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GENERATORS FOR THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

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Abstract

We show that Szepietowski's system of generators for the mapping class group of a non-orientable surface is a minimal generating set by Dehn twists and Y-homemorphisms.

Let N_g be a non-orientable surface which is a connected sum of g projective planes. Let $\mathscr{M}(N_g)$ be the group of isotopy classes of homeomorphisms over N_g, i.e., the *mapping class group* of N_g. In this paper, we assume that $g \geq 4$.

We introduce some elements of $\mathscr{M}(N_g)$. A simple closed curve γ_1 (resp. γ_2) in N_g is *two-sided* (resp. *one-sided*) if a regular neighborhood of γ_1 (resp. γ_2) is an annulus (resp. Möbius band). For a two-sided simple closed curve γ on N_g, we denote by t_γ a Dehn twist about γ. We indicate the direction of a Dehn twist by an arrow beside the curve γ as shown in Figure 1. For a one-sided simple closed curve m and a two-sided simple closed curve a which intersect transversely in one point, let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is homeomorphic to the Klein bottle with one boundary component. Let M be a regular neighborhood of m. We denote by $Y_{m,a}$ a homeomorphism over N_g which is described as the result of pushing M once along a keeping the boundary of K fixed (see Figure 2). We call $Y_{m,a}$ a *Y-homeomorphism*, or *crosscap slide*.

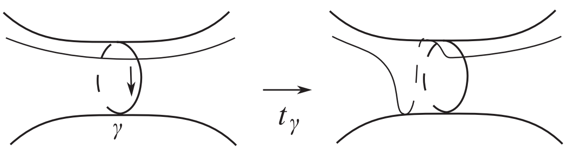

Figure 1. The direction of t_γ is indicated by an arrow beside γ.

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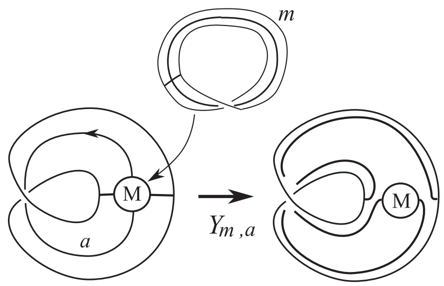

FIGURE 2. A cirlcle with "M" indicates a place where to attach a Möbius band.

Lickorish showed that $\mathcal{M}(N_g)$ is generated by Dehn twists and Y-homeomorphisms [7, Theorem 2], and that $\mathcal{M}(N_g)$ is not generated by Dehn twists [7, p. 310, Note]. Furthermore, Chillingworth [2] found a finite system of generators for $\mathcal{M}(N_g)$. Birman and Chillingworth [1] obtained a finite system of generators by using an argument on the orientable two fold covering of N_g. Szepietowski [9] reduced the system of Chillingworth's generators for $\mathcal{M}(N_g)$ and showed:

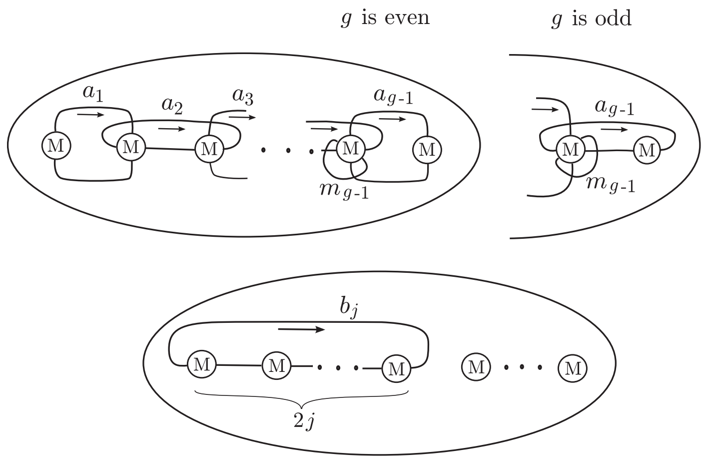

FIGURE 3. Chillingworth's generators for $\mathcal{M}(N_g)$.

THEOREM 1 ([9, Theorem 3.1]). *$\mathcal{M}(N_g)$ is generated by t_{a_i} $(i = 1, \ldots, g-1)$, t_{b_2} and $Y_{m_{g-1},a_{g-1}}$, where a_i, m_{g-1}, b_2 are simple closed curves shown in Figure* 3.

