 - t) + \{\kappa t + \lambda(1 - t)\}A_{\theta}\left\{\frac{\kappa t}{\kappa t + \lambda(1 - t)}\right\}.$$

Note that the dependence in $C_{\lambda,\kappa,\theta}$ is limited since, by the Fréchet–Hoeffding inequality,

$$C_{\lambda,\kappa,\theta}(u, v) \leq u^{1-\lambda}v^{1-\kappa} \min(u^{\lambda}, v^{\kappa}) = \min(uv^{1-\kappa}, vu^{1-\lambda}).$$

As the right-hand term is the Marshall–Olkin copula $MO_{\lambda,\kappa}$, Example 5.5 in [30], implies that

$$\tau(C_{\lambda,\kappa,\theta}) \leq \tau(MO_{\lambda,\kappa}) = \frac{\kappa\lambda}{\kappa + \lambda - \kappa\lambda}.$$

In the present study, the values $\lambda = 0.3, \kappa = 0.8$ were used and, hence, $\tau(C_{\lambda,\kappa,\theta})$ could not exceed 0.279. For each choice of copula family C_{θ} in Group III, the parameter θ was set to make Kendall’s tau equal to 0.20.

Figure 4 shows that the Pickands dependence functions of the copulas in Group III are very similar, though distinct. They are, however, easily distinguished from their symmetric counterparts with the same value of tau. Thus, although these extreme-value copulas would be difficult to tell apart on the basis of S_n in moderate samples, the test may still be reasonably powerful against copulas in Group I.

6.2. Monte Carlo study

The observations in Section 6.1 were confirmed through simulations. To this end, 1000 random samples of size $n = 300$ were generated from 28 different copulas, C , corresponding to the following scenarios:

- (a) C belongs to Group I or II and $\tau(C) \in \{0.25, 0.50, 0.75\}$;
- (b) C belongs to Group III and $\tau(C) = 0.20$.

The statistics S_n^P and S_n^{CFG} were computed for each data set. Four hypotheses of the form $H_0: A \in \mathcal{A}$ were then tested. The choices for \mathcal{A} were the families of Pickands dependence functions for extreme-value copulas in Group I.

All tests were carried out at the 5% level. Each P -value was computed on the basis of $N = 1000$ parametric bootstrap samples. For comparison purposes, goodness of fit was also checked with the general purpose statistic

$$T_n = \sum_{i=1}^n |C_n(U_i, V_i) - C_{\theta_n}(U_i, V_i)|^2.$$

This particular test statistic was chosen because of its good overall performance in the large scale simulation studies of Berg [2] and Genest *et al.* [18].

Tables 1–4 report the percentages of rejection of the four null hypotheses under each scenario. Although this made little difference, these results are for the end-point-corrected versions of A_n^P and A_n^{CFG} , defined for all $t \in [0, 1]$ by

$$1/A_{n,c}^P(t) = 1/A_n^P(t) - (1-t)\{1/A_n^P(0) - 1\} - t\{1/A_n^P(1) - 1\}$$

and

$$\log A_{n,c}^{CFG}(t) = \log\{A_n^{CFG}(t)\} - (1-t)\log\{A_n^{CFG}(0)\} - t\log\{A_n^{CFG}(1)\}.$$

Before commenting on the results, note that for copulas in Groups I and II, the real-valued dependence parameter of each data set was estimated by inversion of Kendall's tau; its implementation relied on the numerical approximation technique of Kojadinovic and Yan [26]. For copulas in Group III, which involve several parameters, maximum pseudo-likelihood estimation was used [14,35].

6.3. Results

It is clear from Table 1 that when $n = 300$, the tests based on T_n , S_n^P and S_n^{CFG} cannot distinguish between copulas in Group I. When $\tau = 0.25$, all rejection rates are within sampling error from the nominal level. There are only small signs of improvement as τ rises to 0.50 and 0.75. The best scores are obtained when testing for the Hüsler–Reiss model with S_n^{CFG} when $\tau = 0.75$. Globally, there is little to choose between the tests.

