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AN ISOPERIMETRIC INEQUALITY FOR DIFFUSED SURFACES

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Abstract

For general varifolds in Euclidean space, we prove an isoperimetric inequality, adapt the basic theory of generalised weakly differentiable functions, and obtain several Sobolev type inequalities. We thereby intend to facilitate the use of varifold theory in the study of diffused surfaces.

1. Introduction

General aim. The isoperimetric inequality is well established in the context of sharp surfaces (e.g., integral currents, sets, or integral varifolds) in Euclidean space, but little appears to be known for diffused surfaces (i.e., for surfaces that are not concentrated on a set of the their own dimension). General varifolds form a very flexible model for the latter case; in fact, for equations of Allen-Cahn type, their utility was established by Ilmanen, Padilla, and Tonegawa (see [6] and [12]) and, for discrete and computational geometry, their unifying use has been recently suggested by Buet, Leonardi, and Masnou (see [3]). The present paper shall contribute to this proposed development by adapting several core tools to the possibly non-rectifiable case. To outline these results, *suppose m and n are positive integers*, $m \leq n$, *V is an m dimensional varifold in* $\mathbf{R}^n$, *and*, to avoid case distinctions, *also* $m > 1$; see Section 2 for the notation.

Isoperimetric inequality, see Section 3. The best result up to now (see the second author [13, 6.11]) did apply to general varifolds, but controlled only their rectifiable parts: *If* $\|V\|(\mathbf{R}^n) < \infty$, *then*

$$\|V\|\{x : \boldsymbol{\Theta}^m(\|V\|, x) \geq d\} \leq \Gamma d^{-1/m}\|V\|(\mathbf{R}^n)^{1/m}\|\delta V\|(\mathbf{R}^n) \quad \textit{for } 0 < d < \infty,$$

where Γ is a positive, finite number determined by m. Following the first author (see [7, 2.2]), it unified the approach of Allard in [2, 7.1] and Michael and Simon in [11, 2.1]. Clearly, if $0 < d < \infty$, and $\boldsymbol{\Theta}^m(\|V\|, x) \geq d$ for $\|V\|$ almost all x,

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the result implies

$$\|V\|(\{x : \boldsymbol{\Theta}^m(\|V\|, x) \geq d\})^{1-1/m} \leq \Gamma d^{-1/m}\|\delta V\|(\mathbf{R}^n).$$

We notice that $\|\delta V\|$ encodes both, the total mass of the variational boundary and the integral of the modulus of the generalised mean curvature of the varifold, see Allard [2, 4.3]; in particular, a more classical form results for varifolds with vanishing mean curvature (i.e., generalised minimal surfaces) and, by Allard [2, 4.8 (4)], the isoperimetric inequality for integral currents with non-optimal constant is a special case. In 3.5 and 3.7, we establish that, *if* $\|V\|(\mathbf{R}^n) < \infty$, *then*

$$\|V\|(A(d))^{1-1/m} \leq \gamma(m)d^{-1/m}\|\delta V\|(\mathbf{R}^n) \quad \textit{for } 0 < d < \infty;$$
$$\textit{where } A(d) = \{x : \|V\|\mathbf{B}(x,r) \geq d\boldsymbol{\alpha}(m)r^m \textit{ for some } 0 < r < \infty\}.$$

By homogeneity considerations, one may not replace $(A(d), d^{-1/m})$ by $(\mathbf{R}^n, 1)$. The sets $A(d)$, for suitable d, naturally describe the region, where the behaviour of the diffused surface resembles the behaviour of an m dimensional sharp surface.

Generalised weakly differentiable functions, see Section 4. We extend the basic theory of generalised weakly differentiable functions (see the first author [9, §§8–9] and [10, 4.1, 2]) from rectifiable varifolds to general varifolds. This theory includes the study of closedness properties (under convergence, composition, addition, and multiplication) and a coarea formula in functional analytic form. The main differences lie in the possible non-existence of decompositions (see 4.12) and the ineffectiveness of $(\|V\|, m)$ approximate differentials (see 4.7). This development allows us to state the Sobolev inequalities in their natural framework, but goes beyond that purpose.

Sobolev inequalities, see Section 5. In view of 4.11, 4.18, and [9, 8.16, 9.2], a version of our Sobolev inequality in 5.6 may be stated as follows, employing (see 4.2) the space of Y valued generalised weakly differentiable functions $\mathbf{T}(V, Y)$ and the derivative $V\mathbf{D}f$ associated to functions f in that space: *If* $\|\delta V\|$ *is a Radon measure*, Y *is a finite dimensional normed vector space*, $f \in \mathbf{T}(V, Y)$, $\|V\|\{x : |f(x)| > 0\} < \infty$, $0 < r < \infty$, *and* $g : \mathbf{R}^n \to \mathbf{R}$ *satisfies*

$$g(a) = \sup\{y : \|V\|(\mathbf{B}(a,r) \cap \{x : |f(x)| \leq y\}) \leq 2^{-1}\|V\|\mathbf{B}(a,r)\}$$

for $a \in \mathbf{R}^n$, *then*, *for* $0 < d < \infty$, *there holds*

$$\left(\int_{B(d)} g^{m/(m-1)} \, \mathrm{d}\|V\|\right)^{1-1/m} \leq \Gamma d^{-1/m}\left(\int |f| \, \mathrm{d}\|\delta V\| + \int \|V\mathbf{D}f\| \, \mathrm{d}\|V\|\right),$$

where $B(d) = \{x : \|V\|\mathbf{B}(x,r) \geq d\boldsymbol{\alpha}(m)r^m\}$, *and* $\Gamma = 2\boldsymbol{\beta}(n)\gamma(m)$. In this theorem, the number r acts as a scale on which both the lower density ratio bound and the averaging process by medians occur; in fact, the width of a diffused surface could be a natural choice for such a scale. More generally, in 5.6, we replace r by a $\|V\|$ measurable function. Simple examples show that one may not replace

$(g, B(d))$ by $(f, A(d))$, see 5.7. Finally, the special case $0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ of the preceding theorem could be derived replacing the use of 4.17 and the coarea formula for generalised weakly differentiable functions (see 4.11 and [9, 8.5, 30]) by Allard's more basic result [2, 4.10].

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2. Notation

Generally, the notation of [9, §1] is employed; the only exception is the usage of $\gamma(m)$, see 3.7 and 3.9. In particular, our notation is largely consistent with that of Federer [5, pp. 669–671] and Allard [2]. While we do not duplicate each definition from [9, §1], for the convenience of the reader, we recall some less commonly used symbols and conventions below.

