

A PRIME GEODESIC THEOREM FOR HIGHER RANK BUILDINGS

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Abstract

We prove a prime geodesic theorem for compact quotients of affine buildings and apply it to get class number asymptotics for global fields of positive characteristic.

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Introduction

The prime geodesic theorem for a compact, negatively curved Riemannian manifold M states that the number $\pi(x)$ of prime closed geodesics of length $\leq x$ on M satisfies

$$\pi(x) \sim \frac{e^{Ax}}{Bx},$$

where $A, B > 0$ are (explicit) constants depending on the manifold. It has been applied in [14] to arithmetic quotients of symmetric spaces to derive class number asymptotics. In case of hyperbolic spaces, remainder terms have been given, see for instance [8, 15]. In [2] this has been extended to a case of higher rank and in [3] to a non-compact situation. The full higher rank case has been explored in [4, 5].

The p -adic counterpart of symmetric spaces are Bruhat-Tits buildings. In the rank one case, these are trees. In [6] we applied the Prime Geodesic Theorem for graphs to get class number asymptotics for orders over imaginary

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quadratic fields. See [12] for a different approach to this case. In the current paper we derive the Prime Geodesic Theorem for quotients of buildings and we use it for class number asymptotics for global fields of positive characteristic. The result is an equidistribution assertion for units in orders over a given global field, where the units are weighted by class numbers and regulators.

1. The building

In this section we recall some results of [7]. Let X be a locally finite, simplicial, thick, affine building of dimension $d = \dim X$. Recall that a *type function* on a chamber is a bijection from the set $\text{vert}(C)$ of vertices of C to the set $\{0, 1, \dots, d\}$. Further, a *type function* on X is a map from the set $\text{vert}(X)$ of vertices of X to $\{0, \dots, d\}$ that restricts to a type function on each chamber.

Given a chamber C the restriction defines a bijection between the set of all type functions on X and the set of all type functions on C . An automorphism g of X is said to be *type preserving*, if g preserves one (and hence any) type on X .

Let $\text{Aut}(X)$ denote the group of automorphisms of the building X , i.e., the group of automorphisms of the complex mapping apartments to apartments. The group $\text{Aut}(X)$ is a totally disconnected locally compact group where a basis of unit-neighborhoods is given by the set of all pointwise stabilizers of finite sets in X . Let

$$G \subset \text{Aut}(X)$$

be an open, finite index subgroup. Note that any open subgroup is closed and that any finite index closed subgroup is open. Let $G^0 \subset G$ be the subgroup of all type preserving automorphisms in G . It is normal in G and G/G^0 is a finite group.

We assume that G^0 acts *strongly transitively*, i.e., G^0 acts transitively on the set of all pairs (\mathfrak{a}, C) , where \mathfrak{a} is an apartment of X and C is a chamber in \mathfrak{a} . See [1], Chapter 6. This is equivalent to saying that G^0 acts chamber-transitively on X and the pointwise stabilizer K_C of a given chamber C acts transitively on the set of all apartments \mathfrak{a} containing C .

Let ∂X be the visibility boundary of X and let $\mathcal{C} \subset \partial X$ be a spherical chamber and fix an apartment $\mathfrak{a} \subset X$ such that $\mathcal{C} \subset \partial \mathfrak{a}$. Let $P = P_{\mathcal{C}}$ be the point-wise stabilizer of \mathcal{C} , then P is called a *minimal parabolic* subgroup of G . In the boundary $\partial \mathfrak{a}$, there is a unique chamber $\bar{\mathcal{C}}$ opposite to \mathcal{C} . Let \bar{P} denote the corresponding parabolic and set

$$L = P \cap \bar{P}.$$

We call L a *Levi-component* of P . Let M denote the point-wise stabilizer of \mathfrak{a} , then M is normal in L and the quotient $A = L/M$ is free abelian of rank d . (Note that in [7] we wrote “rank $\leq d$ ”, this discrepancy is due to the fact that we are now assuming G^0 to act strongly transitively, which we did not assume in [7].) The group A acts on \mathfrak{a} by translations. Let A^- denote the open cone in A of all elements translating towards a point in the open spherical chamber $\bar{\mathcal{C}}$.

Fix a fundamental chamber C_0 in \mathfrak{a} and let $K \subset G$ denote its *pointwise* stabilizer. As $M \subset K$ it makes sense to write KAK for the set KLK .

DEFINITION 1.1. Since X is affine, there exists a metric d on X which is euclidean on each apartment and G -invariant. For $g \in G$ let

$$d(g) = \inf_{x \in X} d(gx, x).$$

As G acts cellularly, this infimum is always attained, hence a minimum. An element $g \in G$ is called *hyperbolic*, if $d(g) > 0$, the minimum is attained on a convex subset which is the union of parallel lines along which g is a translation [7]. This set is called the *minimal set* of g and is written as $\text{Min}(g)$. We call g a *generic* element, if the so defined element of ∂X is generic, i.e., is not in a lower dimensional stratum. This implies that the minimal set $\text{Min}(g)$ consists of exactly one apartment.

2. Closed geodesics

DEFINITION 2.1. A *geodesic curve* in X is a curve $c : \mathbf{R} \rightarrow X$, which in the euclidean structure of each apartment is a straight line which is parametrized at unit speed. Two curves c, c' are called equivalent if there exists $t_0 \in \mathbf{R}$ such that $c'(t) = c(t + t_0)$ holds for all $t \in \mathbf{R}$. A *geodesic* in X is an equivalence class of geodesic curves.

DEFINITION 2.2. Let $\Gamma \subset G$ be a discrete, cocompact subgroup of G . By a *geodesic in $\Gamma \backslash X$* we understand the image of a geodesic in X . This is the proper definition in the case when Γ is allowed to have torsion elements and thus the local euclidean structure is not necessarily preserved in the quotient $\Gamma \backslash X$. A geodesic c in $\Gamma \backslash X$ is called a *closed geodesic*, if $c(t + l) = c(t)$ holds for some $l > 0$ and all $t \in \mathbf{R}$. A closed geodesic c in $\Gamma \backslash X$ thus lifts to a geodesic \tilde{c} in X . The forward direction of \tilde{c} defines a point $b \in \partial X$. We call c a *generic geodesic*, if b is a generic point of the spherical building ∂X , i.e., if b is not contained in any wall of a Weyl chamber.