Table 2 shows what happens when $n = 1000$. Power is on the rise, especially when $\tau = 0.75$. In the latter case, it seems preferable to base the test on S_n^{CFG} rather than on S_n^P – both do better

Table 1. Percentage of rejection of H_0 for copulas in Group I when $n = 300$

H_0	True	$\tau = 0.25$			$\tau = 0.50$			$\tau = 0.75$		
		T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}
GH	GH	4.2	3.8	4.0	4.0	4.8	3.6	4.2	5.3	5.5
	GA	4.8	4.3	4.2	4.4	3.8	3.8	4.8	4.3	4.3
	HR	4.8	4.2	4.0	5.4	3.4	3.9	3.7	3.1	1.7
	t-EV	4.2	3.8	4.5	5.1	5.5	6.4	4.8	7.5	8.9
GA	GH	4.5	4.7	3.9	4.0	5.8	4.7	4.4	5.6	6.8
	GA	4.3	4.6	4.0	5.5	3.9	4.8	4.3	4.7	4.6
	HR	4.6	4.8	4.2	5.0	3.4	3.4	3.7	3.7	1.8
	t-EV	4.6	4.7	4.4	5.3	8.0	7.1	5.7	8.2	10.9
HR	GH	4.6	6.4	4.4	4.3	9.6	7.5	4.5	9.6	15.7
	GA	4.3	5.4	4.5	5.1	6.6	7.2	5.1	8.4	11.7
	HR	4.9	5.2	4.2	5.3	4.3	3.9	4.0	4.3	3.3
	t-EV	4.6	5.9	4.8	5.8	13.7	11.5	6.6	14.9	29.3
t-EV	GH	4.2	3.4	4.0	4.1	3.9	2.9	4.0	3.3	2.4
	GA	4.1	4.3	4.4	4.8	3.4	3.9	4.6	3.0	1.7
	HR	4.7	4.1	4.4	5.4	3.2	3.4	3.8	2.2	1.3
	t-EV	4.6	3.7	4.2	4.7	4.8	5.2	4.1	4.3	4.7

Table 2. Percentage of rejection of H_0 for copulas in Group I when $n = 1000$

H_0	True	$\tau = 0.25$			$\tau = 0.50$			$\tau = 0.75$		
		T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}
GH	GH	4.9	4.7	5.3	4.0	5.9	6.0	3.8	5.2	5.4
	GA	5.1	5.8	4.0	5.9	4.4	5.1	4.8	3.8	4.2
	HR	5.1	6.3	6.3	5.1	6.3	9.0	3.4	3.5	9.2
	t-EV	5.4	4.4	5.4	6.1	6.2	6.9	5.4	9.8	15.9
GA	GH	5.2	7.4	6.1	4.4	8.1	8.4	4.3	5.6	7.3
	GA	5.0	5.6	4.0	5.4	5.1	5.4	4.8	4.4	5.2
	HR	4.4	5.0	5.2	4.5	4.5	6.2	3.5	3.1	6.2
	t-EV	6.1	6.9	6.6	6.7	9.4	12.7	5.5	12.8	23.1
HR	GH	6.2	10.6	8.6	5.1	17.6	17.8	5.5	18.1	40.2
	GA	5.4	6.6	4.1	5.6	8.1	8.8	5.5	12.7	23.4
	HR	4.6	5.9	5.5	4.2	4.9	5.1	3.4	4.7	5.6
	t-EV	6.6	10.1	8.2	8.2	27.0	34.4	6.5	45.2	81.7
t-EV	GH	4.7	4.7	5.3	4.4	4.4	5.7	4.0	3.4	3.0
	GA	4.8	5.6	4.2	5.6	4.8	6.0	4.8	3.0	3.7
	HR	5.3	6.4	6.1	5.5	8.5	12.3	4.3	3.9	28.7
	t-EV	5.1	4.5	5.5	5.6	4.8	5.3	5.2	4.5	4.7

Table 3. Percentage of rejection of H_0 for copulas in Group II when $n = 300$

H_0	True	$\tau = 0.25$			$\tau = 0.50$			$\tau = 0.75$		
		T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}	T_n	S_n^P	S_n^{CFG}
GH	C	98.8	99.5	82.1	100.0	100.0	100.0	100.0	100.0	100.0
	F	36.6	11.0	48.0	82.0	7.1	100.0	92.1	27.2	100.0
	N	26.9	21.9	21.8	43.5	44.9	66.9	37.5	18.7	82.3
	P	34.3	17.3	43.4	68.0	44.6	98.6	65.0	71.6	100.0
GA	C	98.9	99.7	84.0	100.0	100.0	100.0	100.0	100.0	100.0
	F	39.8	15.1	50.1	83.4	10.2	100.0	92.1	29.9	100.0
	N	28.1	25.7	21.9	44.0	49.0	69.5	37.4	21.5	83.2
	P	37.7	23.4	45.0	70.8	57.1	99.0	65.7	76.7	100.0
HR	C	99.1	99.9	84.5	100.0	100.0	100.0	100.0	100.0	100.0
	F	42.3	18.8	52.5	85.2	18.9	100.0	93.7	42.1	100.0
	N	28.3	29.0	22.5	46.0	55.7	73.3	38.9	34.8	89.8
	P	41.1	28.8	48.3	75.1	74.5	99.5	73.1	92.1	100.0
t-EV	C	98.6	99.5	82.6	100.0	100.0	100.0	100.0	100.0	100.0
	F	36.7	11.1	48.3	81.3	5.0	100.0	90.9	18.9	100.0
	N	26.5	21.8	21.7	43.5	42.7	66.2	36.9	10.6	74.2
	P	34.8	17.2	43.7	67.7	35.7	98.1	62.1	53.2	99.8