On the other hand, Lickorish [6] showed that the mapping class group $\mathcal{M}(\Sigma_g)$ of the orientable closed surface Σ_g of genus g is generated by finitely many Dehn twists, and Humphries [4] reduced the number of Dehn twists generating

$\mathcal{M}(\Sigma_g)$ to $2g+1$ and showed that this is the minimum number of Dehn twists generating $\mathcal{M}(\Sigma_g)$. We will show the analogous result for the mapping class group of the non-orientable surface.

THEOREM 2. *We assume* $g \geq 4$. *If Dehn twists* $t_{c_1}, \dots, t_{c_n}$ *and* Y*-homeomorphisms* $Y_1, \dots, Y_k$ *generate* $\mathcal{M}(N_g)$, *then* $n \geq g$ *and* $k \geq 1$. *In particular, any proper subset of* $\{t_{a_i} (i = 1, \dots, g-1), t_{b_2}, Y_{m_{g-1}, a_{g-1}}\}$ *does not generate* $\mathcal{M}(N_g)$.

Remark 3. When $g = 1$, $\mathcal{M}(N_g)$ is trivial. When $g = 2$, $\mathcal{M}(N_2) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ and generated by t_{a_1} and Y_{m_1, a_1} (see [7, Lemma 5]), therefore $\{t_{a_1}, Y_{m_1, a_1}\}$ is a minimal generating set by Dehn twists and Y-homeomorphisms. When $g = 3$, $\mathcal{M}(N_3)$ is generated by t_{a_1}, t_{a_2} and Y_{m_2, a_2} (see [1, Theorem 3] and [9, Theorem 3.1]). If $\mathcal{M}(N_3)$ is generated by one Dehn twist t_a and one Y-homeomorphism, then the group of the action of $\mathcal{M}(N_3)$ on $H_1(N_3; \mathbf{Z}_2)$ should be isomorphic to $\mathbf{Z}_2$ generated by the induced isomorphism $(t_a)_*$ on $H_1(N_3; \mathbf{Z}_2)$. Nevertheless, $(t_{a_1})_*$ is not equal to $(t_{a_2})_*$. Therefore $\{t_{a_1}, t_{a_2}, Y_{m_2, a_2}\}$ is a minimal generating set by Dehn twists and Y-homeomorphisms.

Let $w_1 : H_1(N_g; \mathbf{Z}_2) \to \mathbf{Z}_2$ be the first Stiefel-Whitney class, that is to say, if $x \in H_1(N_g; \mathbf{Z}_2)$ is represented by a one-sided simple closed curve on N_g then $w_1(x) = 1$, otherwise $w_1(x) = 0$. For the basis $\{x_1, \dots, x_g\}$ for $H_1(N_g; \mathbf{Z}_2)$ indicated in Figure 4, $w_1(x_i) = 1$. For each pair of elements x, y of $H_1(N_g; \mathbf{Z}_2)$, the $\mathbf{Z}_2$-intersection form of x and y is denoted by (x, y). For the basis $\{x_1, \dots, x_g\}$, $(x_i, x_j) = \delta_{i,j}$. Let $H_1^+(N_g; \mathbf{Z}_2)$ be the kernel of w_1, then $\dim_{\mathbf{Z}_2} H_1^+(N_g; \mathbf{Z}_2) = g - 1$. If a complement of a two-sided simple closed curve c on N_g is connected and non-orientable, we call c an *admissible* A-circle. For an admissible A-circle c on N_g, $N_g \backslash c$ is homeomorphic to N_{g-2} removed two 2-disks. Therefore, if c_1 and c_2 are admissible A-circles then there is $\phi \in \mathcal{M}(N_g)$ such that $\phi(c_1) = c_2$, by the change of coordinates principle in [3, §1.3].

LEMMA 4. *Let* c *be a two-sided simple closed curves on* N_g. *If* c *is not admissible*, *then* c *represents* 0 *or* $x_1 + \cdots + x_g$ *in* $H_1(N_g; \mathbf{Z}_2)$.

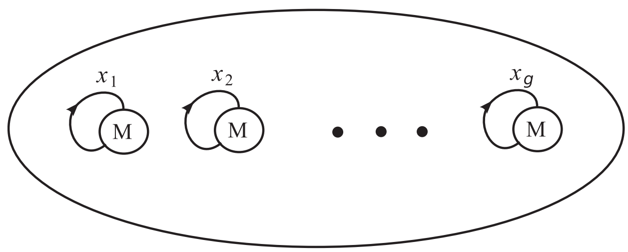

FIGURE 4. A basis for $H_1(N_g; \mathbf{Z}_2)$.