For a given closed geodesic c there exists $\gamma \in \Gamma$ closing it, i.e., if $c(t + l) = c(t)$ for all $t \in \mathbf{R}$ and some $l > 0$, then $\tilde{c}(t + l) = \gamma \tilde{c}(t)$ holds for a uniquely determined $\gamma \in \Gamma$ and all $t \in \mathbf{R}$. This element γ is hyperbolic and \tilde{c} lies in its minimal set $\text{Min}(\gamma)$.

- LEMMA 2.3. Let $A^0 = A \cap G^0$, or rather $A^0 = L^0/M$, where $L^0 = L \cap G^0$.
- (a) The group G^0 is generated by the set KA^0K . Likewise, it is generated by the union of all pointwise stabilizers K_C of chambers C in X .
 - (b) If $\gamma \in \Gamma$ closes a generic geodesic, then γ is hyperbolic and conjugate in G to an element of KAK . In particular, γ is generic (Definition 1.1).
 - (c) Let $\theta : G^0 \rightarrow H$ be a group homomorphism with $\theta(K) = 1$. Then $\theta \equiv 1$.

Proof. (a) Let C_0 be the fundamental chamber and let $g \in G^0$. If $gC_0 = C_0$ then g lies in K already and we are done. Otherwise, we have to show that the group H generated by KAK contains an element h such that $gC_0 = hC_0$, because then $h^{-1}g \in K$ and (c) is proven. So we have to show that HC_0 contains all chambers of X .

We start by showing that it contains all direct neighbors C of C_0 . Choose a labeling v_0, \dots, v_d of the vertices of C_0 . Let C be a neighbor sharing the face F with vertices v_1, \dots, v_d with C_0 . As X is thick, there is another such neighbor D of C_0 . Let α be an apartment containing C_0 and D . Let s be the reflection at F in α . Let T be the Weyl translation with $Tv_0 = w_0 = s(v_0)$ and let E be the chamber $T(C_0)$. The translation T extends to an element of $KAK \subset H$, so that the group TKT^{-1} lies in H . This group, however, is the pointwise stabilizer K_E of the chamber E . Let now α' be an apartment containing E and C , then there is $k \in K_E \subset H$ with $k\alpha = \alpha'$, hence $kC_0 = C$ as claimed. This proof can now be iterated to show that HC_0 contains the direct neighbors of direct neighbors of C_0 and so on. Examining the proof, one finds that we have shown that the group generated by all stabilizers of chambers generates G .

(b) If γ closes a generic geodesic, it must be hyperbolic and its minimal set is an apartment α' , [7]. So γ induces a translation on this apartment, which makes it an element of $K'A'K$ where K' is the stabilizer of a chamber in α' . Now the set $K'A'K'$ is G -conjugate to KAK and the claim follows.

(c) If $\theta(K) = 1$, then $\theta(gKg^{-1}) = 1$ for every $g \in G^0$, but these groups generate G^0 by part (a), so $\theta \equiv 1$. □

3. The zeta function

The zeta function $S(u)$, which is recalled in this section, is a several variable zeta function $S(u)$ constructed from geometrical data of closed geodesics and enjoying an analytic continuation to all of \mathbf{C}^r .

Let $\gamma \in \Gamma$ be a generic element as in Definition 1.1. The minimal set $\text{Min}(\gamma)$ is an apartment α . Let G_α denote the stabilizer of α in G and let G_γ and Γ_γ denote the respective centralizers of γ . Let G^α denote the image of G_α in $\text{Aut}(\alpha)$ and let $\Gamma_\gamma^\alpha \subset G_\gamma^\alpha \subset G^\alpha$ denote the image of the groups Γ_γ and G_γ in G^α . Then the set $G_\gamma^\alpha/\Gamma_\gamma^\alpha$ is finite. Note that, as G^0 acts strongly transitively, G^α contains the Weyl group of α .

For a measurable set $M \subset G$ we set

$$[\gamma : M] = \text{vol}(\{x \in G/G_\gamma : x\gamma x^{-1} \in M\}).$$

In [7] it is shown that the index $[\gamma : KA^{-1}K]$ is a natural number.

LEMMA 3.1. *For an element $\gamma \in G$ the following are equivalent:*

- (a) γ is generic,
- (b) γ is conjugate to an element of KaK with $a \in A^-$.

In this case the element a is uniquely determined by γ and the index satisfies $[\gamma : KA^{-1}K] = [\gamma : KaK] = 1$.

Proof. If γ is generic, then, as G^0 acts strongly transitively, modulo conjugation we can assume $\text{Min}(\gamma) = \mathfrak{a}$ and thus $\gamma = ak$ for some generic $a \in A$ and $k \in K$. As the image of $G^{\mathfrak{a}}$ in $\text{Aut}(\mathfrak{a})$ contains the Weyl group, we can conjugate a into A^- . This establishes (a) \Rightarrow (b).

The assertion (b) \Rightarrow (a) and the uniqueness of a is valid without the assumption of strong transitivity and is proven in Lemma 2.3.8 of [7]. The assertion about the index rests on strong transitivity. With the given normalizations,

$$\begin{aligned} [\gamma : KaK] &= \text{vol}(\{x \in G/G_\gamma : x\gamma x^{-1} \in KaK\}) \\ &= |\{a \in K \setminus G/G_\gamma : x\gamma x^{-1} \in KaK\}|. \end{aligned}$$

