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ON 3-DIMENSIONAL HOMOGENEOUS GENERALIZED m-QUASI-EINSTEIN MANIFOLDS

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Abstract

In this paper, we show that for 3-dimensional homogeneous manifolds only the space form can carry a proper generalized m-quasi-Einstein structure.

1. Introduction

Recently, there has been increasing interest on the quasi-Einstein manifolds, which generalize the notion of Einstein manifolds. Recall that a Riemannian manifold (M^n, g) with a potential function f is called m-quasi-Einstein if its associated m-Bakry-Emery Ricci tensor $\mathrm{Ric}_f^m := \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df$ is a constant multiple of the metric g (cf. [4] and the references therein). An m-quasi-Einstein manifold will be called trivial if f is constant. Otherwise, it will be called nontrivial. One of the motivations to study m-quasi-Einstein manifold is its close relation with warped product Einstein metrics. As a matter of fact, it was shown in [12] that an n-dimensional m-quasi-Einstein manifold is exactly the manifold which is the base of an $(n+m)$-dimensional Einstein warped product.

To extend the notion of m-quasi-Einstein, Catino [5] introduced the concept of generalized quasi-Einstein manifold, and as its particular case, Barros and Ribeiro [2] further proposed to consider the following notion of gradient generalized m-quasi-Einstein manifold, or simply *generalized m-quasi-Einstein manifold*:

Definition 1.1 ([2]). For a positive integer m, we say that a manifold (M^n, g) with a potential function f is generalized m-quasi-Einstein if there exists a smooth function λ on M such that the Ricci tensor Ric of (M^n, g) satisfies

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the relation

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.1}$$

where ∇^2 and $\otimes$ denote the Hessian and the tensorial product, respectively. If, in particular, the function λ in (1.1) is non-constant on M, the generalized m-quasi-Einstein manifold is called proper.

In recent years, generalized m-quasi-Einstein manifolds have been extensively studied by many mathematicians, see e.g. [1, 2, 5, 7, 8, 9, 10, 11, 13, 15], among many others. As usual, one can define the positive function $u = e^{-f/m}$ on M^n so that (1.1) can be rewritten as

$$\mathrm{Ric} - \frac{m}{u} \nabla^2 u = \lambda g. \tag{1.2}$$

Due to such a fact, the generalized m-quasi-Einstein manifold satisfying (1.1) is usually denoted by (M^n, g, u, λ), where $u = e^{-f/m}$. Moreover, (M^n, g, u, λ) will be called *trivial* if the potential function u is constant. Otherwise, it will be called *nontrivial*. It is easy to see that an n-dimensional ($n \geq 3$) proper generalized m-quasi-Einstein manifold must be nontrivial. Obviously, the triviality of (M^n, g, u, λ) implies that (M^n, g) is Einstein. But, generally, the converse is not true. To see this fact more clearly, we would recall that in [2] it was shown that, as trivial Einstein manifold, each of the three kinds of space forms can possess a nontrivial generalized m-quasi-Einstein structure.

In [3], Barros et al. studied m-quasi-Einstein structures on 3-dimensional homogeneous Riemannian manifolds. As the main theorem, they proved that if a 3-dimensional homogeneous manifold carries an m-quasi-Einstein structure then it is either Einstein or $\mathbf{H}^2(k) \times \mathbf{R}$, where $\mathbf{H}^2(k)$ denotes the 2-dimensional hyperbolic space with sectional curvature k.

In [4] Case-Shu-Wei showed that a nontrivial m-quasi-Einstein manifold is Einstein if and only if it is isometric to a hyperbolic space or a special Einstein warped product. That is to say, neither Euclidean space $\mathbf{R}^n$ nor sphere $\mathbf{S}^n$ can carry a nontrivial m-quasi-Einstein structure. Along with the result due to Barros et al. [3], we find that for 3-dimensional homogeneous manifolds only hyperbolic space and $\mathbf{H}^2(k) \times \mathbf{R}$ can carry nontrivial m-quasi-Einstein structures. However, there do exist manifolds that can not carry any nontrivial m-quasi-Einstein structure but can possess a proper generalized m-quasi-Einstein structure. Indeed, according to [2], both $\mathbf{R}^n$ and $\mathbf{S}^n$ can carry a proper generalized m-quasi-Einstein structure. Along this direction, in this work, we will focus on proper generalized m-quasi-Einstein structure on homogeneous 3-manifolds.