The difference of sets A and B is denoted by $A \sim B$. Whenever f is a linear map and v belongs to its domain, the expression $\langle v, f \rangle$ is synonymously used with $f(v)$. The inner product of v and w, by contrast, is denoted by $v \bullet w$. The symbol $P_\natural$ denotes the symmetric linear homomorphism of $\mathbf{R}^n$ whose image is P and whose restriction to P is the identity map of P, whenever P is a linear subspace of $\mathbf{R}^n$. If X is a locally compact Hausdorff space, then $\mathscr{K}(X)$ denotes the vector space of continuous real valued functions on X with compact support. Whenever ϕ measures X, Y is a separable Banach space, f is a ϕ measurable Y valued function, and $1 \leq p \leq \infty$, the value of the Lebesgue seminorm $\phi_{(p)}$ at f satisfies

$$\phi_{(p)}(f) = \left(\int |f|^p \, \mathrm{d}\phi \right)^{1/p} \quad \text{if } p < \infty,$$

$$\phi_{(p)}(f) = \inf\{s : s \geq 0, \phi\{x : |f(x)| > s\} = 0\} \quad \text{if } p = \infty.$$

Whenever U is an open subset of a finite dimensional normed space, and Y is a separable Banach space, $\mathscr{E}(U, Y)$ denotes the space of all functions from U into Y, that are continuously differentiable of every positive integer order, $\mathscr{D}(U, Y)$ denotes the subspace of those functions in $\mathscr{E}(U, Y)$ with compact support, and $\mathscr{D}'(U, Y)$ is the space of distributions in U of type Y. Whenever $T \in \mathscr{D}'(U, Y)$, the symbol $\|T\|$ denotes the largest Borel regular measure over U such that

$$\|T\|(A) = \sup\{T(\theta) : \theta \in \mathscr{D}(U, Y) \text{ with } \operatorname{spt} \theta \subset A \text{ and } |\theta(x)| \leq 1 \text{ for } x \in U\}$$

whenever A is an open subset of U. In case such T is representable by integration (equivalently, if $\|T\|$ is a Radon measure), $T(\theta)$ denotes the value of the unique $\|T\|_{(1)}$ continuous extension of T to $\mathbf{L}_1(\|T\|, Y)$ at $\theta \in \mathbf{L}_1(\|T\|, Y)$, and

$T \llcorner A$ denotes the restriction of T to A, whenever A is $\|T\|$ measurable (i.e., we have $(T \llcorner A)(\theta) = T(\theta_A)$ whenever $\theta \in \mathscr{D}(U, Y)$, where $\theta_A(x) = \theta(x)$ for $x \in A$ and $\theta_A(x) = 0$ for $x \in U \sim A$). Finally, if V is an m dimensional varifold in an open subset U of $\mathbf{R}^n$, $\|\delta V\|$ is a Radon measure, and E is an $\|V\| + \|\delta V\|$ measurable set, then the distributional boundary $V\partial E$ satisfies

$$V\partial E = (\delta V) \llcorner E - \delta(V \llcorner E \times \mathbf{G}(n, m)) \in \mathscr{D}'(U, \mathbf{R}^n).$$

3. General isoperimetric inequality

In this section, we prove a general isoperimetric inequality in 3.5. It involves a maximal-type function corresponding to the density defined in 3.1. Additionally, its proof relies on a simple iteration lemma (see 3.2) and a variant of the "calculus lemma" used by Simon (see 3.3 and 3.4). Finally, in 3.10 and 3.12, we state a version of the isoperimetric inequality in case the varifold is contained in a ball and a version involving the size of the varifold.

3.1 (Maximal-type function). Suppose m and n are positive integers, $m \leq n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, and the function $M : \mathbf{R}^n \to \overline{\mathbf{R}}$ satisfies

$$M(x) = \sup\left\{ \frac{\|V\|\mathbf{B}(a, s)}{\boldsymbol{\alpha}(m)s^m} : a \in \mathbf{R}^n,\ 0 < s < \infty,\ \text{and}\ x \in \mathbf{B}(a, s) \right\}$$

for $x \in \mathbf{R}^n$. Clearly, if $a \in \mathbf{R}^n$, $0 < s < \infty$, $0 < d < \infty$, and $\|V\|\mathbf{B}(a, s) \geq d\boldsymbol{\alpha}(m)s^m$, then $\mathbf{B}(a, s) \subset \{x : M(x) \geq d\}$.

LEMMA 3.2 (Iteration lemma). *Suppose $0 \leq \kappa < \infty$, $0 < \lambda < 1$, $0 < \mu < 1$, the function $a : \{d : 0 < d < \infty\} \to \mathbf{R}$ is nonnegative, $\limsup_{d \to 0+} a(d) < \infty$, and*

$$a(d) \leq \kappa d^{-\mu} a(\lambda d)^{\mu} \quad \textit{for } 0 < d < \infty.$$

Then, $a(d)^{1-\mu} \leq \kappa d^{-\mu} (1/\lambda)^{\mu^2/(1-\mu)}$ for $0 < d < \infty$.

Proof. Assume $\kappa > 0$. Then, induction yields that $\log a(d)$ does not exceed

$$(\log(\kappa) + \mu \log(1/d)) \left(\sum_{i=0}^{j-1} \mu^i \right) + \log(1/\lambda) \left(\sum_{i=1}^{j-1} i\mu^{i+1} \right) + \mu^j \log a(\lambda^j d)$$

whenever $0 < d < \infty$ and j is a positive integer; here $\sum_{i=1}^{0} i\mu^{i+1} = 0$. □

LEMMA 3.3 (Calculus lemma). *Suppose $0 < s < \infty$, $f : \{t : s \leq t < \infty\} \to \mathbf{R}$ is a nonnegative, nondecreasing function, $0 < m < \infty$, $s^{-m} f(s) \geq 3/4$,*

$$r = \sup\{t : s \leq t < \infty \text{ and } t^{-m} f(t) \geq 1/3\} < \infty,$$

$g : \{t : s \leq t \leq r\} \to \mathbf{R}$ is a nonnegative, $\mathscr{L}^1 \llcorner \{t : s \leq t \leq r\}$ measurable function, and $t^{-m} f(t) \leq r^{-m} f(r) + \int_t^r u^{-m} g(u)\, \mathrm{d}\mathscr{L}^1 u$ whenever $s \leq t \leq r$.