So we need to show that if $\gamma, x\gamma x^{-1}$ both lie in KaK , then $x \in KG_\gamma$. If $\gamma, x\gamma x^{-1} \in KaK$, we can replace both with K -conjugates to get them into aK . This means that $\gamma C_0 = x\gamma x^{-1} C_0 = aC_0$, so C_0, aC_0 lie in $\text{Min}(\gamma)$ and in $\text{Min}(x\gamma x^{-1}) = x \text{Min}(\gamma)$. That means that $\mathfrak{b} = \text{Min}(\gamma)$ and $x\mathfrak{b}$ are apartments containing C_0 . By strong transitivity, K acts transitively on these apartments, therefore there are $k_1, k_2 \in K$ with

$$\mathfrak{a} = k_1 \text{Min}(\gamma) = \text{Min}(k_2\gamma k_1^{-1}) = k_2x \text{Min}(\gamma) = \text{Min}(k_2x\gamma x^{-1}k_2^{-1}).$$

Replacing γ with $k_1\gamma k_1^{-1}$ and x with k_2x we can assume that γ and $x\gamma x^{-1}$ preserve \mathfrak{a} and both act as the same translation $y \mapsto ay$ on \mathfrak{a} . Then $\mathfrak{a} = \text{Min}(x\gamma x^{-1}) = x \text{Min}(\gamma) = x\mathfrak{a}$, so x preserves \mathfrak{a} as well. As $x\gamma x^{-1}$ and γ act as the same generic translation on \mathfrak{a} , x itself acts by a translation on \mathfrak{a} . There is an element $y \in G_\gamma$ acting by the same translation as x , so $xy^{-1} \in K$. The claim follows. \square

DEFINITION 3.2. Let Γ^{gen} denote the set of generic elements in Γ and let $[\Gamma^{\text{gen}}]$ denote the set of Γ -conjugacy classes in Γ^{gen} . Let \mathfrak{a} denote the apartment used to define A . Let v_0 be a special vertex in \mathfrak{a} and let C be the unique chamber in \mathfrak{a} with vertex v_0 such that the wall W of C which is opposite to v_0 , faces $\overline{\mathcal{C}}$. Let v_1, \dots, v_d denote the remaining vertices of C . The map $v_j \mapsto j$ extends in a unique way to a map from the set $V(X)$ of vertices of X to $\{0, 1, \dots, d\}$ which is injective on the set $V(D)$ of vertices of any given chamber D . The image of a vertex is called the *type* of the vertex. Then all vertices of type zero are special vertices.

Using v_0 as origin we give \mathfrak{a} the structure of a real vector spaces and v_1, \dots, v_d is a basis. Let $e_j = r_j v_j$, where $r_j > 0$ is the smallest rational number such that all vertices of type zero are contained in

$$Z = \mathbf{Z}e_1 \oplus \dots \oplus \mathbf{Z}e_d.$$

A given $a \in A$ acts on \mathfrak{a} by translation $ax = x + v_a$ where

$$v_a = \lambda_1(a)e_1 + \dots + \lambda_d(a)e_d$$

is the translation vector. Since this translation respects the simplicial structure, the numbers $\lambda_1(a), \dots, \lambda_d(a)$ are integers. Indeed, the map

$$\begin{aligned} \lambda : A &\mapsto \mathbf{Z}^d, \\ a &\mapsto (\lambda_1(a), \dots, \lambda_d(a)) \end{aligned}$$

is an isomorphism of the group A to a lattice $\Lambda = \Lambda_A \subset \mathbf{Z}^d$, which maps the cone A^- to the cone

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda_1, \dots, \lambda_d > 0\}.$$

DEFINITION 3.3. For $u \in \mathbf{C}^d$ and $a \in A^-$ we write

$$u^a = u_1^{\lambda_1(a)} \dots u_d^{\lambda_d(a)},$$

and define the several variable zeta function

$$S(u) = \sum_{[\gamma] \in [\Gamma^{\text{gen}}]} |G_\gamma^a / \Gamma_\gamma^a| u^{a_\gamma},$$

where $a_\gamma \in A^-$ is the unique element such that γ is G -conjugate to an element of $Ka_\gamma K$. Theorem 2.4.2 of [7] states that the series $S(u)$ converges for small u to a rational function. More precisely, there exists a finite set $E \subset A$, elements $a_1, \dots, a_d \in A^-$ and quasi-characters $\eta_1, \dots, \eta_r : A \rightarrow \mathbf{C}^\times \cup 0$ such that

$$S(u) = \sum_{j=1}^r \sum_{e \in E} \frac{\eta_j(e) u^e}{(1 - \eta_j(a_1) u^{a_1}) \dots (1 - \eta_j(a_d) u^{a_d})}.$$

Moreover, the space $L^2(\Gamma \backslash G)^K \cong L^2(\Gamma \backslash G / K)$ has a basis ϕ_1, \dots, ϕ_r such that all $R(\mathbf{1}_{KaK})$ with $a \in A^-$ are in Jordan normal form with respect to this basis. In particular,

$$R(\mathbf{1}_{KaK})\phi_j - \eta_j(a)\phi_j \in \text{span}(\phi_1, \dots, \phi_{j-1})$$

holds for every j and all $a \in A^-$ and this equation defines the quasi-character η_j . Here R is the right translation representation of G on $L^2(\Gamma \backslash G)$, so $R(y)\phi(x) = \phi(xy)$, $x, y \in G$ and for a function f on G (like $f = \mathbf{1}_{KaK}$) we define $R(f)$ by integration:

$$R(f)\phi(x) = \int_G f(y)\phi(xy) dy.$$

Note that the space $L^2(\Gamma \backslash G)^K$ contains the constant function $\phi \equiv 1$. Then

$$R(\mathbf{1}_{KaK})\phi(x) = \int_G \mathbf{1}_{KaK}(y)\phi(xy) dy = \text{vol}(KaK) = |KaK/K|,$$

since we normalize the Haar measure by giving the compact open subgroup K volume 1. We can assume $\phi_1 \equiv 1$.

At this point we note that we have a certain amount of freedom in choosing the group $G \subset \text{Aut}(X)$. By changing G if necessary, we can assume that G is

generated by G^0 together with Γ . Since Γ and G^0 are subgroups and G^0 is a normal subgroup, this means that we have

$$G = \Gamma G^0,$$

i.e., every $g \in G$ can be written as a product $g = \gamma g^0$ of some $\gamma \in \Gamma$ and an element g^0 of G^0 .