As is well known, every space form can carry a proper generalized m-quasi-Einstein structure. In fact, this has been stated as Examples 1, 2 and 3 in [2] with detailed discussions. Now, the following problem becomes interesting.

PROBLEM. Are space forms the only 3-dimensional homogeneous manifolds that can carry a proper generalized m-quasi-Einstein structure?

In this paper, as the main result we will give a positive answer to this problem.

THEOREM 1.1. *Let* (M^3, g) *be a* 3*-dimensional homogeneous manifold that can carry a proper generalized m-quasi-Einstein structure, then* (M^3, g) *must be a space form.*

Remark 1.1. As has been mentioned above, the proper generalized m-quasi-Einstein structure on each space form has been studied in [2]. In fact, as shown in Theorem 1 of [2], the potential function f for that situation can be explicitly determined on space forms up to constant.

Remark 1.2. Due to the relation between the m-quasi-Einstein manifold and the $(n+m)$-dimensional Einstein warped product, in the definition of generalized m-quasi-Einstein manifold, m is usually assumed to be a positive integer (cf. [1, 2]). On the other side, we would like to point out that, the results in this paper also hold for any positive constant m.

2. Preliminary

The classification of simply connected 3-dimensional homogeneous manifolds is well known (see W. P. Thurston [14]). In fact, such a manifold has an isometry group of dimension 3, 4 or 6. If the dimension of the isometry group is 6, then the manifold is a space form. If the dimension of the isometry group is 3, the manifold has the geometry of the Lie group Sol_3. Whereas if the dimension of the isometry group is 4, such a manifold is a Riemannian fibration over a 2-dimensional space form $\mathbf{N}^2(k)$ with constant Gauss curvature k, the fibers are totally geodesic and there exists a one-parameter family of translations along the fibers, generated by a unit Killing vector field ξ. These manifolds can be classified, up to isometry, by k and the so-called bundle curvature τ, the latter is defined by the equation $\overline{\nabla}_X \xi = \tau X \times \xi$ for any vector field X, where $\times$ denotes the vector product and $\overline{\nabla}$ denotes the Riemannian connection. Moreover, k and τ can be any real numbers satisfying $k \neq 4\tau^2$. The manifold with 4-dimensional isometry group as described above is always denoted by $E^3(k,\tau)$. According to the classification, if $\tau \neq 0$ and $k > 0$, $E^3(k,\tau)$ is compact, and it has the isometry group of the Berger sphere $\mathbf{S}^3_{k,\tau}$. If $E^3(k,\tau)$ is non-compact, it has isometry group of one of the following Riemannian manifolds:

$$\begin{cases} \mathbf{S}^2(k) \times \mathbf{R}, & \text{when } \tau = 0,\ k > 0; \\ \mathbf{H}^2(k) \times \mathbf{R}, & \text{when } \tau = 0,\ k < 0; \\ \mathrm{Nil}_3, & \text{when } \tau \neq 0,\ k = 0; \\ \widetilde{\mathrm{PSL}_2}(\mathbf{R}), & \text{when } \tau \neq 0,\ k < 0. \end{cases}$$

Here Nil_3 stands for the classical Heisenberg group endowed with a left invariant metric, $\widetilde{\mathrm{PSL}_2(\mathbf{R})}$ is the universal cover of the Lie group $\mathrm{PSL}_2(\mathbf{R})$ (endowed with a 2-parameter family of homogeneous metrics).

3. Generalized m-quasi-Einstein structure

Based on the dimension of the isometry group, we will discuss 3-dimensional homogeneous generalized m-quasi-Einstein manifolds in different cases. Since homogeneous 3-manifolds with isometry group of dimension 6 are space forms, and the generalized m-quasi-Einstein structure on space forms has been treated in [2], in what follows, we will mainly deal with homogeneous 3-manifolds with isometry group of dimension 3 and 4, respectively, in order to see whether there exists a proper generalized m-quasi-Einstein structure on these manifolds.