Then, there exists t satisfying

$$s \le t \le r \quad \textit{and} \quad f(5t) \le 5^m r g(t).$$

Proof. Abbreviating $\kappa = \sup\{t^{-m} f(t) : s \le t \le r\}$, we note that

$$s < r, \quad 3/4 \le \kappa < \infty, \quad \sup\{t^{-m} f(t) : r \le t < \infty\} \le 1/3,$$

$$\int_s^r t^{-m} f(t)\, \mathrm{d}\mathscr{L}^1 t \le \kappa r, \quad \int_r^{5r} t^{-m} f(t)\, \mathrm{d}\mathscr{L}^1 t \le \frac{4r}{3}.$$

Therefore, if the conclusion were false, we could estimate

$$\int_s^r t^{-m} g(t)\, \mathrm{d}\mathscr{L}^1 t < r^{-1} \int_s^r (5t)^{-m} f(5t)\, \mathrm{d}\mathscr{L}^1 t = \frac{1}{5r} \int_{5s}^{5r} t^{-m} f(t)\, \mathrm{d}\mathscr{L}^1 t \le \frac{\kappa}{5} + \frac{4}{15},$$

$$\kappa \le r^{-m} f(r) + \int_s^r t^{-m} g(t)\, \mathrm{d}\mathscr{L}^1 t < \frac{3}{5} + \frac{\kappa}{5},$$

whence, as $3/4 \le \kappa < \infty$, it would follow $3/5 \le 4\kappa/5 < 3/5$, a contradiction. □

Remark 3.4. The previous lemma and its proof are adapted from [14, 18.7].

THEOREM 3.5 (General isoperimetric inequality). *Suppose m, n, V, and M are as in* 3.1, *and* $\|V\|(\mathbf{R}^n) < \infty$. *Then,*

$$\|V\|(\{x : M(x) \ge d\})^{1-1/m} \le \Gamma d^{-1/m} \|\delta V\|(\mathbf{R}^n) \quad \textit{for } 0 < d < \infty,$$

where $\Gamma = 2^{-1}$ *if* $m = 1$, $\Gamma = 5^m 3^{1/(m-1)} \boldsymbol{\alpha}(m)^{-1/m}$ *if* $m > 1$, *and* $0^0 = 0$.

Proof. Assume $\|\delta V\|(\mathbf{R}^n) < \infty$. In view of [9, 4.8 (1)], we may assume that $m > 1$. We abbreviate $\kappa = 5^m 3^{1/m} \boldsymbol{\alpha}(m)^{-1/m} \|\delta V\|(\mathbf{R}^n)$. By 3.2 applied with λ, μ, and $a(d)$ replaced by $1/3$, $1/m$, and $\|V\|\{x : M(x) \ge d\}$, respectively, it is sufficient to prove

$$\|V\|\{x : M(x) \ge d\} \le \kappa d^{-1/m} \|V\|(\{x : M(x) \ge d/3\})^{1/m} \quad \text{for } 0 < d < \infty.$$

For this purpose we define

$$r = \sup\{s : a \in \mathbf{R}^n,\ 0 < s < \infty, \text{ and } \|V\|\mathbf{B}(a,s) \ge (d/3)\boldsymbol{\alpha}(m) s^m\}$$

and note that $r \le 3^{1/m} d^{-1/m} \boldsymbol{\alpha}(m)^{-1/m} \|V\|(\{x : M(x) \ge d/3\})^{1/m} < \infty$ by 3.1. Moreover, whenever $x \in \mathbf{R}^n$ and $M(x) \ge d$, there exist $a \in \mathbf{R}^n$ and $0 < t \le r$ satisfying

$$x \in \mathbf{B}(a,t), \quad \|V\|\mathbf{B}(a,5t) \le 5^m r \|\delta V\|\mathbf{B}(a,t);$$

in fact, taking $a \in \mathbf{R}^n$ and $0 < s < \infty$ satisfying the conditions $x \in \mathbf{B}(a,s)$ and $\|V\|\mathbf{B}(a,s) \ge (3d/4)\boldsymbol{\alpha}(m) s^m$, in view of [9, 4.5, 6], one may apply 3.3 with $f(t)$ and $g(t)$ replaced by $d^{-1}\boldsymbol{\alpha}(m)^{-1}\|V\|\mathbf{B}(a,t)$ and $d^{-1}\boldsymbol{\alpha}(m)^{-1}\|\delta V\|\mathbf{B}(a,t)$, respectively. Finally, Vitali's covering theorem (see [5, 2.8.5, 8]) yields the conclusion. □

Remark 3.6. Apart of a possibly unnecessarily large number Γ, the preceding isoperimetric inequality comprises, firstly, that of Allard in [2, 7.1], secondly, that of the first author in [7, 2.2], and, thirdly, those of the second author in [13, 6.5, 11]. The last two items as well as the present inequality employ the strategy introduced by Michael and Simon in [11, 2.1] (see also Simon [14, 18.6]) in the context of Sobolev inequalities.

DEFINITION 3.7 (Best isoperimetric constant). Whenever m is a positive integer, we denote by $\gamma(m)$ the smallest nonnegative real number with the following property: if n, V, and M are related to m as in 3.1, and $\|V\|(\mathbf{R}^n) < \infty$, then

$$\|V\|(\{x : M(x) \geq d\})^{1-1/m} \leq \gamma(m) d^{-1/m} \|\delta V\|(\mathbf{R}^n) \quad \text{for } 0 < d < \infty;$$

here $0^0 = 0$.

Remark 3.8. Considering a unit disc, we notice $\boldsymbol{\alpha}(m)^{-1/m}/m \leq \gamma(m)$; in particular, $\gamma(1) = 2^{-1}$ by 3.5. Also, if $m > 1$, then $\gamma(m) \leq 5^m 3^{1/(m-1)} \boldsymbol{\alpha}(m)^{-1/m}$ by 3.5, but the precise value of $\gamma(m)$ is unknown.

Remark 3.9. Notice that $\gamma(m)$ is greater or equal to the number bearing that name in [9, §1]; if $m > 1$, it is unknown whether these numbers agree.

COROLLARY 3.10 (General isoperimetric inequality in a ball). *Suppose m and n are positive integers, $m \leq n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, $a \in \mathbf{R}^n$, $0 < r < \infty$, and* $\operatorname{spt}\|V\| \subset \mathbf{B}(a, r)$.

Then,

$$\boldsymbol{\alpha}(m)^{-1/m} r^{-1} \|V\|(\mathbf{R}^n) \leq \gamma(m) \|\delta V\|(\mathbf{R}^n).$$

Proof. Letting $d = \boldsymbol{\alpha}(m)^{-1} r^{-m} \|V\|(\mathbf{R}^n)$ and assuming $d > 0$, it is sufficient to apply 3.7, since $M(x) \geq d$ for $x \in \mathbf{B}(a, r)$, see 3.1. □

3.11 (Embeddings of weak Lebesgue spaces)**.** If ϕ measures X, f is a ϕ measurable $\{t : 0 \leq t \leq \infty\}$ valued function, $\phi\{x : f(x) > 0\} < \infty$, $1 \leq q < p < \infty$, and

$$\kappa = \sup\{d\phi(\{x : f(x) \geq d\})^{1/p} : 0 < d < \infty\} < \infty,$$

then we have $\phi_{(q)}(f) \leq (1 - q/p)^{-1/q} \phi(\{x : f(x) > 0\})^{1/q - 1/p} \kappa$.