LEMMA 3.4. *For $j = 1, \dots, r$ we have*

$$|\eta_j(a)| \leq |KaK/K|, \quad a \in A^-.$$

For every $j \geq 2$ there exists $a \in A^-$ with

$$|\eta_j(a)| < |KaK/K|.$$

Proof. For $1 \leq j \leq r$ we have

$$|R(\mathbf{1}_{KaK})\phi(x)| \leq \int_{KaK} |\phi(xy)| \, dy \leq |KaK/K| \|\phi\|_{\Gamma \backslash G},$$

where $\|\phi\|_{\Gamma \backslash G} = \sup_{x \in \Gamma \backslash G} |\phi(x)|$. Suppose that $R(\mathbf{1}_{KaK})\phi = \eta(a)\phi$ holds for every $a \in A^-$, then, taking supremum over x yields

$$|\eta(a)| \|\phi\|_{\Gamma \backslash G} \leq |KaK/K| \|\phi\|_{\Gamma \backslash G}.$$

For the second assertion assume additionally $|\eta(a)| = |KaK/K| = \eta_1(a)$ for every $a \in A^-$. We have to show that ϕ is constant. Let $a \in A^-$ and write $KaK = \bigsqcup_{j=1}^s k_j a K$, then $s = |KaK/K| = \eta_1(a)$ and

$$R(\mathbf{1}_{KaK})\phi = \sum_{j=1}^s R(k_j)R(a)\phi.$$

For the L^2 -norm we have, since $R(k_j)R(a)$ is unitary,

$$\begin{aligned} s\|\phi\|_2 &= |\eta(a)| \|\phi\|_2 = \left\| \sum_{j=1}^s R(k_j)R(a)\phi \right\|_2 \\ &\leq \sum_{j=1}^s \|\phi\|_2 = s\|\phi\|_2. \end{aligned}$$

We get equality everywhere, but this can only happen if the $R(k_j)R(a)\phi$ are all \mathbf{R}_+ -collinear and by unitarity this means that $R(k_j)R(a)\phi = R(a)\phi$ for every j . Since $s = \eta_1(a)$, follows

$$R(a)\phi = \frac{\eta(a)}{\eta_1(a)} \phi.$$

As ϕ is K -invariant, we have for $k_1, k_2 \in K$,

$$R(k_1 a k_2) \phi = R(k_1) R(a) R(k_2) \phi = R(k_1) R(a) \phi = \frac{\eta(a)}{\eta_1(a)} R(k_1) \phi = \frac{\eta(a)}{\eta_1(a)} \phi.$$

Since R is a representation, the map $\theta = \frac{\eta}{\eta_1}$ therefore extends to a character on the group generated by KA^-K , which contains G^0 . By Lemma 2.3 we get $\theta(G^0) = 1$, so for every $g^0 \in G^0$ we have

$$\phi(g^0) = R(g_0) \phi(1) = \theta(g_0) \phi(1) = \phi(1).$$

Finally, as $G = \Gamma G^0$ we write any given $g \in G$ as $g = \gamma g^0$ accordingly and we get

$$\phi(g) = \phi(\gamma g^0) = \phi(g^0) = \phi(1).$$

So ϕ is constant, as claimed. □

4. The prime geodesic theorem

DEFINITION 4.1. For $\gamma \in \Gamma^{\text{gen}}$ we write $\text{Ind}(\gamma) = |G_\gamma^\alpha / \Gamma_\gamma^\alpha|$, where $\alpha = \text{Min}(\gamma)$. For $k \in \mathbf{N}^d$ let

$$N(k) = \sum_{[\gamma]: \lambda(a_\gamma) = k} \text{Ind}(\gamma),$$

where the sum runs over all conjugacy classes $[\gamma]$ in Γ^{gen} such that $\lambda(a_\gamma) = k$. So $N(k) = 0$ if $k \notin \Lambda$, the lattice of Definition 3.2.

In the following, for $k \in \mathbf{N}^d$ and $c \in \mathbf{R}^d$ we write

$$c^k = c_1^{k_1} \dots c_d^{k_d}.$$

THEOREM 4.2 (Prime Geodesic Theorem). *Let $\Lambda \subset \mathbf{Z}^d$ be the lattice of Definition 3.2. There exists a sub-lattice $\Lambda' \subset \Lambda$ and a function $C_{\Lambda/\Lambda'} : \Lambda/\Lambda' \rightarrow (0, \infty)$, and constants $c_1, \dots, c_d > 1$ such that for $k_j \rightarrow \infty$ independently, we have*

$$N(k) \sim \mathbf{1}_\Lambda(k) C_{\Lambda/\Lambda'}(\bar{k}) c^k.$$

Explicit formulae for the constants and the function $C_{\Lambda/\Lambda'}$, are given below.

Proof. We have

$$\begin{aligned} S(u) &= \sum_{k \in \mathbf{N}^d} N(k) u^k \\ &= \sum_{j=1}^r \sum_{e \in E} \sum_{m \in \mathbf{N}_0^d} \eta_j(e) \eta_j(a_1)^{m_1} \dots \eta_j(a_d)^{m_d} u^{\lambda(m-a+e)}, \end{aligned}$$

where $m \cdot a$ stands for $m_1 a_1 + \dots + m_d a_d$. This implies

$$N(k) = \sum_{j=1}^r \underbrace{\sum_{\substack{e \in E \\ m \in \mathbf{N}^d \\ \lambda(a \cdot m + e) = k}} \eta_j(e) \eta_j(a_1)^{m_1} \dots \eta_j(a_d)^{m_d}}_{=: N_j(k)}.$$