Proof. If c is not admissible, then either $N_g \backslash c$ is not connected or $N_g \backslash c$ is connected and orientable. In the former case, c is 0 in $H_1(N_g; \mathbf{Z}_2)$. In the latter case, g is even, and there is a homeomorphism which brings c to $b_{g/2}$ in Figure 3, since their complements are homeomorphic to $\Sigma_{g/2-1}$ removed two 2-disks. The simple closed curve $b_{g/2}$ represents $x_1 + \cdots + x_g$ and the action of any homeomorphism of N_g on $H_1(N_g; \mathbf{Z}_2)$ preserves $x_1 + \cdots + x_g$. Therefore c represents $x_1 + \cdots + x_g$, which is the Poincaré dual of the first Stiefel-Whitney class of N_g. $\square$

LEMMA 5. *Let $c_1, \dots, c_n$ be arbitrary two-sided simple closed curves such that $t_{c_1}, \dots, t_{c_n}$ and Y-homeomorphisms $Y_1, \dots, Y_k$ generate $\mathcal{M}(N_g)$. Then at least one of $c_1, \dots, c_n$ is admissible.*

Proof. For $y \in H_1(N_g; \mathbf{Z}_2)$, we define an isomorphism τ_y of $H_1(N_g; \mathbf{Z}_2)$ by $\tau_y(x) = x + (x, y)y$. By Lemma 4 and the fact that Y-homeomorphisms act on $H_1(N_g; \mathbf{Z}_2)$ trivially [8, Theorem 5.5], if $c_1, \dots, c_n$ are not admissible, then the action of each elements of $\mathcal{M}(N_g)$ on $H_1(N_g; \mathbf{Z}_2)$ is a power of $\tau_{x_1+\cdots+x_g}$. On the other hand, $(t_{a_1})_* = \tau_{x_1+x_2}$ is not a power of $\tau_{x_1+\cdots+x_g}$. $\square$

LEMMA 6. *If $t_{c_1}, \dots, t_{c_n}$ and Y-homeomorphisms $Y_1, \dots, Y_k$ generate $\mathcal{M}(N_g)$ then $[c_1], \dots, [c_n]$ generate $H_1^+(N_g; \mathbf{Z}_2)$. In particular, $n \geq g - 1$.*

Proof. By Lemma 5, we may assume c_1 is an admissible A-circle. For any $x \in H_1^+(N_g; \mathbf{Z}_2)$, we can write $x = x_{i_1} + x_{i_2} + \cdots + x_{i_{2k}}$. We can represent $x_{i_{2j-1}} + x_{i_{2j}}$ by an admissible A-circle γ_j as in Figure 5. Hence, x is represented by a union of admissible A-circles, that is, $x = [\gamma_1] + \cdots + [\gamma_k]$ in $H_1^+(N_g; \mathbf{Z}_2)$. For each γ_j, there is an element $\phi_j \in \mathcal{M}(N_g)$ such that $\phi_j(c_1) = \gamma_j$. By the assumption of this lemma, ϕ_j is a product of $t_{c_1}, \dots, t_{c_n}$ and $Y_1, \dots, Y_k$. We see that Y_i acts on $H_1(N_g; \mathbf{Z}_2)$ trivially, and, for each $x \in H_1(N_g; \mathbf{Z}_2)$, $(t_{c_i})_*(x) = x + (x, [c_i])[c_i]$. Therefore, $[\gamma_j] \in H_1^+(N_g; \mathbf{Z}_2)$ is a sum of $[c_1], \dots, [c_n]$, hence x is a sum of $[c_1], \dots, [c_n]$. This shows that $H_1^+(N_g; \mathbf{Z}_2)$ is generated by $[c_1], \dots, [c_n]$. $\square$

Let $2\times : \mathbf{Z}_2 \to \mathbf{Z}_4$ be an injection defined by $2 \times ([n]) = [2n]$. A map $q : H_1(N_g; \mathbf{Z}_2) \to \mathbf{Z}_4$ is called a $\mathbf{Z}_4$-*quadratic form*, if $q(x + y) = q(x) + q(y) + 2 \times (x, y)$ for any $x, y \in H_1(N_g; \mathbf{Z}_2)$. This map q is determined by values of q for elements in a $\mathbf{Z}_2$-basis of $H_1(N_g; \mathbf{Z}_2)$. Putting $x = y = 0$ in the above formula,

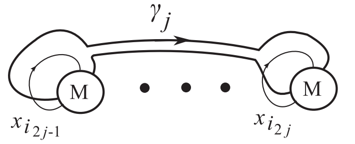

FIGURE 5. An element $x_{i_{2j-1}} + x_{i_{2j}} \in H_1(N_g; \mathbf{Z}_2)$ is represented by an admissible A-circle γ_i.

we have $q(0) = 0$. If $x \in H_1(N_g; \mathbf{Z}_2)$ is represented by a one-sided simple closed curve, in other word x is represented by a core of a Möbius band embedded in N_g, then $(x, x) = 1$. Since $2x = 0$ in $H_1(N_g; \mathbf{Z}_2)$, we have $0 = q(x + x) = q(x) + q(x) + 2 \times (x, x) = 2q(x) + 2$. Therefore, we have $q(x) = \pm 1$. By the same argument, if $x \in H_1(N_g; \mathbf{Z}_2)$ is represented by a two-sided simple closed curve, then we have $q(x) = 0$ or 2.