Table 4. Percentage of rejection of H_0 for copulas in Group III when $n = 300$

H_0	True	T_n	S_n^P	S_n^{CFG}
GH	a-GH	32.7	40.9	86.5
	a-GA	33.5	42.8	86.7
	a-HR	28.4	37.5	83.5
	a-t-EV	33.1	41.4	88.6
GA	a-GH	33.4	40.8	89.2
	a-GA	34.0	42.3	89.3
	a-HR	28.4	38.1	86.5
	a-t-EV	32.7	40.6	90.5
HR	a-GH	36.2	37.5	93.3
	a-GA	31.7	39.1	89.7
	a-HR	32.6	40.9	90.3
	a-t-EV	40.5	42.8	92.3
t-EV	a-GH	32.0	41.2	87.1
	a-GA	33.2	43.3	87.8
	a-HR	27.3	38.4	83.6
	a-t-EV	31.3	40.7	88.7

than the test based on T_n . Overall, the results remain disappointingly low, except when testing for the Hüsler–Reiss model with $\tau \geq 0.50$.

These observations are in line with Figure 2, which shows striking similarities between the Gumbel–Hougaard, Galambos, Hüsler–Reiss and t-EV copula with four degrees of freedom. While S_n^P and S_n^{CFG} still have difficulty telling them apart when the sample size is 1000, their power eventually rises when $n \rightarrow \infty$, as explained in Section 5. To illustrate this point, samples of various sizes were generated from the Gumbel–Hougaard copula with $\tau = 0.50$ and the statistic S_n^{CFG} was used to test for the Galambos family. The following results, based on 1000 repetitions and $N = 1000$ bootstrap samples, give an idea of the sample sizes needed to differentiate models in Group I:

Sample size n	5 000	10 000	20 000	40 000
Percentage of rejection of H_0	10.8	22.6	60.2	97.3

Returning to the case $n = 300$, we can see from Table 3 that the test based on S_n^{CFG} is quite good at detecting non-extreme-value LTD alternatives from Group II. Its power is higher than those of S_n^P and T_n , except when the data are generated from the Clayton or the Normal copula with $\tau = 0.25$. Interestingly, the general purpose test based on T_n is often second best. The statistic S_n^P has the edge only for the Clayton when $\tau = 0.25$; it does very poorly against the Frank, and against the Normal when $\tau = 0.75$.

These results are in close agreement with the plots displayed in Figure 3. Consider, for instance, the case where S_n^P is used to test for the Gumbel–Hougaard copula from weakly dependent data ($\tau = 0.25$). From Table 3, the alternatives can be ranked as follows in decreasing order of power:

$$\text{Clayton} > \text{Normal} > \text{Plackett} > \text{Frank}.$$

Looking at Figure 3, we find that this ordering is concordant with the overall degree of dissimilarity between A_C^P and A . In this case, as in others, it is found that at fixed sample size, curves that look alike are harder to distinguish than others.

Finally, Table 4 shows that the statistic S_n^{CFG} is much better than the other two at detecting asymmetric extreme-value alternatives. The overall good performance of this test is consistent with evidence from [19] that A_n^{CFG} is generally a better nonparametric estimator of the Pickands dependence function than A_n^P . When the margins are known, this phenomenon is well documented; see, for example, [4,22] or [32].

7. Conclusion

Copula models are now common. As illustrated, for instance, by Ben Ghorbal *et al.* [1], so are situations in which the dependence structure of a random pair (X, Y) is well represented by an extreme-value copula, even though X and Y themselves do not necessarily exhibit extreme-value behavior. In such cases, the statistics considered here can be used to test the goodness of fit of specific parametric copula families of the form (2) such as the Gumbel–Hougaard, Galambos, Hüsler–Reiss or Student extreme-value copula.

Theoretical and empirical evidence presented here shows that the nonparametric tests based on the Cramér–von Mises statistic S_n are generally consistent and that they are an effective tool for distinguishing between symmetric and asymmetric extreme-value copulas, as well as for detecting other left-tail decreasing (LTD) dependence structures.