We shall show that the term $N_1(k)$ dominates as $k \rightarrow \infty$. To simplify the notation we write $\eta(x) = \eta_1(x) = |KxK/K|$. The proof of Lemma 2.4.4 in [7] reveals that each a_j is a multiple of v_j and hence the equation $\lambda(a \cdot m + e) = k$ implies $\lambda_v(a_v)m_v + \lambda_v(e) = k_v$, $v = 1, \dots, d$, so that

$$m_v = \frac{k_v - \lambda_v(e)}{\lambda_v(a_v)}.$$

Let Λ' be the sub-lattice of Λ generated by $\lambda(a_1), \dots, \lambda(a_d)$. For each $k \in \mathbf{N}^d$ there is at most one $e \in E$ with $\lambda(a \cdot m + e) = k$. This element e only depends on k up to Λ' . We get

$$N_1(k) = \eta(a_1)^{k_1/\lambda_1(a_1)} \dots \eta(a_d)^{k_d/\lambda_d(a_d)} \sum_{\substack{e \in E \\ m \in \mathbf{N}^d \\ \lambda(a \cdot m + e) = k}} \eta(e) \eta(a_1)^{\lambda_1(e)/\lambda_1(a_1)} \dots \eta(a_d)^{\lambda_d(e)/\lambda_d(a_d)}.$$

The sum

$$C_{\Lambda/\Lambda'}(k) = \sum_{\substack{e \in E \\ m \in \mathbf{N}^d \\ \lambda(a \cdot m + e) = k}} \eta(e) \eta(a_1)^{\lambda_1(e)/\lambda_1(a_1)} \dots \eta(a_d)^{\lambda_d(e)/\lambda_d(a_d)}$$

has at most one summand and depends on k only up to Λ' and is non-zero if and only if $k \in \Lambda$. Setting $c_v = \eta(a_v)^{1/\lambda_v(a_v)}$ we get the desired asymptotic for $N_1(k)$ instead of $N(k)$.

By Lemma 3.4, for each $j = 2, \dots, r$ there exists $v(j)$ such that $|\eta_j(a_{v(j)})| < \eta(a_{v(j)})$. Set

$$\theta = \max_{j \geq 2} \frac{|\eta_j(a_{v(j)})|}{\eta(a_{v(j)})} < 1.$$

It then follows that

$$|N_2(k) + \dots + N_r(k)| \leq \left(\sum_{j=2}^r \theta^{k_{v(j)}} \right) N_1(k).$$

This implies that $N_1(k)$ dominates the asymptotic and thus we get the claim for $N(k)$. □

5. Division algebras

Let R be an integral domain and K its field of fractions. Let A be a finite-dimensional K -algebra with unit. An R -order in A is an R -sub-algebra Λ of A , which is finitely generated as R -module and spans the K -vector space A , i.e., $K\Lambda = A$.

Now assume that K is a global field of positive characteristic and $R \subset K$ is a Dedekind domain with fraction field K . Then K is the function field of a curve \mathcal{C} over a finite field k . An example of a possible ring R would be the coordinate ring of the affine curve $\mathcal{C} \setminus \{\infty\}$, where $\infty \in \mathcal{C}$ is a rational closed point.

If the K -algebra A is a global field F over K and \mathcal{O} is an R -order in F , let $I(\mathcal{O})$ denote the set of all finitely generated \mathcal{O} -submodules of F . By the Jordan-Zassenhaus Theorem [13], the set $[I(\mathcal{O})]$ of isomorphism classes of elements of $I(\mathcal{O})$ is finite. Let $h(\mathcal{O})$ be its cardinality, called the *class number* of the order \mathcal{O} .

If F is a global field over K , there is a maximal R -order \mathcal{O}_F , which is the integral closure of R in F , and which contains any other order in F . The same applies in the local situation, if F is a finite extension of K_v for a place v , the integral closure \mathcal{O}_F of R_v is a maximal R_v -order containing every other order. We say that an R -order \mathcal{O} of F is *maximal at v* , if $\mathcal{O} \otimes_R R_v$ is the maximal order of F_v .

Let $d \in \mathbb{N}$ be such that $d + 1$ is a prime number and let D denote a division algebra over K [11] of dimension $(d + 1)^2$. Let $D(R)$ denote a fixed maximal R -order in D . Note that all maximal orders in D are conjugate [13]. For any R -algebra A we define

$$D(A) = D(R) \otimes_R A.$$

Then $D(K)$ is canonically isomorphic to D . For almost all places v , one has $D(K_v) \cong M_d(K_v)$, where $M_d(E)$ denotes the algebra of $d \times d$ -matrices over a field E , [11]. If $D(K_v) \cong M_d(K_v)$, we say that D *splits at p* . If D doesn't split at p , then $D(K_v)$ is a division algebra over K_v . This latter fact rests on the choice of the degree d to be a prime.

Let S be the finite set of all places, at which D doesn't split.

LEMMA 5.1. *Let $A \subset D$ be a K -subalgebra. Then the dimension of A is 1, d or d^2 . In the first case $A = K$, in the last $A = D$. In the remaining case A is a field extension of K of degree d , such that every place $v \in S$ is non-decomposed in A , i.e., there is only one place of A above v . Every field of degree d over K , satisfying these conditions occurs as a subalgebra of D .*

Proof. This lemma is standard. It can be pieced together from the information in Pierce's book [11]. □

Let v be a place not in S . Let F/K be a field extension of degree d which embeds into $D(K)$. Then for any embedding $\sigma : F \hookrightarrow D(K)$ the set

$$\mathcal{O}_\sigma = \sigma^{-1}(D(R))$$

is an R -order in F .

LEMMA 5.2. *Let v be a place not in S . Let $\sigma : F \hookrightarrow D(K)$ be a K -embedding of the degree d field F/K . Then for any $w \in S$, the order $\mathcal{O}_{\sigma,w} = \mathcal{O}_\sigma \otimes_R R_w$ is maximal in the local field F_w . Conversely, let $\mathcal{O} \subset F$ be an R -order such that for any $w \in S$ the order \mathcal{O}_w is maximal, then there exists an embedding σ such that $\mathcal{O} = \mathcal{O}_\sigma$.*

Proof. Analogous to the proof of Lemma 2.2 of [2]. □

For a given degree d field extension F/K which embeds into $D(K)$, let $S_{\text{inert}}(F)$ be the set of all $w \in S$ which are inert in F . Define the S -inertia degree by

$$f_S(F) = \prod_{w \in S} f_w(F) = d^{|S_{\text{inert}}(F)|},$$

where $f_w(F)$ is the inertia degree of w in F . For any order \mathcal{O} of F let

$$f_S(\mathcal{O}) = f_S(F).$$

Let \mathcal{O} be a R -order in F , which is maximal at all $w \in S$. By Lemma 5.2 there exists an embedding $\sigma : F \rightarrow D(K)$ such that $\mathcal{O} = \mathcal{O}_\sigma$. Let $u \in D(R)^\times$ and let ${}^u\sigma$ be the embedding given by ${}^u\sigma(x) = u\sigma(x)u^{-1}$. Then $\mathcal{O}_{{}^u\sigma} = \mathcal{O}_\sigma$, so the group $D(R)^\times$ acts on the set $\Sigma(\mathcal{O})$ of all σ with $\mathcal{O}_\sigma = \mathcal{O}$.