COROLLARY 3.12 (General isoperimetric inequality with size). *Suppose m and n are positive integers, $2 \leq m \leq n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, and $(\|V\| + \|\delta V\|)(\mathbf{R}^n) < \infty$.*

Then,

$$d\mathscr{H}^m(\{x : \boldsymbol{\Theta}^m(\|V\|, x) \geq d\})^{1-1/m} \leq \gamma(m) \|\delta V\|(\mathbf{R}^n) \quad \textit{for } 0 < d < \infty;$$

in particular, if $V \in \mathbf{RV}_m(\mathbf{R}^n)$ and $\mathscr{H}^m\{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\} < \infty$, then

$$\|V\|(\mathbf{R}^n) \leq m\gamma(m) \mathscr{H}^m(\{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\})^{1/m} \|\delta V\|(\mathbf{R}^n).$$

Proof. The principal conclusion is a consequence of 3.7, as [5, 2.10.19 (3)] yields

$$\mathscr{H}^m\{x : \boldsymbol{\Theta}^m(\|V\|, x) \geq d\} \leq d^{-1}\|V\|\{x : M(x) \geq d\}.$$

The postscript follows from [2, 3.5 (1b)] and 3.11 with $p = \dfrac{m}{m-1}$ and $q = 1$. □

4. Generalised weakly differentiable functions

This section extends the basic theory of generalised weakly differentiable functions on rectifiable varifolds (see [9, §§8–9] and [10, 4.1, 2]) to general varifolds. In fact, most of it extends almost verbatim (see 4.4, 4.10, 4.11, 4.18, and 4.20) once suitable approximation procedures for Lipschitzian functions (see 4.6 and 4.19) are available to replace the usages of the $(\|V\|, m)$ approximate differential in [9, §§8–9]. It is unknown (see 4.8), whether the role of that approximate differential could be taken, for Lipschitzian functions, by the notion of differentiability introduced by Alberti and Marchese in [1].

A part that does not extend is the existence of decompositions (see 4.12), hence the same holds for the characterisation of functions with vanishing derivative (see 4.13). Nevertheless, a generalised weakly differentiable function may, under the natural summability hypothesis, be defined using a partition of the varifold induced by sets with vanishing distributional boundary (see 4.14).

Lemma 4.1 (Disintegration for varifolds). *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, and $V \in \mathbf{V}_m(U)$. Then,* (*see* [2, 3.3])

$$\int k \, \mathrm{d}V = \iint k(x, P) \, \mathrm{d}V^{(x)}P \, \mathrm{d}\|V\|x$$

whenever k is an $\overline{\mathbf{R}}$ valued V integrable function.

Proof. The case $k \in \mathscr{K}(U \times \mathbf{G}(n, m))$ is treated by Allard in [2, 3.3]. Noting [5, 2.5.13, 14], successive approximation by the method of [5, 2.5.3] yields the case that k is a characteristic function of a V measurable set. Finally, we employ [5, 2.3.3, 4.8, 4.4 (6)] to deduce the general case. □

Definition 4.2 (Generalised V weakly differentiable functions). Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and Y is a finite dimensional normed vector space.

Then, a Y valued $\|V\| + \|\delta V\|$ measurable function f with $\operatorname{dmn} f \subset U$ is called *generalised V weakly differentiable* if and only if for some $\|V\|$ measurable $\operatorname{Hom}(\mathbf{R}^n, Y)$ valued function F, the following two conditions hold:

(1) If K is a compact subset of U and $0 \leq s < \infty$, then

$$\int_{K \cap \{x : |f(x)| \leq s\}} \|F\| \, \mathrm{d}\|V\| < \infty.$$

(2) If $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} \mathrm{D}\gamma$ is compact, then

$$(\delta V)((\gamma \circ f)\theta) = \int \gamma(f(x)) P_\natural \bullet \mathrm{D}\theta(x) + \langle \theta(x), \mathrm{D}\gamma(f(x)) \circ F(x) \rangle \, \mathrm{d} V(x, P).$$

The function F is $\|V\|$ almost unique. Therefore, one may define the *generalised V weak derivative of f* to be the function $V\mathbf{D}f$ characterised (see [5, 2.8.9, 14, 9.13]) by $a \in \operatorname{dmn} V\mathbf{D}f$ if and only if

$$(\|V\|, C) \operatorname*{ap\,lim}_{x \to a} F(x) = \sigma \quad \text{for some } \sigma \in \operatorname{Hom}(\mathbf{R}^n, Y),$$

$$\text{where } C = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^n,\ 0 < r < \infty, \text{ and } \mathbf{B}(a, r) \subset U\},$$

and, in this case, $V\mathbf{D}f(a) = \sigma$. The set of all Y valued generalised V weakly differentiable functions will be denoted by $\mathbf{T}(V, Y)$. Finally, $\mathbf{T}(V) = \mathbf{T}(V, \mathbf{R})$.

Remark 4.3. This definition is in accordance with [9, 8.3], where it is introduced under the additional hypothesis that V is rectifiable.

Remark 4.4. The closedness results [10, 4.1, 2] hold (with the same proof) when the condition $V \in \mathbf{RV}_m(U)$ in their statements is replaced by $V \in \mathbf{V}_m(U)$.

Example 4.5. If $f : U \to Y$ is of class 1, then $f \in \mathbf{T}(V, Y)$ and (see [2, 3.3])

$$V\mathbf{D}f(x) = \mathrm{D}f(x) \circ \int P_\natural \, \mathrm{d} V^{(x)} P \quad \text{for } \|V\| \text{ almost all } x;$$

hence, $\|V\mathbf{D}f(x)\| \le \|\mathrm{D}f(x) | T(x)\|$, where $T(x) = \operatorname{im} \int P_\natural \, \mathrm{d} V^{(x)} P$, for such x.

Lemma 4.6 (Lipschitzian functions I). *Suppose m, n, U, V, and Y are as in 4.2. Then, the following two statements hold.*

(1) *If $f : U \to Y$ is a locally Lipschitzian function, then $f \in \mathbf{T}(V, Y)$ and*

$$\|V\mathbf{D}f(x)\| \le \lim_{r \to 0+} \operatorname{Lip}(f | \mathbf{B}(x, r)) \quad \textit{for } \|V\| \textit{ almost all } x.$$

(2) *If $f_i : U \to Y$ is a sequence of locally Lipschitzian functions converging to $f : U \to Y$ locally uniformly as $i \to \infty$, and*

$$\kappa = \sup\{\|V\|_{(\infty)}(V\mathbf{D}f_i) : i = 1, 2, 3, \ldots\} < \infty,$$

then $f \in \mathbf{T}(V, Y)$, $\|V\|_{(\infty)}(V\mathbf{D}f) \le \kappa$, and

$$\lim_{i \to \infty} \int \langle V\mathbf{D}f_i, G \rangle \, \mathrm{d}\|V\| = \int \langle V\mathbf{D}f, G \rangle \, \mathrm{d}\|V\|$$

whenever $G \in \mathbf{L}_1(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)^*)$.