Except in the presence of massive data, however, it seems very difficult to discriminate between extreme-value copulas whose Pickands dependence functions are close. This may come as something of a disappointment, but, on reflection, we may wonder whether, in the light of Figure 2, there is any *practical* difference between, say, the Gumbel–Hougaard and the Galambos copula when they have the same value of Kendall’s tau.

For example, many studies have concluded that a Gumbel–Hougaard copula structure is adequate for the insurance data mentioned in the Introduction; see, for instance, [5,8,9,11,15,16] or [27]. In these papers, comparisons were made between the Gumbel–Hougaard model and non-extreme-value copulas that were either Archimedean or meta-elliptical.

As Ben Ghorbal *et al.* [1] conclude that the data exhibit extreme-value dependence, it may be worth comparing the Gumbel–Hougaard structure with other extreme-value copulas from Groups I and III. This is done in Table 5 using the statistics S_n^P and S_n^{CFG} and the inversion of Kendall’s tau to estimate θ . Because the test is yet to be adapted to the case of censoring, the analysis ignored the 34 claims for which the policy limit was reached. Each P -value in the table is based on $N = 2500$ bootstrap samples. Given the comparatively small sample size, $n = 1466$, it is little wonder that no model is rejected at the 5% level.

Figure 5 displays the end-point-corrected estimates $A_{n,c}^P$ and $A_{n,c}^{CFG}$ for the data at hand. For comparison, the best-fitting symmetric and asymmetric Galambos extreme-value copulas are superimposed. Although these two models yield the highest P -values, they are not significantly better than the alternatives listed in Table 5. Given the estimators’ sampling variability, the data set is simply too small to distinguish between them. This is not a major concern, however, as predictions derived from these various models would be roughly the same. To paraphrase Box and Draper ([3], page 424), it may be that all these models are false, but they are nearly equivalent and probably equally useful.

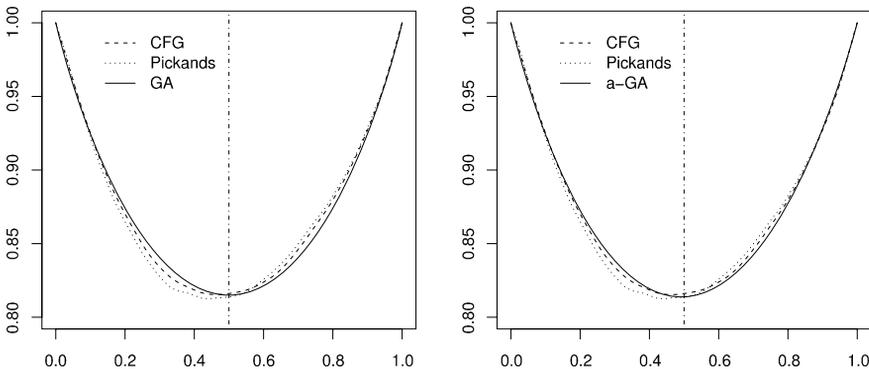


Figure 5. Nonparametric estimates $A_{n,c}^P$ and $A_{n,c}^{CFG}$, and fitted Pickands dependence function, for the Galambos copula (left) and asymmetric Galambos copula (right).

Table 5. Values of the statistics S_n^P, S_n^{CFG} and approximate P -values computed using $N = 2500$ parametric bootstrap samples for the insurance data

Model	S_n^P	P -value	S_n^{CFG}	P -value
GH	0.087	0.073	0.048	0.171
GA	0.084	0.074	0.045	0.184
HR	0.088	0.067	0.049	0.157
t-EV	0.088	0.069	0.048	0.166
a-GH	0.052	0.274	0.012	0.152
a-GA	0.046	0.325	0.009	0.244
a-HR	0.051	0.272	0.011	0.174
a-t-EV	0.062	0.204	0.015	0.122

Appendix A: Proof of Proposition 1

Let \mathbb{A}_n denote either \mathbb{A}_n^P or \mathbb{A}_n^{CFG} and write $\mathbb{A}_{n,\theta_n} = \mathbb{A}_n - \mathbb{B}_{n,\theta_0}$, where $\mathbb{B}_{n,\theta_0} = \sqrt{n}(A_{\theta_n} - A_{\theta_0})$. As the sequence Θ_n is assumed to converge weakly, it is tight. Thus, for given $\delta > 0$, there exists $L = L(\delta)$ such that $\Pr(\|\Theta_n\| > L) < \delta$ holds for every integer n . Therefore, for given $\zeta > 0$,