LEMMA 5.3. *The quotient $\Sigma(\mathcal{O})/D(R)^\times$ is finite and has cardinality*

$$|\Sigma(\mathcal{O})/D(R)^\times| = f_S(\mathcal{O})h(\mathcal{O}).$$

Compare Lemma 2.3 in [2].

Proof. Fix an embedding $F \hookrightarrow D(K)$ and consider F as a subfield of $D(K)$ such that $\mathcal{O} = F \cap D(R)$. For $u \in D(K)^\times$ let

$$\mathcal{O}_u = F \cap u^{-1}D(R)u.$$

Let U be the set of all $u \in D(K)^\times$ such that $\mathcal{O}_u = \mathcal{O}$, i.e.,

$$F \cap D(R) = F \cap u^{-1}D(R)u.$$

Then F^\times acts on U by multiplication from the right and $D(R)^\times$ acts by multiplication from the left. One has

$$|D(R)^\times \backslash U / F^\times| = |D(R)^\times \backslash \Sigma(\mathcal{O})|.$$

So we have to show that the left hand side equals $f_S(\mathcal{O})h(\mathcal{O})$. For $u \in U$ let

$$I_u = F \cap D(R)u.$$

Then I_u is a finitely generated \mathcal{O} -module in F . We claim that the map

$$\begin{aligned} \psi : D(R)^\times \setminus U/F^\times &\rightarrow I(\mathcal{O})/F^\times, \\ u &\mapsto I_u \end{aligned}$$

is surjective and $h(\mathcal{O})$ to one. We show this by localization and strong approximation. For any place $w \neq v$ let U_w be the set of all $u_w \in D(K_w)$ such that $\mathcal{O}_w = F_w \cap D(R_w) = F_w \cap u_w^{-1}D(R_w)u_w$. We have to show the following:

- (a) For $w \notin S$, the localized map $\psi_w : D(R_w)^\times \setminus U_w/F_w^\times \rightarrow I(\mathcal{O}_w)/F_w^\times$ is injective,
- (b) for $w \in S$, the map ψ_w is $f_w(F)$ to one,
- (c) the map ψ is surjective.

For (a) let $w \notin S$, $u_w, v_w \in U_w$ and assume

$$F_w \cap D(R_w)u_w = F_w \cap D(R_w)v_w.$$

Let $z_w = v_w u_w^{-1}$. Elementary divisor theory implies that there exist $x, y \in D(R_w)^\times = M_d(R_w)^\times$ such that

$$z_w = x \operatorname{diag}(\pi^{k_1}, \pi^{k_2})y$$

holds, where $k_1 \leq k_2$ and π is a uniformizer for w . Replacing u_w by yu_w and v_w by $x^{-1}v_w$ we may assume that z_w equals the diagonal matrix. The assumptions then imply $k_1 = 0 = k_2$, which gives the first claim. For (b) let $w \in S$ and recall that F_w is a local field, so $h(\mathcal{O}_w) = 1$. Hence the claim is equivalent to

$$|D(R_w)^\times \setminus D(K_w)^\times / F_w^\times| = f_w(F).$$

Taking the valuation w of the reduced norm, one sees that the left hand side equals d if F_w is unramified over K_w and 1 otherwise, i.e., it equals the inertia degree $f_w(F)$ as claimed.

Finally, for the surjectivity of ψ let $I \subset \mathcal{O}$ be an ideal. We show that there is $u \in D(K)^\times$ such that

$$F \cap u^{-1}D(R)u = F \cap D(R)$$

and

$$I = I_u = F \cap D(R)u.$$

We do this locally. First note that, since I is finitely generated, there is a finite set T of places with $T \cap S = \emptyset$ and $v \notin T$ such that for any $w \notin T \cup S$ the completion I_w equals \mathcal{O}_w which is the maximal order of F_w . For these w set $\tilde{u}_w = 1$. Next let $w \in S$ and write v_w for the unique place of F over w . Then \mathcal{O}_w is maximal, so is the valuation ring to v_w and $I_w = \pi_w^k \mathcal{O}_w$ for some $k \geq 0$, where π_w is a uniformizer at v_w . In this case set $\tilde{u}_w = \pi_w^k$.

Next let $w \in T$. Then $D(R_w) = M_d(R_w)$. Let $\overline{\mathcal{O}_w} = \mathcal{O}_w/\pi_w\mathcal{O}_w$ and $\overline{I_w} = I_w/\pi_w I_w$. Then $\overline{\mathcal{O}_w}$ is a commutative algebra over the field $\mathbf{F}_w = R_w/\pi_w R_w$, which implies that $\overline{\mathcal{O}_w} \cong \bigoplus_{i=1}^s F_i$, where each F_i is a finite field extension of \mathbf{F}_w . Let n_i be its degree. Then there is an embedding $\overline{\mathcal{O}_w} \hookrightarrow M_d(\mathbf{F}_w)$ whose image lies in $M_{n_1}(\mathbf{F}_w) \times \cdots \times M_{n_s}(\mathbf{F}_w)$. By the Skolem-Noether Theorem there is a matrix $\overline{S} \in \text{GL}_d(\mathbf{F}_w)$ such that $\overline{S}\overline{\mathcal{O}_w}\overline{S}^{-1} \subset M_{n_1}(\mathbf{F}_w) \times \cdots \times M_{n_s}(\mathbf{F}_w)$. The $\overline{\mathcal{O}_w}$ -ideal $\overline{I_w}$ must be of the form

