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AN ADDITION TYPE FORMULA FOR THE DOUBLE COTANGENT FUNCTION

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Abstract

In this paper, we prove an addition type formula for the double cotangent function. Furthermore, we see that the addition theorem of the usual cotangent function, the reciprocity laws of (classical and higher) Dedekind sums, Lerch's functional equation and Ramanujan's formula can be deduced from it.

Contents

1. Introduction

An explicit construction of class fields over an algebraic number field is one of the attractive problems in number theory. The problem is, for a given algebraic number field K, to find a function F_K whose special values generate abelian extensions such that we can describe the reciprocity laws of them explicitly. By the Kronecker-Weber theorem, special values of the exponential function generate the maximal abelian extension of the rational number field. The theory of complex multiplication shows that, when K is an imaginary quadratic field, a special value of the j-function and special values of the elliptic function whose period lattice is the ring of integers of K generate the maximal abelian extension of K. It is also known that, when K is a CM-field, the complex multiplication of abelian varieties gives rise to certain family of abelian

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extensions over K. However, for other number fields, we don't know much about this problem.

The double sine function is a function considered to be a candidate for F_K when K is a real quadratic field. In fact, Shintani [14] conjectured that a product of division values of the double sine function is a unit in a certain abelian extension over K, and proved it in some special cases.

The double sine function is defined as follows. Let ω_1, ω_2 be two nonzero complex numbers. Assume that ω_1/ω_2 is not a negative real number. Put $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and define the double Hurwitz zeta function as

$$\zeta_2(s, x, \boldsymbol{\omega}) = \sum_{n_1, n_2 = 0}^{\infty} (x + n_1\omega_1 + n_2\omega_2)^{-s},$$

where $(x + n_1\omega_1 + n_2\omega_2)^{-s}$ means $\exp[-s \log(x + n_1\omega_1 + n_2\omega_2)]$ and the logarithm is taken as in [6, §16]. $\zeta_2(s, x, \boldsymbol{\omega})$ is absolutely convergent when $\mathrm{Re}(s) > 2$, meromorphically continued to the whole complex plane and holomorphic at $s = 0$. The double gamma and sine functions are defined respectively as

$$\Gamma_2(x, \boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_2(s, x, \boldsymbol{\omega})\bigg|_{s=0}\right),$$

$$\mathrm{Sin}_2(x, \boldsymbol{\omega}) = \Gamma_2(x, \boldsymbol{\omega})^{-1}\Gamma_2(\omega_1 + \omega_2 - x, \boldsymbol{\omega}).$$

It is known that the double gamma and sine functions have similar properties to that of the usual gamma and sine functions (for example, see [4], [5], [6] and [11]). Koyama and Kurokawa [10] investigated the addition formula for the double sine function from the view point of formal group laws, but an analogue of the addition theorem for the usual sine function is not known. Since a suitable addition theorem for the double sine function would imply the algebraicity of the division value of the double sine function, finding it is regarded as quite important.

The double cotangent function is defined by

$$\mathrm{Cot}_2(x, \boldsymbol{\omega}) = \frac{d}{dx} \log \mathrm{Sin}_2(x, \boldsymbol{\omega}).$$

Since the addition theorem of the usual sine function can be written as that of the cotangent function, we expect that there is a close relation between the addition formulas for the double sine and cotangent functions. In this paper, we study the addition formula for the double cotangent function.

Let's put $c_1(x) = \pi \cot(\pi x)$. Then the addition theorem for the usual cotangent function may be represented as follows:

$$c_1'(x)c_1(y) - c_1(y)c_1'(x+y) - c_1(x)c_1'(x+y) - c_1'(x)c_1(x+y) = 0. \tag{1.1}$$

The purpose of this paper is to find a formula similar to this, which $\mathrm{Cot}_2(x, \boldsymbol{\omega})$ satisfies.