$$\begin{aligned} & \Pr\left\{ \sup_{t \in [0,1]} |\mathbb{B}_{n,\theta_0}(t) - \dot{A}_{\theta_0}^\top(t)\Theta_n| > \zeta \right\} \\ & \leq \Pr\left\{ \sup_{t \in [0,1]} |\mathbb{B}_{n,\theta_0}(t) - \dot{A}_{\theta_0}^\top(t)\Theta_n| > \zeta, \|\Theta_n\| \leq L \right\} + \Pr(\|\Theta_n\| > L) \\ & \leq \Pr\left\{ \sup_{t \in [0,1]} |\mathbb{B}_{n,\theta_0}(t) - \dot{A}_{\theta_0}^\top(t)\Theta_n| > \zeta, \|\Theta_n\| \leq L \right\} + \delta. \end{aligned}$$

An application of the mean value theorem then implies that for every realization ω of the process and every $t \in [0, 1]$, $\mathbb{B}_{n,\theta_0}(t, \omega) = \dot{A}_{\Theta_n^*(t, \omega)}^\top(t)\Theta_n(\omega)$, where $\Theta_n^*(t, \omega) = \theta_0 + \epsilon(t, \omega)n^{-1/2}\Theta_n(\omega)$ for some $\epsilon(t, \omega) \in [0, 1]$. It then follows from condition (6) that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \Pr\left\{ \sup_{t \in [0,1]} |\mathbb{B}_{n,\theta_0}(t) - \dot{A}_{\theta_0}^\top(t)\Theta_n| > \zeta, \|\Theta_n\| \leq L \right\} \\ & \leq \lim_{n \rightarrow \infty} \Pr\left\{ \|\Theta_n\| \sup_{t \in [0,1]} |\dot{A}_{\Theta_n^*(t)}^\top(t) - \dot{A}_{\theta_0}^\top(t)| > \zeta, \|\Theta_n\| \leq L \right\} \\ & \leq \lim_{n \rightarrow \infty} \Pr\left\{ \sup_{\|\theta - \theta_0\| \leq n^{-1/2}L} \sup_{t \in [0,1]} |\dot{A}_\theta(t) - \dot{A}_{\theta_0}(t)| > \zeta/L \right\} = 0. \end{aligned}$$

This completes the argument.

Appendix B: Validity of the parametric bootstrap

To avoid repetitions, let \mathbb{A}_n denote either \mathbb{A}_n^P or $\mathbb{A}_n^{\text{CFG}}$ and let \mathbb{A} stand for either \mathbb{A}^P or \mathbb{A}^{CFG} . The following conditions, adapted from [17], ensure the validity of the parametric bootstrap for computing P -values for the proposed tests.

- (a) The family $\{C_\theta : \theta \in \mathcal{O}\}$ of extreme-value copulas must be such that:
 - (i) the parameter space \mathcal{O} is an open subset of \mathbb{R}^p ;
 - (ii) members of the family are identifiable, that is, for every $\epsilon > 0$,

$$\inf \left\{ \sup_{t \in [0,1]} \|A_\theta(t) - A_{\theta_0}(t)\| : \theta \in \mathcal{O} \text{ and } \|\theta - \theta_0\| > \epsilon \right\} > 0;$$

- (iii) the mapping $\theta \mapsto A_\theta$ is Fréchet differentiable with derivative $\theta \mapsto \dot{A}_\theta$, that is, for all $\theta_0 \in \mathcal{O}$,

$$\lim_{\|h\| \downarrow 0} \sup_{t \in [0,1]} \frac{\|A_{\theta_0+h}(t) - A_{\theta_0}(t) - \dot{A}_{\theta_0}^\top(t)h\|}{\|h\|} = 0;$$

- (iv) C_θ has a Lebesgue density c_θ for all $\theta \in \mathcal{O}$;
- (v) the density c_θ admits first- and second-order derivatives with respect to all components of $\theta \in \mathcal{O}$; the gradient (row) vector with respect to θ is denoted \dot{c}_θ and the Hessian matrix is denoted \ddot{c}_θ ;
- (vi) for arbitrary $(u, v) \in (0, 1)^2$ and every $\theta_0 \in \mathcal{O}$, $\theta \mapsto \dot{c}_\theta(u, v)/c_\theta(u, v)$ and $\theta \mapsto \ddot{c}_\theta(u, v)/c_\theta(u, v)$ are continuous at θ_0 , C_{θ_0} almost surely;
- (vii) for every $\theta_0 \in \mathcal{O}$, there exist a neighborhood \mathcal{N} of θ_0 and a Lebesgue integrable function $h : (0, 1)^2 \rightarrow \mathbb{R}$ such that $\sup_{\theta \in \mathcal{N}} \|\dot{c}_\theta(u, v)\| \leq h(u, v)$ holds for all $(u, v) \in (0, 1)^2$;
- (viii) for every $\theta_0 \in \mathcal{O}$, there exist a neighborhood \mathcal{N} of θ_0 and C_{θ_0} -integrable functions $h_1, h_2 : (0, 1)^2 \rightarrow \mathbb{R}$ such that for all $(u, v) \in (0, 1)^2$,