Proof. Let (2)$'$ denote the statement resulting from (2) by adding the hypothesis, that the sequence f_i belongs to $\mathbf{T}(V, Y)$. To prove (2)$'$, in view of

[5, 2.5.7 (ii)], [10, 2.1], and [4, V.4.2, 5.1], we may assume that, for some function $F \in \mathbf{L}_\infty(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$ with $\|V\|_{(\infty)}(F) \le \kappa$, we have

$$\lim_{i\to\infty} \int \langle V\mathbf{D}f_i, G\rangle \,\mathrm{d}\|V\| = \int \langle F, G\rangle \,\mathrm{d}\|V\| \quad \text{for } G \in \mathbf{L}_1(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)^*).$$

Then, 4.4 and [10, 4.1] yield (2)′ which, as we observe, implies (1) by means of convolution and 4.5; in particular, (2)′ and (2) are equivalent. □

Remark 4.7. (1) partly generalises [9, 8.7]. In the remaining part thereof (i.e., in the characterisation of $V\mathbf{D}f$ in terms of the $(\|V\|, m)$ approximate differential), the hypothesis $V \in \mathbf{RV}_m(U)$ may not be weakened to $V \in \mathbf{V}_m(U)$; in fact, 4.5 and [2, 4.8 (2)] readily yield examples.

Remark 4.8. Here, we formulate two open questions which relate the preceding remark to the differentiability theory of Lipschitzian functions by Alberti and Marchese (see [1, 1.1]). For this purpose, suppose m and n are positive integers, $m < n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, and $\|\delta V\|$ is a Radon measure.

(1) Does it follow that, for $\|V\|$ almost all x, the image of $q(x) = \int P_\natural \,\mathrm{d}V^{(x)}P \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is contained in the "decomposability bundle of $\|V\|$" at x introduced by Alberti and Marchese in [1, 2.6]?

(2) If so, does it follow that, for Lipschitzian functions $f : \mathbf{R}^n \to Y$,

$$V\mathbf{D}f(x) = \mathrm{D}(f|Q(x))(x) \circ q(x) \quad \text{for } \|V\| \text{ almost all } x,$$

where $Q(x) = \{x + \langle v, q(x)\rangle : v \in \mathbf{R}^n\}$?

Remark 4.9. 4.6 (2) is analogous to [8, 4.5 (3)].

Remark 4.10. Referring to 4.5 and 4.6 (2) in place of [8, 4.5 (3)], and noting 4.1, the result in [9, 8.6] takes the following form: *If $f \in \mathbf{T}(V, Y)$, $\theta : U \to \mathbf{R}^n$ is Lipschitzian with compact support, $\gamma : Y \to \mathbf{R}$ is of class* 1, *and either* $\operatorname{spt} \mathrm{D}\gamma$ *is compact or f is locally bounded, then*

$$(\delta V)((\gamma \circ f)\theta) = \int \gamma(f(x)) \operatorname{trace}(V\mathbf{D}\theta(x)) + \langle \theta(x), \mathrm{D}\gamma(f(x)) \circ V\mathbf{D}f(x)\rangle \,\mathrm{d}\|V\|x.$$

Consequently, if $f \in \mathbf{T}(V, Y)$ is locally bounded, Z is a finite dimensional normed vector space, and $g : Y \to Z$ is of class 1, *then $g \circ f \in \mathbf{T}(V, Z)$ with*

$$V\mathbf{D}(g \circ f)(x) = \mathrm{D}g(f(x)) \circ V\mathbf{D}f(x) \quad \textit{for } \|V\| \textit{ almost all } x.$$

We notice that, by [9, 8.7], this is a generalisation of [9, 8.6].

Remark 4.11. The results [9, 8.4, 5, 12, 15, 16, 18, 20, 29, 30, 33] remain valid when the references to "Definition 8.3" in [9] in their statements and proofs are replaced by references to the present, more general definition in 4.2; in fact, it is sufficient to additionally replace the references to "Remark 8.6" in their proofs in [9] by references to 4.10 in the present paper and use (instead of "Example 8.7" in [9]) an approximation based on convolution, 4.5, and 4.6, to justify the second ingredient to the equality on page 1029, line 25 in [9]: namely, the equation

$$\langle u, V\mathbf{D}((v \circ g)\theta)(x)\rangle = \langle u, \mathrm{D}v(g(x)) \circ V\mathbf{D}g(x)\rangle\theta(x) + v(g(x))\langle u, V\mathbf{D}\theta(x)\rangle$$

whenever $u \in \mathbf{R}^n$, for $\|V\|$ almost all x.

Example 4.12 (Nonexistence of decompositions). Suppose m and n are positive integers, $m < n$, and $T \in \mathbf{G}(n, m)$. Then the following three statements hold.

(1) If μ is a Radon measure over $\mathbf{R}^n$, $V = \mu \times \boldsymbol{\delta}_T \in \mathbf{V}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, $f : \mathbf{R}^n \to \mathbf{R}$ is of class 1 with $\mathrm{D}f(x)|T = 0$ for $x \in \mathbf{R}^n$, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$, then $V\partial E(y) = 0$ for $y \in \mathbf{R}$.
(2) If E is an $\mathscr{L}^n$ measurable set, $V = (\mathscr{L}^n \llcorner E) \times \boldsymbol{\delta}_T \in \mathbf{V}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, and V is indecomposable, then $V = 0$.
(3) If $V = \mathscr{L}^n \times \boldsymbol{\delta}_T \in \mathbf{V}_m(\mathbf{R}^n)$, then $\delta V = 0$ and there does not exist a decomposition of V.

To prove (1), we notice $f \in \mathbf{T}(V)$ and $V\mathbf{D}f(x) = 0$ for $\|V\|$ almost all x by 4.5, whence we deduce the assertion by means of 4.11 and [9, 8.29]. Moreover, (1) yields (2) by taking f to be a nonzero member of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$ with $T \subset \ker f$. Finally, Allard [2, 4.8 (2)] and (2) imply (3).

Remark 4.13. 4.12 (3) shows that the rectifiability hypotheses in [9, 6.12, 8.34] may not be omitted.