3.1. Homogeneous 3-manifold with isometry group of dimension 3. In this subsection, we will show that homogeneous 3-manifolds with isometry group of dimension 3 do not carry any generalized m-quasi-Einstein structure. As mentioned above, such a manifold possesses the geometry modeled of the Lie group Sol_3. We will prove the following

PROPOSITION 3.1. Sol_3 *does not carry any generalized m-quasi-Einstein structure.*

We first recall the geometry of the space Sol_3, for details see section 2 in [6]. Exactly, Sol_3 can be viewed as $\mathbf{R}^3$ endowed with the metric

$$\hat{g} = e^{2z}\,dx^2 + e^{-2z}\,dy^2 + dz^2, \tag{3.1}$$

where (x, y, z) are canonical coordinates of $\mathbf{R}^3$. It is worthy to note that Sol_3 has a Lie group structure with respect to which the above metric is left-invariant. A canonical orthonormal frame with respect to $\hat{g}$ is given by

$$\{E_1 = e^{-z}\partial_x,\ E_2 = e^{z}\partial_y,\ E_3 = \partial_z\}. \tag{3.2}$$

By using this frame we get the following lemma.

LEMMA 3.1 (see also [3]). *Let us consider on* Sol_3 *the metric and the frame given by* (3.1) *and* (3.2), *respectively. Then we have*

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = -E_2.$$

The Riemannian connection $\hat{\nabla}$ *of* Sol_3 *can be expressed by*:

$$\begin{aligned}
&\hat{\nabla}_{E_1}E_1 = -E_3, && \hat{\nabla}_{E_1}E_2 = 0, && \hat{\nabla}_{E_1}E_3 = E_1,\\
&\hat{\nabla}_{E_2}E_1 = 0, && \hat{\nabla}_{E_2}E_2 = E_3, && \hat{\nabla}_{E_2}E_3 = -E_2,\\
&\hat{\nabla}_{E_3}E_1 = 0, && \hat{\nabla}_{E_3}E_2 = 0, && \hat{\nabla}_{E_3}E_3 = 0.
\end{aligned}$$

Moreover, the Ricci tensor $\widehat{\mathrm{Ric}}$ *of* Sol_3 *satisfies*

$$\hat{R}_{11} = \hat{R}_{22} = 0, \quad \hat{R}_{33} = -2, \quad \hat{R}_{ij} = 0 \quad \textit{for } i \neq j,$$

where $\hat{R}_{ij} = \widehat{\mathrm{Ric}}(E_i, E_j)$.

To prove Proposition 3.1, we also need the next lemma.

Lemma 3.2. *Suppose that* $(\mathrm{Sol}_3, \hat{g}, u, \lambda)$ *is a generalized* m*-quasi-Einstein structure on* Sol_3. *Then the function* $u = u(x, y, z)$ *satisfies the following equations*:
(1) $u_{xy} = 0$,
(2) $u_{xz} = u_x$,
(3) $u_{yz} = -u_y$,
(4) $e^{-2z}u_{xx} + u_z = -\dfrac{\lambda u}{m}$,
(5) $e^{2z}u_{yy} - u_z = -\dfrac{\lambda u}{m}$,
(6) $u_{zz} = -\dfrac{\lambda + 2}{m}u$.

Proof. With respect to the orthonormal frame (3.2), we can rewrite (1.2) as:

$$\hat{\nabla}^2 u(E_i, E_j) = \frac{u}{m}(\hat{R}_{ij} - \lambda\delta_{ij}).$$

From Lemma 3.1 we get $\hat{R}_{12} = 0$, and then $\hat{\nabla}^2 u(E_1, E_2) = 0$.
On the other hand, using $\hat{\nabla}_{E_1} E_2 = 0$, we can calculate

$$\begin{aligned}\hat{\nabla}^2 u(E_1, E_2) &= \hat{g}(\hat{\nabla}_{E_1}\hat{\nabla} u, E_2) = E_1(E_2(u)) - \hat{g}(\hat{\nabla} u, \hat{\nabla}_{E_1} E_2) \\ &= E_1 E_2(u) = u_{xy}.\end{aligned}$$