$$\sup_{\theta \in \mathcal{N}} \left\| \frac{\dot{c}_\theta(u, v)}{c_\theta(u, v)} \right\|^2 \leq h_1(u, v) \quad \text{and} \quad \sup_{\theta \in \mathcal{N}} \left\| \frac{\ddot{c}_\theta(u, v)}{c_\theta(u, v)} \right\| \leq h_2(u, v).$$

- (b) In addition, the estimators A_n and θ_n satisfy the following:

- (i) $(\mathbb{A}_n, \Theta_n, \mathbb{W}_n) \rightsquigarrow (\mathbb{A}, \Theta, \mathbb{W})$ in $\mathcal{D}([0, 1], \mathbb{R}) \times \mathbb{R}^{p \otimes 2}$ as $n \rightarrow \infty$, where the limit is a centered Gaussian process. Here,

$$\mathbb{W}_n = n^{-1/2} \sum_{i=1}^n \frac{\dot{c}_{\theta_0}^\top(U_i^*, V_i^*)}{c_{\theta_0}(U_i^*, V_i^*)}$$

for a random sample $(U_1^*, V_1^*), \dots, (U_n^*, V_n^*)$ from C_{θ_0} and \mathbb{W} is $\mathcal{N}(0, I_p)$, where I_p is the Fisher information matrix; see [17], page 1101.

- (ii) $E_{\theta_0}(\Theta \mathbb{W}^\top) = J$, where J is the $p \times p$ identity matrix. Further, $E_{\theta_0}\{\mathbb{A}(t)\mathbb{W}\} = \dot{A}_{\theta_0}(t)$ for every $t \in (0, 1)$.

Condition (b) can be checked as follows, under the assumption that $(\mathbb{A}_n, \Theta_n) \rightsquigarrow (\mathbb{A}, \Theta)$ as $n \rightarrow \infty$. First, results from Chapter 5 of [12] can be combined with the functional delta method (see, e.g., [37], Section 3.9) to see that as $n \rightarrow \infty$, $(\mathbb{A}_n, \Theta_n, \mathbb{C}_n, \mathbb{W}_n) \rightsquigarrow (\mathbb{A}, \Theta, \mathbb{C}, \mathbb{W})$.

Next, observe that $E_{\theta_0}\{\mathbb{C}(u, v)\mathbb{W}\} = \dot{C}_{\theta_0}(u, v)$ for all $u, v \in [0, 1]$; see [17], page 1108. Given that, for all $t \in [0, 1]$,

$$\mathbb{A}^P(t) = -A_{\theta_0}^2(t) \int_0^1 \mathbb{C}_{\theta_0}(x^{1-t}, x^t) \frac{dx}{x}$$

and

$$\mathbb{A}^{CFG}(t) = A_{\theta_0}(t) \int_0^1 \mathbb{C}_{\theta_0}(x^{1-t}, x^t) \frac{dx}{x \log(x)},$$

we can see that

$$E_{\theta_0}\{\mathbb{A}^P(t)\mathbb{W}\} = -A_{\theta_0}^2(t) \int_0^1 \dot{C}_{\theta_0}(x^{1-t}, x^t) \frac{dx}{x}$$

and

$$E_{\theta_0}\{\mathbb{A}^{CFG}(t)\mathbb{W}\} = A_{\theta_0}(t) \int_0^1 \dot{C}_{\theta_0}(x^{1-t}, x^t) \frac{dx}{x \log(x)}.$$

Interchanging the order of differentiation and integration, we get $E_{\theta_0}\{\mathbb{A}^P(t)\mathbb{W}\} = E_{\theta_0}\{\mathbb{A}^{CFG}(t)\mathbb{W}\} = \dot{A}_{\theta_0}(t)$ for all $t \in (0, 1)$.

As for the condition $E_{\theta_0}(\Theta\mathbb{W}) = J$, it can be verified using [17], Proposition 4, for the estimators based on maximum pseudo-likelihood and on the inversion of Spearman's rho. To handle the estimator based on Kendall's tau, Proposition 5 in [17] must be used instead.

Appendix C: Proof of Proposition 2

The proof closely mimics the argument presented in [19], Appendix B. To avoid duplication, the same notation is used and only the critical differences are highlighted. This also offers an opportunity to correct minor typographical errors in the original source.