$$\overline{I_w} = \bigoplus_{i=1}^s \varepsilon_i F_i,$$

where $\varepsilon_i \in \{0, 1\}$. Let S be a matrix in $\text{GL}_d(R_w)$ which reduces to \overline{S} modulo π_w and let $\tilde{u}_w = S^{-1}(l^{\varepsilon_1} \text{Id}_{n_1} \times \cdots \times l^{\varepsilon_s} \text{Id}_{n_s})S$ in $M_d(R_w)$. By abuse of notation we also write \tilde{u}_w for its reduction modulo l . Then we have

$$\overline{I_w} = \overline{\mathcal{O}_w} \cap M_d(\mathbf{F}_w)\tilde{u}_w.$$

Let

$$I_{\tilde{u}_w} = F \cap D(R_w)\tilde{u}_w.$$

Then it follows that

$$\overline{I_w} \cong \overline{I_{\tilde{u}_w}} = I_{\tilde{u}_w}/\pi_w I_{\tilde{u}_w}$$

and by Theorem 18.6 of [11] we get that $I_w \cong I_{\tilde{u}_w}$, which implies that there is some $\lambda \in F_w$ with $I_w = I_{\tilde{u}_w}\lambda$. Replacing \tilde{u}_w by $\tilde{u}_w\lambda$ and setting $\tilde{u} = (\tilde{u}_w)_w \in D(\mathbf{A}_{\text{fin}})$ we get

$$I = F \cap D(R)\tilde{u}.$$

By strong approximation there is an element $u \in D(K)^\times$ such that $D(\hat{R})u = D(\hat{R})\tilde{u}$ and therefore $I = I_u$. □

6. Class numbers

For any ring R we write $\det : D(R) \rightarrow R$ for the reduced norm. Note that this convention is compatible with the determinant, as for every field F , over which D splits, the reduced norm equals the determinant. We want to construct a group scheme \mathcal{G} over R such that $\mathcal{G}(F) = D(F)^\times / F^\times$ holds for every field. Note that $D(R)$ is a free R -module of rank d^2 . Let v_1, \dots, v_{d^2} be a basis and note that the reduced norm $\det(X_1 v_1 + \cdots + X_{d^2} v_{d^2})$ is a homogeneous polynomial of degree d in the variables X_1, \dots, X_{d^2} . The group scheme D^\times is given by the coordinate ring

$$\mathcal{O}_{D^\times} = R[X_1, \dots, X_{d^2}, Y]/(\det(X_1 v_1 + \cdots + X_{d^2} v_{d^2})Y - 1).$$

Now GL_1 acts on \mathcal{O}_{D^\times} by

$$\alpha f(X_1, \dots, X_{d^2}, Y) = f(\alpha X_1, \alpha X_2, \dots, \alpha X_{d^2}, \alpha^{-d} Y)$$

and the coordinate ring we need is the ring of invariants

$$\mathcal{O}_{\mathcal{G}} = (\mathcal{O}_{D^\times})^{\mathrm{GL}_1},$$

which is the subring generated by the elements $X_{i_1} X_{i_2} \cdots X_{i_d} Y$ with $1 \leq i_v \leq d^2$.

LEMMA 6.1. *The ring $\mathcal{O}_{\mathcal{G}}$ is the coordinate ring of an affine group scheme \mathcal{G} over R such that for every factorial ring S/R one has*

$$\mathcal{G}(S) = D(S)^\times / S^\times.$$

Proof. The first claim is clear. We prove the second first in case of a field. Consider the exact sequence of group schemes

$$1 \rightarrow \mathrm{GL}_1 \rightarrow D^\times \rightarrow \mathcal{G} \rightarrow 1.$$

For any field K this gives an exact sequence of groups

$$1 \rightarrow \mathrm{GL}_1(K) \rightarrow D(K)^\times \rightarrow \mathcal{G}(K) \rightarrow H^1(K, \mathrm{GL}_1),$$

where the last item is the Galois-cohomology, which vanishes by Hilbert’s Theorem 90. This implies the claim for fields. Now let R be a factorial ring, so R is integral and has unique factorization. Write K for its quotient field and let $\chi \in \mathcal{G}(R)$, so χ is a ring homomorphism from $\mathcal{O}_{\mathcal{G}}$ to R . By the first part of the proof, χ extends to a ring homomorphism $\tilde{\chi} : \mathcal{O}_{D^\times} \rightarrow K$. We show that this lift can be modified so as to have values in R . For $1 \leq i \leq d^2$ we have

$$\tilde{\chi}(X_i)^d \tilde{\chi}(Y) = \chi(X_i^d Y) \in R.$$

If p is an irreducible element of R which divides the denominator of any of the $\chi(X_i)$, we replace any $\tilde{\chi}(X_j)$ by $p\tilde{\chi}(X_j)$ and $\tilde{\chi}(Y)$ by $\frac{1}{p^d}\tilde{\chi}(Y)$ without changing χ , so we can assume each $\tilde{\chi}(X_i)$ to lie in R . Now if $\tilde{\chi}(Y)$ does not lie in R there must be an irreducible p dividing its denominator. But as the product is in R , p also divides $\tilde{\chi}(X_i)^d$ hence $\tilde{\chi}(X_i)$. We therefore can replace $\tilde{\chi}(X_i)$ by $\frac{1}{p}\tilde{\chi}(X_i)$ and $\tilde{\chi}(Y)$ by $p^d\tilde{\chi}(Y)$ and by repeating this procedure we arrive at $\tilde{\chi}(X_i)$ and $\tilde{\chi}(Y)$ both lying in R . \square

We set $\Gamma = \mathcal{G}(R)$. By Theorem 3.2.4 in [9], the group Γ is a uniform lattice in $G = \mathcal{G}(K_v) \cong \mathrm{PGL}_2(K_v)$, i.e., Γ is a discrete subgroup of G such that $\Gamma \backslash G$ is compact.

Let v_1, \dots, v_d denote the standard basis of \mathbf{R}^d .