It follows that $u_{xy} = 0$.
Similarly, direct calculations of the following terms

$$\hat{\nabla}^2 u(E_1, E_3),\ \hat{\nabla}^2 u(E_2, E_3),\ \hat{\nabla}^2 u(E_1, E_1),\ \hat{\nabla}^2 u(E_2, E_2) \text{ and } \hat{\nabla}^2 u(E_3, E_3)$$

will verify all other assertions. □

Proof of Proposition 3.1. If Sol_3 carries a generalized m-quasi-Einstein structure $(\mathrm{Sol}_3, \hat{g}, u, \lambda)$, we first use (4) and (6) of Lemma 3.2 to deduce that the function u satisfies

$$e^{-2z}u_{xx} + u_z = u_{zz} + \frac{2}{m}u,$$

and then

$$e^{-2z}u_{xxy} + u_{zy} = u_{zzy} + \frac{2}{m}u_y. \tag{3.3}$$

According to (1) and (3) of Lemma 3.2, we have

$$u_{xxy} = u_{xyx} = 0, \quad u_{zzy} = u_{yzz} = -u_{yz} = u_y.$$

Putting the above results into (3.3) we obtain

$$\left(\frac{2}{m} + 2\right) u_y = 0,$$

which implies $u_y = 0$.

Similarly, using (1), (2), (5) and (6) of Lemma 3.2, we can also get $u_x = 0$. Moreover, (4) and (5) of Lemma 3.2 give

$$2u_z = e^{2z} u_{yy} - e^{-2z} u_{xx}.$$

It follows from $u_x = u_y = 0$ that we further get $u_z = 0$. Therefore, u is a constant and thus Sol_3 is Einstein.

This is a contradiction, by which the proof of Proposition 3.1 is completed. $\square$

3.2. Homogeneous 3-manifold with isometry group of dimension 4. In this subsection, we concentrate on the problem whether there exists a proper generalized m-quasi-Einstein structure on homogeneous 3-manifolds with isometry group of dimension 4. The main result is the following

PROPOSITION 3.2. Nil_3 *and* $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$ *do not carry any generalized m-quasi-Einstein structure.*

To begin with, we recall that a noncompact homogeneous 3-manifold with isometry group of dimension 4 can be viewed as $\mathbf{R}^3$ endowed with the metric

$$g_{k,\tau} = \begin{cases} dx^2 + dy^2 + [\tau(y\,dx - x\,dy) + dt]^2, & \text{if } k = 0, \\ \rho^2(dx^2 + dy^2) + [2k\tau\rho(x\,dy - y\,dx) + dt]^2, & \text{if } k \neq 0, \end{cases} \tag{3.4}$$

where $\rho = \dfrac{2}{1 + k(x^2 + y^2)}$. An orthonormal frame with respect to $g_{k,\tau}$ is given by

$$\begin{aligned} &\{E_1 = \partial_x - \tau y \partial_t,\ E_2 = \partial_y + \tau x \partial t,\ E_3 = \partial_t\}, && \text{if } k = 0, \\ &\left\{E_1 = \frac{1}{\rho}\partial_x + 2k\tau y \partial_t,\ E_2 = \frac{1}{\rho}\partial_y - 2k\tau x \partial_t,\ E_3 = \partial_t\right\}, && \text{if } k \neq 0. \end{aligned} \tag{3.5}$$

Next, we need the following lemma whose proof can be found in [3].

LEMMA 3.3 ([3]). *Let $E^3(k,\tau)$ be a noncompact homogeneous* 3*-manifold with isometry group of dimension* 4*, whose metric and its associated orthonormal frame are given by* (3.4) *and* (3.5)*, respectively. Then we have*

$$[E_1, E_2] = -kyE_1 + kxE_2 + 2\tau E_3, \quad [E_1, E_3] = [E_2, E_3] = 0.$$

With $\overline{\nabla}$ the Riemannian connection we have the following calculations:

$$\begin{aligned}
&\overline{\nabla}_{E_1}E_1 = kyE_2, &&\overline{\nabla}_{E_1}E_2 = -kyE_1 + \tau E_3, &&\overline{\nabla}_{E_1}E_3 = -\tau E_2,\\
&\overline{\nabla}_{E_2}E_1 = -kxE_2 - \tau E_3, &&\overline{\nabla}_{E_2}E_2 = kxE_1, &&\overline{\nabla}_{E_2}E_3 = \tau E_1,\\
&\overline{\nabla}_{E_3}E_1 = -\tau E_2, &&\overline{\nabla}_{E_3}E_2 = \tau E_1, &&\overline{\nabla}_{E_3}E_3 = 0.
\end{aligned}$$