The main theorem of this paper is as follows. We say that a real number α is *generic* if and only if

$$\lim_{m\to\infty} \|m\alpha\|^{1/m} = 1,$$

where we put $\|x\| := \min\{|x-n|; n \in \mathbf{Z}\}$ for $x \in \mathbf{R}$. Furthermore, we set

$$\begin{aligned} R(x_1, x_2, \boldsymbol{\omega}) = &-\frac{\pi}{\omega_1}\left(\sum_{k=0}^{\infty} \cot\frac{\pi}{\omega_1}(x_1 + k\omega_2)\zeta_2(4, x_2 + k\omega_2, \boldsymbol{\omega})\right. \\ &\qquad \left. + \sum_{k=1}^{\infty} \cot\frac{\pi}{\omega_1}(x_1 - k\omega_2)\zeta_2(4, x_2 - \omega_1 - \omega_2 - k\omega_2, -\boldsymbol{\omega})\right) \\ &-\frac{\pi}{\omega_2}\left(\sum_{k=0}^{\infty} \cot\frac{\pi}{\omega_2}(x_1 + k\omega_1)\zeta_2(4, x_2 + k\omega_1, \boldsymbol{\omega})\right. \\ &\qquad \left. + \sum_{k=1}^{\infty} \cot\frac{\pi}{\omega_2}(x_1 - k\omega_1)\zeta_2(4, x_2 - \omega_1 - \omega_2 - k\omega_1, -\boldsymbol{\omega})\right). \end{aligned}$$

THEOREM 1.1 (Main theorem). *Assume that one of the following conditions holds*:

(i) $\omega_2/\omega_1 \notin \mathbf{R}$.

(ii) $\omega_2/\omega_1 \in \mathbf{Q}_{>0}$.

(iii) ω_2/ω_1 *and* ω_1/ω_2 *are both generic and* $y \notin \mathbf{R}$.

Then we have

$$\begin{aligned} &\mathrm{Cot}_2^{(3)}(x, \boldsymbol{\omega})\,\mathrm{Cot}_2(y, \boldsymbol{\omega}) + \mathrm{Cot}_2^{(3)}(x+y, \boldsymbol{\omega})\,\mathrm{Cot}_2(y, \boldsymbol{\omega}) \\ &\quad - \sum_{k=0}^{3}\binom{3}{k}\mathrm{Cot}_2^{(k)}(x+y, \boldsymbol{\omega})\,\mathrm{Cot}_2^{(3-k)}(x, \boldsymbol{\omega}) \\ &= -6R(y, \omega_1 + \omega_2 - x, \boldsymbol{\omega}) + 6R(y, x+y, \boldsymbol{\omega}). \end{aligned} \tag{1.2}$$

Remark 1.2. Under any one of the three conditions of Theorem 1.1, the absolute values of $\cot\dfrac{\pi}{\omega_i}(y \pm k\omega_j)$ are bounded as $k \to \infty$. Thus the infinite series appearing in the definition of $R(y, \omega_1 + \omega_2 - x, \boldsymbol{\omega})$ and $R(y, x+y, \boldsymbol{\omega})$ are all absolutely convergent. However, when y is a real number and ω_2/ω_1, ω_1/ω_2 are both *generic*, convergence of $R(y, \omega_1 + \omega_2 - x, \boldsymbol{\omega})$ or $R(y, x+y, \boldsymbol{\omega})$ is ambiguous.

We show that the main theorem in fact implies (1.1). Moreover, we prove that the reciprocity laws of (classical and higher) Dedekind sums, Lerch's functional equation and Ramanujan's formula are obtained from the main theorem. In this sense, Theorem 1.1 includes all these formulas.

Although it is natural to ask whether $R(x_1, x_2, \boldsymbol{\omega})$ can be represented in a simpler way, we are currently unable to find such a representation. However, if it exists, we may gain more formulas of the double cotangent and sine functions. Meanwhile, by letting $\tau = \omega_2/\omega_1$ tend to $i\infty$ in (1.2), certain identities for Euler's double zeta values are obtained, where $R(x_1, x_2, \boldsymbol{\omega})$ plays important role in the argument. The detail of this result will be given in a forthcoming paper.

This paper is organized as follows. In section 2, we review the proof of the addition theorem for the usual cotangent function based only on the partial fractional decomposition. In section 3, we summarize the basic properties of the double cotangent function. In section 4, we give the proof of the main theorem. In section 5, we see that the addition theorem of the usual cotangent function, the reciprocity laws of (classical and higher) Dedekind sums, Lerch's functional equation and Ramanujan's formula can be deduced from the main theorem.