THEOREM 6.2. *Let S be a set of places of K with $|S| \geq 2$. Let $\mathcal{F}(S)$ denote the set of field extensions F/K such that every place $v \in S$ is non-decomposed in F . Let $\mathcal{O}(S)$ denote the set of all orders $\mathcal{O} \subset F$ where $F \in \mathcal{F}(S)$ such that \mathcal{O} is*

maximal at every $v \in S$. For $\mathcal{O} \in \mathcal{O}(S)$ let $R(\mathcal{O})$ be its regulator. Then the sum

$$N(k) = \sum_{\substack{\mathcal{O} \in \mathcal{O}(S) \\ \gamma \in \mathcal{O}^\times \\ \lambda(\mathfrak{a}_\gamma) = k}} R(\mathcal{O})h(\mathcal{O})f_S(\mathcal{O}), \quad k \in \mathbf{N}^d,$$

satisfies

$$N(k) \sim \mathbf{1}_\Lambda(k)C_{\Lambda/\Lambda'}(k)c^k$$

for $k_j \rightarrow \infty$ for every $j = 1, \dots, d$ independently. Here we have

$$c_j = \left(\prod_{v=0}^{j-1} \frac{q^{d+1} - q^v}{q^j - q^v} \right)^{d+1}, \quad j = 1, \dots, d,$$

where q is the residue cardinality of R_v . Further $C_{\Lambda/\Lambda'} : \Lambda/\Lambda' \rightarrow (0, \infty)$. The lattice $\Lambda \subset \mathbf{R}^d$ is generated by

$$f_0 = \frac{2}{d}(v_1 + \dots + v_d), \quad f_j = f_0 - 2\frac{d+1}{d}v_j,$$

where $j = 1, \dots, d$ and $\Lambda' \subset \Lambda$ is the sub-lattice generated by $2v_1, \dots, 2v_d$.

Proof. The group G_γ equals F_γ^\times , where F_γ is the field extension given by the centralizer of γ . Therefore one sees that $|G_\gamma^\alpha/\Gamma_\gamma^\alpha|$ equals the regulator of the order \mathcal{O}_γ . The theorem now follows from the Prime Geodesic Theorem 4.2 together with Lemma 5.2 and Lemma 5.3.

The computation of the c_j follows their definition in [7], they are given as $c_j = |Ka_jK/K|$. For this computation one can assume that $K = \mathbf{SL}_{d+1}(R_v)$. Let m_j be the diagonal matrix with entries $(\pi^{-1}, \dots, \pi^{-1}, 1, \dots, 1)$ with j -times π^{-1} , where π is a uniformizer of the discrete valuation ring R_v . Then two elements b, b' of Km_jK lie in the same K -coset if and only if the R_v -span of be_1, \dots, be_{d+1} equals the R_v -span of $b'e_1, \dots, b'e_{d+1}$, where e_1, \dots, e_{d+1} is the standard basis of K_v^{d+1} . Taking this modulo the R_v -submodule spanned by e_1, \dots, e_{d+1} we end up determining the number of j -dimensional \mathbf{F}_q sub vector spaces of \mathbf{F}_q^{d+1} , which is $\prod_{v=0}^{j-1} \frac{q^{d+1} - q^v}{q^j - q^v}$ (The enumerator gives the number of bases and the denominator the number of invertible matrices for a given bases.) This number is $|Km_jK/K|$. But now $a_j = m_j^{d+1}$ and the map $a \mapsto |KaK/K|$ is a quasi-character on A^- , see Lemma 2.4.5 of [7]. This concludes the computation of c_j and finishes the proof of the theorem. □

REFERENCES

[1] PETER ABRAMENKO AND KENNETH S. BROWN, Buildings, Graduate texts in mathematics **248**, Springer, New York, 2008.
 [2] ANTON DEITMAR, Class numbers of orders in cubic fields, J. Number Theory **95** (2002).

- [3] ANTON DEITMAR AND WERNER HOFFMANN, Asymptotics of class numbers, *Invent. Math.* **160** (2005), 647–675.
- [4] A. DEITMAR, A prime geodesic theorem for higher rank spaces, *Geom. Funct. Anal.* **14** (2004), 1238–1266.
- [5] ANTON DEITMAR, A prime geodesic theorem for higher rank II: Singular geodesics, *Rocky Mountain J. Math.* **39** (2009), 485–507.
- [6] ANTON DEITMAR, Ihara zeta functions and class numbers, *Adv. Studies in Contemp. Math.* **24** (2014), 439–450.
- [7] ANTON DEITMAR, MING-HSUAN KANG AND RUPERT MCCALLUM, Building lattices and zeta functions, <http://arxiv.org/abs/1412.3327>.
- [8] J. S. FRIEDMAN, J. JORGENSON AND J. KRAMER, An effective bound for the Huber constant for cofinite Fuchsian groups, *Math. Comp.* **80** (2011), 1163–1196.
- [9] G. A. MARGULIS, Discrete subgroups of semisimple Lie groups, *Ergebnisse der mathematik und ihrer grenzgebiete (3)* [Results in mathematics and related areas (3)] **17**, Springer-Verlag, Berlin, 1991.
- [10] JÜRGEN NEUKIRCH, Algebraic number theory, *Grundlehren der mathematischen wissenschaften* [Fundamental principles of mathematical sciences] **322**, Springer-Verlag, Berlin, 1999.
- [11] RICHARD S. PIERCE, Associative algebras, *Graduate texts in mathematics* **88**, Springer-Verlag, New York-Berlin, 1982.
- [12] NICOLE RAULF, Trace formulae and applications to class numbers, *Cent. Eur. J. Math.* **12** (2014), 824–847.
- [13] IRVING REINER, Maximal orders, *London Mathematical Society monographs, new series* **28**, The Clarendon Press, Oxford University Press, Oxford, 2003.
- [14] PETER SARNAK, Class numbers of indefinite binary quadratic forms, *J. Number Theory* **15** (1982), 229–247.
- [15] K. SOUNDARARAJAN AND MATTHEW P. YOUNG, The prime geodesic theorem, *J. Reine Angew. Math.* **676** (2013), 105–120.

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