First, consider the process given by $\mathbb{B}_n^P(t) = n^{1/2}\{1/A_n^P(t) - 1/A_C^P(t)\}$ for all $t \in [0, 1]$ and show that $\mathbb{B}_n^P \rightsquigarrow \mathbb{B} = -\mathbb{A}_C^P/(A_C^P)^2$ as $n \rightarrow \infty$. Then

$$\sqrt{n}(A_n^P - A_C^P) = \frac{-(A_C^P)^2 \mathbb{B}_n^P}{1 + n^{-1/2} \mathbb{B}_n^P A_C^P} \rightsquigarrow \mathbb{A}_C^P,$$

as a consequence of the functional version of Slutsky's lemma.

Put $k_n = 2 \log(n + 1)$ and write

$$\mathbb{B}_n^P(t) = \int_0^1 \mathbb{C}_n(x^{1-t}, x^t) \frac{dx}{x} = \int_0^\infty \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) ds = I_{1,n} + I_{2,n},$$

where, for each $t \in [0, 1]$,

$$I_{1,n}(t) = \int_{k_n}^{\infty} \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \, ds, \quad I_{2,n}(t) = \int_0^{k_n} \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \, ds.$$

The contribution of $I_{1,n}(t)$ is asymptotically negligible because the fact that $s > k_n$ implies that $\min(e^{-s(1-t)}, e^{-st}) < 1/(n + 1)$ and hence that

$$|\mathbb{C}_n(e^{-s(1-t)}, e^{-st})| = n^{1/2} C(e^{-s(1-t)}, e^{-st}) \leq n^{1/2} \min(e^{-s(1-t)}, e^{-st}) \leq n^{1/2} e^{-s/2}.$$

Thus, for all $t \in [0, 1]$,

$$|I_{1,n}(t)| \leq n^{1/2} \int_{k_n}^{\infty} C(e^{-s(1-t)}, e^{-st}) \, ds \leq n^{1/2} \int_{k_n}^{\infty} e^{-s/2} \, ds \leq \frac{2}{n^{1/2}}. \tag{A.1}$$

Consequently, the asymptotic behavior of \mathbb{B}_n^P is determined entirely by $I_{2,n}$. Invoking the Stute representation given by Genest and Segers [19], we may write $I_{2,n} = J_{1,n} + J_{2,n} + J_{3,n} + o(1)$, where, for each $t \in [0, 1]$,

$$\begin{aligned} J_{1,n}(t) &= \int_0^{k_n} \alpha_n(e^{-s(1-t)}, e^{-st}) \, ds, \\ J_{2,n}(t) &= - \int_0^{k_n} \alpha_n(e^{-s(1-t)}, 1) \dot{C}_1(e^{-s(1-t)}, e^{-st}) \, ds, \\ J_{3,n}(t) &= - \int_0^{k_n} \alpha_n(1, e^{-st}) \dot{C}_2(e^{-s(1-t)}, e^{-st}) \, ds. \end{aligned}$$

Here, $\dot{C}_1(u, v) = \partial C(u, v) / \partial u$, $\dot{C}_2(u, v) = \partial C(u, v) / \partial v$ and α_n is the empirical process associated with the pairs $(F(X_1), G(Y_1)), \dots, (F(X_n), G(Y_n))$.

Fix $\omega \in (0, 1/2)$ and write $q_\omega(t) = t^\omega(1 - t)^\omega$ for all $t \in [0, 1]$. Also, let

$$\begin{aligned} K_1(s, t) &= q_\omega \{ \min(e^{-s(1-t)}, e^{-st}) \}, \\ K_2(s, t) &= q_\omega(e^{-s(1-t)}) \dot{C}_1(e^{-s(1-t)}, e^{-st}), \\ K_3(s, t) &= q_\omega(e^{-st}) \dot{C}_2(e^{-s(1-t)}, e^{-st}) \end{aligned}$$

for all $s \in (0, \infty)$ and $t \in [0, 1]$. The proof that $J_{1,n} + J_{2,n} + J_{3,n}$ has the stated limit then proceeds exactly as in Appendix B of [19], provided that for $i = 1, 2, 3$, there exists an integrable function $K_i^* : (0, \infty) \rightarrow \mathbb{R}$ such that $K_i(s, t) \leq K_i^*(s)$ for all $s \in (0, \infty)$ and $t \in [0, 1]$.