Moreover, the Ricci tensor $\overline{\mathrm{Ric}}$ of $E^3(k,\tau)$ satisfies

$$\bar{R}_{11} = \bar{R}_{22} = k - 2\tau^2, \quad \bar{R}_{33} = 2\tau^2, \quad \bar{R}_{ij} = 0 \quad \textit{for } i \neq j,$$

where $\bar{R}_{ij} = \overline{\mathrm{Ric}}(E_i, E_j)$.

To prove Proposition 3.2, we also need the following lemma.

Lemma 3.4. *Let $E^3(k,\tau)$ be a noncompact homogeneous 3-manifold with isometry group of dimension 4. Suppose that $(E^3(k,\tau), g_{k,\tau}, u, \lambda)$ is a generalized m-quasi-Einstein structure on $E^3(k,\tau)$. Then, with respect to the orthonormal frame given by* (3.5), *the functions u and λ satisfy the following equations*:
(1) $E_1E_2(u) + kyE_1(u) - \tau E_3(u) = 0$,
(2) $E_1E_3(u) + \tau E_2(u) = 0$,
(3) $E_2E_3(u) - \tau E_1(u) = 0$,
(4) $E_1E_1(u) - kyE_2(u) = \dfrac{u}{m}(k - 2\tau^2 - \lambda)$,
(5) $E_3E_3(u) = \dfrac{u}{m}(2\tau^2 - \lambda)$.

Proof. Since $(E^3(k,\tau), g_{k,\tau}, u, \lambda)$ is a generalized m-quasi-Einstein structure, from (1.2) we obtain

$$\overline{\nabla}^2 u(E_1, E_2) = \frac{u}{m}\bar{R}_{12} = 0.$$

On the other hand, by definition and Lemma 3.3, we have

$$\begin{aligned}
\overline{\nabla}^2 u(E_1, E_2) &= g_{k,\tau}(\overline{\nabla}_{E_1}\overline{\nabla}u, E_2) = E_1(E_2(u)) - g_{k,\tau}(\overline{\nabla}u, \overline{\nabla}_{E_1}E_2)\\
&= E_1E_2(u) + kyE_1(u) - \tau E_3(u)
\end{aligned}$$

and the first assertion (1) follows.

Similarly, direct calculations of the following terms

$$\overline{\nabla}^2 u(E_1, E_3),\ \overline{\nabla}^2 u(E_2, E_3),\ \overline{\nabla}^2 u(E_1, E_1) \text{ and } \overline{\nabla}^2 u(E_3, E_3)$$

will verify the assertions (2), (3), (4) and (5), respectively. □

Proof of Proposition 3.2. If Nil_3 or $\widetilde{\mathrm{PSL}_2}(\mathbf{R})$ carries a generalized m-quasi-Einstein structure, then according to (4) and (5) of Lemma 3.4, the potential function u satisfies

$$E_3E_3(u) + \frac{u}{m}(k - 4\tau^2) = E_1E_1(u) - kyE_2(u). \tag{3.6}$$

Moreover, because of item (1) of Lemma 3.4 along with $E_3(y) = 0$, we have

$$E_3E_1E_2(u) + kyE_3E_1(u) - \tau E_3E_3(u) = 0,$$

and then it follows from Lemma 3.3 and items (1)–(3) of Lemma 3.4 that

$$\begin{aligned} \tau E_3E_3(u) &= E_3E_1E_2(u) + kyE_3E_1(u) \\ &= E_1E_3E_2(u) + kyE_1E_3(u) \\ &= E_1E_2E_3(u) + kyE_1E_3(u) \\ &= \tau E_1E_1(u) - \tau kyE_2(u). \end{aligned} \tag{3.7}$$

Comparing (3.6) with (3.7), noting that $u > 0$ and $\tau \neq 0$ for both Nil_3 and $\widetilde{\mathrm{PSL}_2}(\mathbf{R})$, we get $k = 4\tau^2$. This is a contradiction to the assumption that the non-compact homogeneous 3-manifold has an isometry group of dimension 4.