THEOREM 4.14 (Weakly differentiable functions by partitions). *Suppose m, n, U, V, and Y are as in* 4.2, *Ξ is a countable subset of $\mathbf{V}_m(U)$, ξ maps Ξ into the class of all Borel subsets of U such that distinct members of Ξ are mapped onto disjoint sets,* $(\|V\| + \|\delta V\|)(U \sim \bigcup \operatorname{im} \xi) = 0$,

$$W = V \llcorner \xi(W) \times \mathbf{G}(n, m) \quad \textit{and} \quad V\partial\xi(W) = 0 \quad \textit{for } W \in \Xi,$$

$f_W \in \mathbf{T}(W, Y)$ for $W \in \Xi$, and

$$f = \bigcup\{f_W \mid \xi(W) : W \in \Xi\}, \quad F = \bigcup\{(W\mathbf{D}f_W) \mid \xi(W) : W \in \Xi\}.$$

Then, the following three statements hold:
(1) *The function f is $\|V\| + \|\delta V\|$ measurable.*
(2) *The function F is $\|V\|$ measurable.*
(3) *If $\int_{K \cap \{x : |f(x)| \le s\}} \|F\| \,\mathrm{d}\|V\| < \infty$ whenever K is a compact subset of U and $0 \le s < \infty$, then $f \in \mathbf{T}(V, Y)$ and*

$$V\mathbf{D}f(x) = F(x) \quad \textit{for } \|V\| \textit{ almost all } x.$$

Proof. The proof of [9, 8.24] applies unchanged. □

Remark 4.15. In view of 4.13, it is important that the preceding generalisation of [9, 8.24] does not assume the members of Ξ to be indecomposable.

Definition 4.16 (Zero boundary values). Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and G is a relatively open subset of Bdry U. Then the space $\mathbf{T}_G(V)$ is defined to be the set of all nonnegative functions $f \in \mathbf{T}(V)$ such that, abbreviating $B = (\operatorname{Bdry} U) \sim G$ and $E(y) = \{x : f(x) > y\}$ for $0 < y < \infty$, the following conditions hold for $\mathscr{L}^1$ almost all $0 < y < \infty$:

$$(\|V\| + \|\delta V\|)(E(y) \cap K) + \|V \partial E(y)\|(U \cap K) < \infty,$$

$$\int_{E(y) \times \mathbf{G}(n,m)} P_\natural \bullet \mathrm{D}\theta(x) \, \mathrm{d}V(x,P) = ((\delta V) \llcorner E(y))(\theta|U) - V\partial E(y)(\theta|U)$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$ and $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$.

Remark 4.17. Defining $W_y \in \mathbf{V}_m(\mathbf{R}^n \sim B)$ by

$$W_y(k) = \int_{E(y) \times \mathbf{G}(n,m)} k \, \mathrm{d}V \quad \text{for } k \in \mathscr{K}((\mathbf{R}^n \sim B) \times \mathbf{G}(n,m))$$

for $0 < y < \infty$, we see that, whenever y satisfies the conditions of 4.16, we have

$$\|\delta W_y\|(A) \leq \|\delta V\|(E(y) \cap A) + \|V \partial E(y)\|(U \cap A) \quad \text{for } A \subset \mathbf{R}^n \sim B;$$

in particular, $\|\delta W_y\|$ is a Radon measure for such y.

Remark 4.18. The definition in 4.16 is in accordance with [9, 9.1], where it is stated under the additional hypothesis that V is rectifiable. Moreover, the results of [9, 9.2, 4, 5, 9, 12, 13, 14] remain valid if the references to "Definition 9.1" in their statements and proofs in [9] are replaced by references to the present, more general definition in 4.16; in fact, taking 4.11 into account, the proofs remain otherwise unchanged.

Lemma 4.19 (Lipschitzian functions II). *Suppose m, n, U, V, and G are as in* 4.16, $c : U \cup G \to \mathbf{R}$, $\operatorname{Lip}(c|K) < \infty$ *whenever K is a compact subset of* $\mathbf{R}^n \sim B$, $g = c|U$, *D is a $\|V\|$ measurable set*, $W \in \mathbf{V}_m(\mathbf{R}^n \sim B)$, $W(k) = \int_{D \times \mathbf{G}(n,m)} k \, \mathrm{d}V$ *for* $k \in \mathscr{K}((\mathbf{R}^n \sim B) \times \mathbf{G}(n,m))$, *and $\|\delta W\|$ is a Radon measure.*

Then, there holds $c \in \mathbf{T}(W)$, $g \in \mathbf{T}(V)$, *and*

$$W\mathbf{D}c(x) = V\mathbf{D}g(x) \quad \textit{for } \|V\| \textit{ almost all } x \in D.$$

Proof. Assuming $\kappa = \operatorname{Lip} c < \infty$, we extend c to $\zeta : \mathbf{R}^n \to \mathbf{R}$ such that $\operatorname{Lip} \zeta = \kappa$ by means of [5, 2.10.43]. Then, using convolution, we construct a sequence $\zeta_i \in \mathscr{E}(\mathbf{R}^n, \mathbf{R})$ with $\operatorname{Lip} \zeta_i \leq \kappa$ for every positive integer i and

$$\zeta_i(x) \to \zeta(x), \quad \text{uniformly for } x \in \mathbf{R}^n, \text{ as } i \to \infty.$$

Since $W^{(x)} = V^{(x)}$ for $\|V\|$ almost all $x \in D$ by [5, 2.8.9, 18, 9.11], we note

$$W\mathbf{D}(\zeta_i \,|\, \mathbf{R}^n \sim B)(x) = V\mathbf{D}(\zeta_i|U)(x) \quad \text{for } \|V\| \text{ almost all } x \in D$$

for every positive integer i by 4.5. Therefore, passing to the limit $i \to \infty$ with the help of 4.6, we deduce the conclusion. □

Remark 4.20. The result of [9, 9.16] remains valid if the references to "Definition 9.1" in its statement and its proof are replaced by references to the present, more general definition in 4.16; in fact, taking 4.11 and 4.18 into account, it is sufficient to additionally replace the occurrences of "$\mathbf{RV}_m$" on page 1044, lines 15 and 29 in [9] by "$\mathbf{V}_m$" and the words "Example 8.7 in conjunction with [5, 2.10.19 (4), 2.10.43]" on page 1044, lines 26–27 in [9] by a reference to 4.19 in the present paper.

5. Sobolev inequalities

In this section, we present Sobolev inequalities for generalised weakly differentiable functions with zero boundary values, that are entailed by the general isoperimetric inequalities in 3.5 and 3.10. As the formal analogue to 3.5 does not hold (see 5.3), two alternative formulations are offered. The first version (see 5.6) involves an averaging process based on medians and a scale (possibly depending on the point). The second version (see 5.8) implies control only on the rectifiable part. For both statements, we isolate a classical technique due to Federer in 5.4 and 5.5. The analogue for 3.10, in contrast, is immediate (see 5.9). Finally, the negative results of this section (see 5.3 and 5.7) are entailed by examples (see 5.1 and 5.2) based on known scaling properties of derivatives in Euclidean space.