2. Addition theorems for the cotangent function

In this section, we consider the usual cotangent function. For simplicity, we put

$$c_1(x) = \pi \cot(\pi x).$$

Then the function c_1 satisfies a following theorem:

THEOREM 2.1.

$$c_1'(x)c_1(y) - c_1(y)c_1'(x+y) - c_1(x)c_1'(x+y) - c_1'(x)c_1(x+y) = 0.$$

Theorem 2.1 is easily obtained by clearing the fraction in the addition theorem

$$c_1(x+y) = \frac{c_1(x)c_1(y) - \pi^2}{c_1(x) + c_1(y)}. \tag{2.1}$$

and differentiating the both sides with respect to x, but Eisenstein showed this based only on the partial fractional decomposition of c_1. Since the idea of Eisenstein will also be used in the proof of the main theorem, we review it.

Eisenstein's proof is as follows. Recall that $c_1(x)$ has the following partial fractional decomposition:

$$c_1(x) = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n} + \frac{1}{x-n}\right). \tag{2.2}$$

By differentiating this repeatedly, for $k \geq 2$, we have

$$\frac{(-1)^{k-1}}{(k-1)!} c_1^{(k-1)}(x) = \sum_{n \in \mathbf{Z}} \frac{1}{(x+n)^k}. \tag{2.3}$$

The key to the proof is the following lemma:

Lemma 2.2. *Put $r = p + q$. Then we have, for two positive integers l_1, l_2,*

$$\frac{1}{p^{l_1} q^{l_2}} = \sum_{k=0}^{l_1-1} \binom{l_2+k-1}{k} \frac{1}{p^{l_1-k} r^{l_2+k}} + \sum_{l=0}^{l_2-1} \binom{l_1+l-1}{l} \frac{1}{q^{l_2-l} r^{l_1+l}}.$$

Lemma 2.2 is obtained by dividing the both sides of $r = p + q$ by pqr, differentiating $l_1 - 1$ times with respect to p and $l_2 - 1$ times with respect to q.

Especially when $l_1 = l_2 = 2$, Lemma 2.2 becomes

$$\frac{1}{p^2 q^2} = \frac{1}{p^2 r^2} + \frac{2}{pr^3} + \frac{1}{q^2 r^2} + \frac{2}{qr^3}. \tag{2.4}$$

By adding together the two identities obtained by putting

$$(p, q) = (x + m, y + n - m), (x - m - 1, y + n + m + 1)$$

in (2.4) and applying $\sum_{m=0}^{\infty}$ to the both sides, we have

$$\sum_{m \in \mathbf{Z}} \left(\frac{1}{(x+m)^2 (y+n-m)^{-2}} \right) + \frac{c_1^{(1)}(x)}{(x+y+n)^2} + \frac{c_1^{(1)}(y+n)}{(x+y+n)^2} = 2\frac{c_1(x) + c_1(y+n)}{(x+y+n)^3}.$$

Because of the periodicity, we may replace $c_1(y+n)$ and $c_1^{(1)}(y+n)$ by $c_1(y)$ and $c_1^{(1)}(y)$, respectively. Applying $\sum_{n \in \mathbf{Z}}$ to the both sides yields

$$c_1^{(1)}(x) c_1^{(1)}(y) - c_1^{(1)}(x) c_1^{(1)}(x+y) - c_1^{(1)}(y) c_1^{(1)}(x+y) = c_1^{(2)}(x+y)(c_1(x) + c_1(y))$$

because of the absolute convergence of the series in the left hand side. By integrating the both sides of this with respect y, we obtain the theorem (for the proof of the constant of integration being 0, see [15, Chapter I§4]).

Remark 2.3. Conversely it is possible to deduce (2.1) from Theorem 2.1 by integrating with respect to x. In fact the constant of integration is determined by using the identity $c_1'(x) + c_1(x)^2 + \pi^2 = 0$, which can be obtained from Theorem 2.1. Thus Theorem 2.1 and (2.1) are equivalent and Theorem 2.1 is regarded as the addition theorem for the cotangent function.

3. Properties of the double cotangent function

In this section, we summarize the basic properties of the double cotangent function. These will be used in Section 4 and Section 5.