Lemma 7. *There is no $\mathbf{Z}_4$-quadratic form over $H_1(N_g; \mathbf{Z}_2)$ which is preserved by every non-trivial element of $\mathcal{M}(N_g)$.*

Proof. For any $\mathbf{Z}_4$-quadratic form q over $H_1(N_g; \mathbf{Z}_2)$, there is a non-trivial element $x \in H_1^+(N_g; \mathbf{Z}_2)$ such that $q(x) = 0$; even if $q([a_1]) = q([a_3]) = 2$ then $q([a_1] + [a_3]) = q([a_1]) + q([a_3]) + 2 \times ([a_1], [a_3]) = 0$. We can write $x = x_{i_1} + \cdots + x_{i_{2n}}$, let $y = x_{i_1}$, then $(y, x) = 1$. Let γ be a simple closed curve on N_g representing x then $q \circ (t_\gamma)_*(y) = q(y + (y, x)x) = q(y) + q((y, x)x) + 2 \times (y, (y, x)x) = q(y) + q(x) + 2 = q(y) + 2 \neq q(y)$. Therefore $q \circ (t_\gamma)_* \neq q$. □

Lemma 8. *Let $c_1, \ldots, c_{g-1}$ be two-sided simple closed curves such that $[c_1], \ldots, [c_{g-1}]$ generate $H_1^+(N_g; \mathbf{Z}_2)$, then there is a $\mathbf{Z}_4$-quadratic form over $H_1(N_g; \mathbf{Z}_2)$ preserved by any t_{c_i}.*

Proof. Let α be a one-sided simple closed curve on N_g, then $\{[c_1], \ldots, [c_{g-1}], [\alpha]\}$ is a $\mathbf{Z}_2$-basis of $H_1(N_g; \mathbf{Z}_2)$. We define a $\mathbf{Z}_4$-quadratic form q over $H_1(N_g; \mathbf{Z}_2)$ by $q([c_1]) = \cdots = q([c_{g-1}]) = 2$ and $q([\alpha]) = 1$. For any $i = 1, \ldots, g-1$ and $x \in H_1(N_g; \mathbf{Z}_2)$, we see $q \circ (t_{c_i})_*(x) = q(x + (x, [c_i])[c_i]) = q(x) + q((x, [c_i])[c_i]) + 2 \times (x, (x, [c_i])[c_i])$. If $(x, [c_i]) = 0$, $q \circ (t_{c_i})_*(x) = q(x)$. If $(x, [c_i]) = 1$, $q \circ (t_{c_i})_*(x) = q(x) + q([c_i]) + 2 \times (x, [c_i]) = q(x) + 2 + 2 = q(x)$. Therefore, $q \circ (t_{c_i})_* = q$ for any $i = 1, \ldots, g-1$. □

We assume that Dehn twists $t_{c_1}, \ldots, t_{c_n}$ and Y-homeomorphisms $Y_1, \ldots, Y_k$ generate $\mathcal{M}(N_g)$. In [7], Lickorish showed that $\mathcal{M}(N_g)$ is not generated by Dehn twists, therefore we see $k \geq 1$. By Lemma 6, $[c_1], \ldots, [c_n]$ generate $H_1^+(N_g; \mathbf{Z}_2)$, in particular $n \geq g-1$. We assume that $n = g-1$. By Lemma 8, there is a $\mathbf{Z}_4$-quadratic form over $H_1(N_g; \mathbf{Z}_2)$ preserved by Dehn twists $t_{c_1}, \ldots, t_{c_n}$ and Y-homeomorphisms $Y_1, \ldots, Y_k$, which contradicts Lemma 7. Hence, we see $n \geq g$. This completes the proof of Theorem 2.

Remark 9. The proof of Theorem 2 is inspired by the master thesis [5] by Shigehisa Ishimura, in which he proved Humphries' result by using $\mathbf{Z}_2$-quadratic form over $H_1(\Sigma_g; \mathbf{Z}_2)$.

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