Example 5.1. Suppose n is an integer, $n \geq 2$, and $n/(n-1) < p \leq \infty$. Then,

$$\sup\left\{ (\mathscr{L}^n)_{(p)}(f) : 0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}), \operatorname{spt} f \subset \mathbf{U}(0,1), \int |\mathrm{D}f| \,\mathrm{d}\mathscr{L}^n \leq 1 \right\} = \infty;$$

in fact, we fix $0 \leq g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ with $\operatorname{spt} g \subset \mathbf{U}(0,1)$ and $\int |\mathrm{D}g| \,\mathrm{d}\mathscr{L}^n = 1$, and consider $f_\varepsilon \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ with $f_\varepsilon(x) = \varepsilon^{1-n} g(\varepsilon^{-1}x)$ for $x \in \mathbf{R}^n$ and $0 < \varepsilon \leq 1$.

Example 5.2. Suppose m and n are positive integers, $m < n$, and Φ is the set of (V, f) such that $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\|V\|\mathbf{U}(0,1) = \boldsymbol{\alpha}(n)$, $\delta V = 0$, $0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $\operatorname{spt} f \subset \mathbf{U}(0,1)$, and $\int |V\mathbf{D}f| \,\mathrm{d}\|V\| \leq 1$. Then, we will prove that

$$\sup\{\|V\|_{(p)}(f) : (V, f) \in \Phi\} = \infty \quad \text{for } n/(n-1) < p \leq \infty.$$

We first pick $T \in \mathbf{G}(n, m)$, define $W = \mathscr{L}^n \times \boldsymbol{\delta}_T \in \mathbf{V}_m(\mathbf{R}^n)$, and recall $\delta W = 0$ from 4.12 (3); in particular, the assertion resulting from replacing "$\mathbf{RV}_m$" by "$\mathbf{V}_m$" is a consequence of 4.5 and 5.1. Then, we approximate W by varifolds $V \in \mathbf{RV}_m(\mathbf{R}^n)$ with $\|V\|\mathbf{U}(0,1) = \boldsymbol{\alpha}(n)$, $\delta V = 0$, and $V^{(x)} = \boldsymbol{\delta}_T$ for $x \in \mathbf{R}^n$. (Geometrically, each approximating varifold V corresponds to a positive multiple of the union of a countable collection of affine planes parallel to T.)

Example 5.3 (Sobolev inequality vs. general isoperimetric inequality). If m and n are positive integers, $m < n$, $\beta = \infty$ if $m = 1$, $\beta = m/(m-1)$ if $m > 1$, and $0 < d < \infty$, then the supremum of the set of all numbers

$$(\|V\| \llcorner \{x : M(x) \geq d\})_{(\beta)}(f)$$

corresponding to $V \in \mathbf{RV}_m(\mathbf{R}^n)$ and $f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ satisfying $\delta V = 0$, $f \geq 0$, and $\int |V \mathbf{D} f| \, \mathrm{d}\|V\| \leq 1$, where M is associated to m, n, and V as in 3.1, equals ∞; in fact, assuming $d = \boldsymbol{\alpha}(m)^{-1}\boldsymbol{\alpha}(n)$, one may take $p = \beta$ in 5.2.

LEMMA 5.4 (Integrating superlevel sets). *Suppose ϕ measures X, f is a nonnegative ϕ measurable function, $1 \leq p \leq \infty$, and $E(y) = \{x : f(x) > y\}$ for $0 \leq y < \infty$. Then,*

$$\phi_{(p)}(f) \leq \int_0^\infty \phi(E(y))^{1/p} \, \mathrm{d}\mathscr{L}^1 y;$$

here $0^{1/p} = 0$ and $\infty^{1/p} = \infty$.

Proof. Assume $p < \infty$ and $\int_0^\infty \phi(E(y))^{1/p} \, \mathrm{d}\mathscr{L}^1 y < \infty$. Then, possibly replacing $f(x)$ by $\sup\{0, f(x) - \varepsilon\}$ for $0 < \varepsilon < \infty$, we may also assume $\phi(E(0)) < \infty$. Abbreviating $f_y = \inf\{f, y\}$, we define $g : \{y : 0 \leq y < \infty\} \to \mathbf{R}$ by

$$g(y) = \phi_{(p)}(f_y) \quad \text{for } 0 \leq y < \infty.$$

Minkowski's inequality (see [5, 2.4.15]) yields

$$0 \leq g(y + \upsilon) - g(y) \leq \phi_{(p)}(f_{y+\upsilon} - f_y) \leq \upsilon \phi(E(y))^{1/p}$$

for $0 \leq y < \infty$ and $0 \leq \upsilon < \infty$. Therefore, $\operatorname{Lip} g < \infty$ and, by [5, 2.9.19],

$$0 \leq g'(y) \leq \phi(E(y))^{1/p} \quad \text{for } \mathscr{L}^1 \text{ almost all } 0 \leq y < \infty,$$

hence, by [5, 2.4.7, 9.20], we infer $\phi_{(p)}(f) = \lim_{y \to \infty} g(y) = \int_0^\infty g' \, \mathrm{d}\mathscr{L}^1$. □

Remark 5.5. The method of the proof is taken from [5, 4.5.9 (18)].

THEOREM 5.6 (Sobolev inequality—with averaging). *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V)$, $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$, $\|V\|(E(y)) < \infty$ for $0 < y < \infty$,*

$$\beta = \infty \quad \textit{if } m = 1, \qquad \beta = m/(m-1) \quad \textit{if } m > 1,$$

$0 < d < \infty$, *r is a* $\{t : 0 < t < \infty\}$ *valued* $\|V\|$ *measurable function*, $\operatorname{dmn} r \subset U$, $0 < \lambda < 1$, $g : \operatorname{dmn} r \to \overline{\mathbf{R}}$ *satisfies*

$$g(a) = \sup\{y : \|V\|(U \cap \mathbf{B}(a, r(a)) \sim E(y)) \le \lambda \|V\|(U \cap \mathbf{B}(a, r(a)))\}$$

for $a \in \operatorname{dmn} r$, *and* $A = \{a : \infty > \|V\|(U \cap \mathbf{B}(a, r(a))) \ge d\boldsymbol{\alpha}(m) r(a)^m\}$.

Then, g is $\|V\|$ *measurable and there holds*

$$(\|V\| \llcorner A)_{(\beta)}(g) \le \Gamma\left(\int f \,\mathrm{d}\|\delta V\| + \int |V\mathbf{D}f| \,\mathrm{d}\|V\|\right),$$

where $\Gamma = (1 - \lambda)^{-1} \boldsymbol{\beta}(n)^{1-1/m} \boldsymbol{\gamma}(m) d^{-1/m}$.