This completes the proof of Proposition 3.2. □

3.3. Proof of Theorem 1.1. Propositions 3.1, 3.2 have shown that Sol_3, Nil_3 and $\widetilde{\mathrm{PSL}_2}(\mathbf{R})$ do not carry any generalized m-quasi-Einstein structure. In addition, it has been proved in [1] that a compact generalized m-quasi-Einstein manifold with constant scalar curvature must be isometric to the standard sphere. Hence, based on the classification of homogeneous 3-manifolds, all remaining is to verify the following:

CLAIM. Both $\mathbf{S}^2(k) \times \mathbf{R}$ and $\mathbf{H}^2(k) \times \mathbf{R}$ do not carry any proper generalized m-quasi-Einstein structure.

Note that both $\mathbf{S}^2(k) \times \mathbf{R}$ and $\mathbf{H}^2(k) \times \mathbf{R}$ are of parallel Ricci tensor. On the other hand, generalized m-quasi-Einstein manifolds with parallel Ricci tensor have been classified in [8] and can be stated as follows:

THEOREM 3.1 ([8]). *Let (M^n, g, u, λ) be a complete n-dimensional $(n \geq 3)$ nontrivial generalized m-quasi-Einstein manifold which possesses parallel Ricci tensor. Then (M^n, g) is isometric to one of the following manifolds*:

(1) *a space form*,

(2) $\mathbf{D}_c^n$,

(3) $\mathbf{R} \times N^{n-1}(b)$,

(4) $\mathbf{H}^p(a) \times N^{n-p}(b)$, $b = \dfrac{m+p-1}{p-1}a$,

(5) $\mathbf{D}_c^p \times N^{n-p}(b)$, $b = (1-m-p)c^2$,

where a, b are negative constants, $N^k(b)$ denotes a k-dimensional Einstein manifold with scalar curvature kb, here we recall that traditionally one also calls b the

Einstein constant of $N^k(b)$; $\mathbf{H}^p(a)$ *denotes the* p*-dimensional hyperbolic space with Einstein constant* a; $\mathbf{D}_c^k$ *denotes a* k*-dimensional Einstein warped product* $\mathbf{R} \times_{c^{-1}e^{cr}} F^{k-1}$, *i.e.* $\mathbf{R} \times F^{k-1}$ *endowed with the metric* $dr^2 + (c^{-1}e^{cr})^2 g_F$, c *is a positive constant,* F^{k-1} *with metric* g_F *is a Ricci flat manifold.*

It follows from Theorem 3.1 that $\mathbf{S}^2(k) \times \mathbf{R}$ do not carry any nontrivial generalized m-quasi-Einstein structure. Furthermore, Remark 2.2 in [8] shows that, if $\mathbf{R} \times N^{n-1}(b)$ in Theorem 3.1 possesses a generalized m-quasi-Einstein structure $(\mathbf{R} \times N^{n-1}(b), g, u, \lambda)$, then λ must be a negative constant. That is to say, though $\mathbf{H}^2(k) \times \mathbf{R}$ possesses an m-quasi-Einstein structure, as generalized m-quasi-Einstein manifold, it must be not proper.

This verifies the claim. Accordingly, we complete the proof of Theorem 1.1. □

Actually, the above arguments also imply the following corollary, which extends Theorem 1.1 without the assumption of properness.

Corollary 3.1. *Let* (M^3, g) *be a* 3*-dimensional homogeneous manifold that can carry a generalized m-quasi-Einstein structure, then* (M^3, g) *is either a space form or* $\mathbf{H}^2(k) \times \mathbf{R}$.

Remark 3.1. Theorem 2 in [3] shows that, if a 3-dimensional homogeneous manifold carries a nontrivial m-quasi-Einstein structure, then it is either hyperbolic space or $\mathbf{H}^2(k) \times \mathbf{R}$. In this sense, Corollary 3.1 also generalizes Theorem 2 in [3] from m-quasi-Einstein to generalized m-quasi-Einstein structure.

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