Proposition 3.1.

$$\mathrm{Cot}_2(x + \omega_1, \boldsymbol{\omega}) = \mathrm{Cot}_2(x, \boldsymbol{\omega}) - \frac{1}{\omega_2} c_1\left(\frac{x}{\omega_2}\right).$$

$$\mathrm{Cot}_2(x + \omega_2, \boldsymbol{\omega}) = \mathrm{Cot}_2(x, \boldsymbol{\omega}) - \frac{1}{\omega_1} c_1\left(\frac{x}{\omega_1}\right).$$

Proposition 3.1 follows immediately from the quasiperiodicity of the double sine function ([11, Theorem 2.1 (a)]). By using Proposition 3.1 repeatedly, for non-negative integers n_1 and n_2, we have the following:

$$\begin{aligned}
&\mathrm{Cot}_2(x+n_1\omega_1+n_2\omega_2,\boldsymbol{\omega})\\
&\qquad=\mathrm{Cot}_2(x,\boldsymbol{\omega})-\frac{1}{\omega_1}\sum_{r=0}^{n_2-1}c_1\left(\frac{x+r\omega_2}{\omega_1}\right)-\frac{1}{\omega_2}\sum_{r=0}^{n_1-1}c_1\left(\frac{x+r\omega_1}{\omega_2}\right).\\
&\mathrm{Cot}_2(x-n_1\omega_1-n_2\omega_2,\boldsymbol{\omega})\\
&\qquad=\mathrm{Cot}_2(x,\boldsymbol{\omega})+\frac{1}{\omega_1}\sum_{r=1}^{n_2}c_1\left(\frac{x-r\omega_2}{\omega_1}\right)+\frac{1}{\omega_2}\sum_{r=1}^{n_1}c_1\left(\frac{x-r\omega_1}{\omega_2}\right).
\end{aligned}\tag{3.1}$$

Proposition 3.2. *For $c\neq 0$, we have*

$$\mathrm{Cot}_2(cx,c\boldsymbol{\omega})=\frac{1}{c}\,\mathrm{Cot}_2(x,\boldsymbol{\omega}).$$

Proposition 3.2 is deduced from the homogeneity of the double sine function ([11, Theorem 2.1 (e)]). In [11], this formula is proved for $c>0$, but the same proof works for $c\in\mathbf{C}^{\times}$.

Proposition 3.3. *$\mathrm{Cot}_2(x,\boldsymbol{\omega})$ has the following partial fractional decomposition*:

$$\begin{aligned}
\mathrm{Cot}_2(x,\boldsymbol{\omega})&=\gamma(\boldsymbol{\omega})+\frac{1}{x}-\frac{1}{x-\omega_1-\omega_2}\\
&\quad+\sum_{\substack{n_1,n_2\geq 0\\(n_1,n_2)\neq(0,0)}}\left(\frac{1}{x+n_1\omega_1+n_2\omega_2}-\frac{1}{x-(n_1+1)\omega_1-(n_2+1)\omega_2}\right.\\
&\qquad\left.-\frac{2}{n_1\omega_1+n_2\omega_2}+\frac{\omega_1+\omega_2}{(n_1\omega_1+n_2\omega_2)^2}\right),
\end{aligned}\tag{3.2}$$

where $\gamma(\boldsymbol{\omega})$ is a constant depends only on $\boldsymbol{\omega}$, and the infinite series on the right hand side are absolutely convergent. Therefore, when $k\geq 1$, we have

$$\begin{aligned}
&\frac{(-1)^k}{k!}\,\mathrm{Cot}_2^{(k)}(x,\boldsymbol{\omega})\\
&\quad=\sum_{n_1,n_2\geq 0}\left(\frac{1}{(x+n_1\omega_1+n_2\omega_2)^{k+1}}-\frac{1}{(x-(n_1+1)\omega_1-(n_2+1)\omega_2)^{k+1}}\right).
\end{aligned}$$

Proof. From the definition of the double sine function, it follows that

$$\mathrm{Cot}_2(x,\boldsymbol{\omega}) = -\frac{\Gamma_2'(x,\boldsymbol{\omega})}{\Gamma_2(x,\boldsymbol{\omega})} - \frac{\Gamma_2'(\omega_1+\omega_2-x,\boldsymbol{\omega})}{\Gamma_2(\omega_1+\omega_2-x,\boldsymbol{\omega})}. \tag{3.3}$$