Proof. Firstly, we use the facts, that the supremum equalling $g(a)$ remains unchanged when y therein is restricted to be rational and that

$$(U \times U) \cap \{(a, x) : |a - x| \le r(a)\}$$

is $\|V\| \times \|V\|$ measurable, to deduce the $\|V\|$ measurability of g from Fubini's theorem (see [5, 2.6.2]). Next, we define $W_y \in \mathbf{V}_m(\mathbf{R}^n)$ as in 4.17 and let M_y denote the function resulting from replacement of V by W_y in the definition of the function M in 3.1. Whenever $0 < y < \infty$, $a \in A$, and $g(a) > y$, we note

$$\|V\|(U \cap \mathbf{B}(a, r(a))) \le \|W_y\| \mathbf{B}(a, r(a)) + \lambda \|V\|(U \cap \mathbf{B}(a, r(a))) < \infty,$$
$$\|V\|(U \cap \mathbf{B}(a, r(a))) \le (1 - \lambda)^{-1} \|W_y\| \mathbf{B}(a, r(a)),$$
$$\mathbf{B}(a, r(a)) \subset \{x : M_y(x) \ge (1 - \lambda) d\}$$

by 3.1. Therefore, the Besicovich-Federer covering theorem yields

$$\begin{aligned}\|V\|(A \cap \{a : g(a) > y\}) &= \lim_{i \to \infty} \|V\|(A \cap \{a : g(a) > y \text{ and } r(a) \le i\}) \\ &\le \boldsymbol{\beta}(n)(1 - \lambda)^{-1} \|W_y\| \{x : M_y(x) \ge (1 - \lambda) d\}\end{aligned}$$

for $0 < y < \infty$, whence we infer, as $\|W_y\|(\mathbf{R}^n) < \infty$, that

$$\|V\|(A \cap \{a : g(a) > y\})^{1/\beta} \le \Gamma \|\delta W_y\|(\mathbf{R}^n) \le \Gamma(\|\delta V\|(E(y)) + \|V \partial E(y)\|(U))$$

for $\mathscr{L}^1$ almost all $0 < y < \infty$ by 3.5, 3.7, and 4.17; here $0^0 = 0$. Since g is nonnegative, integrating this inequality with respect to $\mathscr{L}^1$ yields the conclusion by means of 5.4, [5, 2.6.2], 4.11, and [9, 8.5, 30]. □

Remark 5.7. If $m < n$, one may not replace g by f in the preceding estimate; in fact, in view of 4.5, 4.18, and [9, 9.4], it is sufficient to consider $U = \mathbf{R}^n \cap \mathbf{U}(0, 1)$, $d = 2^{-m} \boldsymbol{\alpha}(m)^{-1} \boldsymbol{\alpha}(n)$, and $r(a) = 2$ for $a \in U$, and take $p = \infty$ if $m = 1$ and $p = m/(m - 1)$ if $m > 1$ in 5.2.

THEOREM 5.8 (Sobolev inequality—rectifiable part). *Suppose m and n are positive integers, $m \le n$, U is an open subset of* $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ *is a Radon* *measure*, $f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V)$, $\|V\|\{x : f(x) > y\} < \infty$ *for* $0 < y < \infty$,

$$\beta = \infty \quad \text{if} \quad m = 1, \qquad \beta = m/(m-1) \quad \text{if} \quad m > 1,$$

$0 < d < \infty$, *and* $A = \{a : \boldsymbol{\Theta}^m(\|V\|, a) \geq d\}$.

Then, there holds

$$(\|V\| \llcorner A)_{(\beta)}(f) \leq \Gamma\left(\int f \,\mathrm{d}\|\delta V\| + \int |V\mathbf{D}f| \,\mathrm{d}\|V\|\right),$$

where $\Gamma = \boldsymbol{\gamma}(m) d^{-1/m}$.

Proof. We define $E(y)$ as in 4.16. Moreover, we define $W_y \in \mathbf{V}_m(\mathbf{R}^n)$ as in 4.17 and let M_y denote the function resulting from replacement of V by W_y in the definition of the function M in 3.1. Since $\boldsymbol{\Theta}^m(\|W_y\|, x) \geq d$ for $\|V\|$ almost all $x \in A \cap E(y)$ by [5, 2.8.9, 18, 9.11], we conclude

$$\|V\|(A \cap E(y)) \leq \|W_y\|\{x : M_y(x) \geq d\} \quad \text{for } 0 < y < \infty.$$

In conjunction with 3.5, 3.7, and 4.17, we infer, as $\|W_y\|(\mathbf{R}^n) < \infty$, that

$$\|V\|(A \cap E(y))^{1/\beta} \leq \Gamma\|\delta W_y\|(\mathbf{R}^n) \leq \Gamma(\|\delta V\|(E(y)) + \|V \partial E(y)\|(U))$$

for $\mathscr{L}^1$ almost all $0 < y < \infty$; here $0^0 = 0$. Integrating this inequality yields the conclusion by means of 5.4, [5, 2.6.2], 4.11, and [9, 8.5, 30]. □

THEOREM 5.9 (Poincaré inequality in a ball—zero boundary values). *Suppose m and n are positive integers, $m \leq n$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{V}_m(\mathbf{U}(a, r))$, $\|\delta V\|$ is a Radon measure, and $f \in \mathbf{T}_{\mathrm{Bdry}\,\mathbf{U}(a,r)}(V)$.*

Then, there holds

$$\boldsymbol{\alpha}(m)^{-1/m} r^{-1} \int f \,\mathrm{d}\|V\| \leq \boldsymbol{\gamma}(m)\left(\int f \,\mathrm{d}\|\delta V\| + \int |V\mathbf{D}f| \,\mathrm{d}\|V\|\right).$$

Proof. Define $E(y) = \{x : f(x) > y\}$ for $0 < y < \infty$. In view of 4.17, we apply 3.10 with V replaced by W_y to obtain

$$\boldsymbol{\alpha}(m)^{-1/m} r^{-1} \|V\|(E(y)) \leq \boldsymbol{\gamma}(m)(\|\delta V\|(E(y)) + \|V \partial E(y)\| \mathbf{U}(a, r))$$

for $\mathscr{L}^1$ almost all $0 < y < \infty$. Integrating this inequality with respect to $\mathscr{L}^1$ yields the conclusion by means of Fubini's theorem (see [5, 2.6.2]) and the coarea formula (see 4.11 and [9, 8.5, 30]). □

Remark 5.10. In view of 4.5, 4.18, and [9, 9.4], there is no similar control of $\|V\|_{(p)}(f)$ involving a number depending only on m and p, for any $p > 1$ by 5.2.

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