For K_1 , this is immediate because $K_1(s, t) \leq e^{-\omega s/2}$ for all $s \in (0, \infty)$ and $t \in [0, 1]$. For K_2 , the facts that C is LTD and smaller than the Fréchet–Hoeffding upper bound imply that

$$\dot{C}_1(e^{-s(1-t)}, e^{-st}) \leq e^{s(1-t)} C(e^{-s(1-t)}, e^{-st}) \leq e^{s(1-t)} \min(e^{-s(1-t)}, e^{-st}).$$

Now, set $m(t) = \max(t, 1 - t)$ and note that $q_\omega(e^{-s(1-t)}) \leq e^{-\omega s(1-t)}$ for all $s \in (0, \infty)$ and $t \in [0, 1]$. Therefore,

$$K_2(s, t) \leq e^{s(1-\omega)(1-t)} e^{-sm(t)} \leq e^{s(1-\omega)m(t)} e^{-sm(t)} = e^{-s\omega m(t)} \leq e^{-\omega s/2}$$

because $m(t) \geq 1/2$ for all $t \in [0, 1]$. The argument for K_3 is similar.

Turning to the A_n^{CFG} estimator, observe that

$$\begin{aligned} \mathbb{B}_n^{\text{CFG}}(t) &= n^{1/2} \{ \log A_n^{\text{CFG}}(t) - \log A_C^{\text{CFG}}(t) \} \\ &= \int_0^1 \mathbb{C}_n(x^{1-t}, x^t) \frac{dx}{x \log(x)} = - \int_0^\infty \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \frac{ds}{s} \end{aligned}$$

for all $t \in [0, 1]$. This process can be written as $-(I_{1,n} + I_{2,n} + I_{3,n})$, where

$$\begin{aligned} I_{1,n}(t) &= \int_{k_n}^\infty \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \frac{ds}{s}, \\ I_{2,n}(t) &= \int_{\ell_n}^{k_n} \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \frac{ds}{s}, \\ I_{3,n}(t) &= \int_0^{\ell_n} \mathbb{C}_n(e^{-s(1-t)}, e^{-st}) \frac{ds}{s} \end{aligned}$$

with $k_n = 2 \log(n + 1)$ as above and $\ell_n = 1/(n + 1)$.

Arguing as in (A.1), we see that $|I_{1,n}| \leq n^{-1/2}$. Similarly, $I_{3,n}$ is negligible asymptotically, for if $s \in (0, \ell_n)$ and $t \in [0, 1]$, then we have

$$\min(e^{-s(1-t)}, e^{-st}) \geq e^{-1/(n+1)} > \frac{n}{n+1}$$

and hence $\mathbb{C}_n(e^{-s(1-t)}, e^{-st}) = 1$. Furthermore, the fact that C is LTD implies that $C(e^{-s(1-t)}, e^{-st}) \geq e^{-s}$ for all $s \in (0, \infty)$ and $t \in [0, 1]$. Therefore,

$$|\mathbb{C}_n(e^{-s(1-t)}, e^{-st})| \leq n^{1/2}(1 - e^{-s}) \leq n^{1/2}s.$$

Consequently, $|I_{3,n}| \leq n^{1/2}\ell_n \leq n^{-1/2}$. As a result, the asymptotic behavior of $\mathbb{B}_n^{\text{CFG}}$ is determined entirely by $I_{2,n}$. Following Genest and Segers [19], we can further write $I_{2,n} = J_{1,n} + J_{2,n} + J_{3,n} + o(1)$, where, for all $t \in [0, 1]$,

$$\begin{aligned} J_{1,n}(t) &= \int_{\ell_n}^{k_n} \alpha_n(e^{-s(1-t)}, e^{-st}) \frac{ds}{s}, \\ J_{2,n}(t) &= - \int_{\ell_n}^{k_n} \alpha_n(e^{-s(1-t)}, 1) \dot{C}_1(e^{-s(1-t)}, e^{-st}) \frac{ds}{s}, \\ J_{3,n}(t) &= - \int_{\ell_n}^{k_n} \alpha_n(1, e^{-st}) \dot{C}_2(e^{-s(1-t)}, e^{-st}) \frac{ds}{s}. \end{aligned}$$

The joint asymptotic behavior of these terms can be determined in the same way as before. The only difference is that the integration measure is now ds/s . For $s \in [1, \infty)$, the same upper bounds K_1^* , K_2^* , K_3^* apply and they have already been shown to be integrable on this domain. To obtain an integrable bound for K_1 on $(0, 1)$, it suffices to use the fact that $K_1(s, t) \leq (1 - e^{-sm(t)})^\omega \leq \{sm(t)\}^\omega \leq s^\omega$. The same bound works for both K_2 and K_3 because $\dot{C}_i \in [0, 1]$ for $i = 1, 2$. This completes the argument.

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