By the infinite product representation of the double gamma function

$$\Gamma_2^{-1}(x,\boldsymbol{\omega}) = \rho_2(\boldsymbol{\omega}) \exp\left(\frac{x^2}{2}\gamma_{22} + x\gamma_{21}\right) \cdot x \cdot \prod_{\substack{n_1,n_2\geq 0\\(n_1,n_2)\neq(0,0)}} \left(1+\frac{x}{n_1\omega_1+n_2\omega_2}\right)$$
$$\cdot \exp\left(-\frac{x}{n_1\omega_1+n_2\omega_2} + \frac{x^2}{2(n_1\omega_1+n_2\omega_2)^2}\right),$$

which was obtained by Barnes [6, §25] (γ_{21}, γ_{22}, and $\rho_2(\boldsymbol{\omega})$ are constants), we have

$$-\frac{\Gamma_2'(x,\boldsymbol{\omega})}{\Gamma_2(x,\boldsymbol{\omega})} = \gamma_{22}x + \gamma_{21} + \frac{1}{x} \tag{3.4}$$
$$+ \sum_{n_1,n_2}\left(\frac{1}{x+n_1\omega_1+n_2\omega_2} - \frac{1}{n_1\omega_1+n_2\omega_2} + \frac{x}{(n_1\omega_1+n_2\omega_2)^2}\right).$$

Since

$$\frac{1}{x+n_1\omega_1+n_2\omega_2} - \frac{1}{n_1\omega_1+n_2\omega_2} + \frac{x}{(n_1\omega_1+n_2\omega_2)^2}$$
$$= \frac{x^2}{(x+n_1\omega_1+n_2\omega_2)(n_1\omega_1+n_2\omega_2)^2},$$

the infinite series on the right hand side of (3.4) is absolutely convergent. By (3.4) and (3.3), we have (3.2). Absolute convergence of the right hand side of (3.2) follows that of (3.4). □

Remark 3.4. Proposition 3.1 can be also deduced from Proposition 3.3, as the periodicity of $c_1(x)$ can be deduced from the partial fractional decomposition (2.2).

PROPOSITION 3.5. *Assume that one of the following conditions holds*:
(i) $\omega_1, \mathrm{Im}(\omega_2), \mathrm{Im}(x), \mathrm{Im}(x/\omega_2) > 0$.
(ii) *Both* ω_1/ω_2 *and* ω_2/ω_1 *are generic and* $\mathrm{Im}(x) > 0$.
Then we have

$$\mathrm{Cot}_2(x,\boldsymbol{\omega}) = \frac{1}{\omega_1}\sum_{k=1}^{\infty}\left(c_1\left(\frac{k\omega_2}{\omega_1}\right) - \pi i\right)e^{2\pi i k(x/\omega_1)} + \frac{1}{\omega_2}\sum_{n=1}^{\infty}\left(c_1\left(\frac{n\omega_1}{\omega_2}\right) - \pi i\right)e^{2\pi i n(x/\omega_2)}$$
$$+ \frac{\pi i x}{\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right).$$

Proposition 3.5 is proved by taking logarithmic derivative of the expressions in [9, Theorem 2] and [13, Proposition 5].

Corollary 3.6. *Assume one of the conditions in Proposition* 3.5 *holds. As* $|x| \to \infty$, *we have the following*:

$$\begin{aligned}
\mathrm{Cot}_2(x, \boldsymbol{\omega}) &= \frac{\pi i x}{\omega_1 \omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + o(|x|^{-1}), \\
\mathrm{Cot}_2^{(1)}(x, \boldsymbol{\omega}) &= \frac{\pi i}{\omega_1 \omega_2} + o(|x|^{-1}), \\
\mathrm{Cot}_2^{(k)}(x, \boldsymbol{\omega}) &= o(|x|^{-1}) \quad (k \geq 2).
\end{aligned}$$

4. Proof of the main theorem

We are now in a position to prove Theorem 1.1. Throughout this section,

$$\begin{aligned}
z &= x + y, \\
\boldsymbol{m} &= (m_1, m_2), \\
\boldsymbol{n} &= (n_1, n_2), \\
\mathbf{1} &= (1, 1).
\end{aligned}$$

In this section, we use Proposition 3.3 without previous notice.

First, we prove the theorem when the condition (i) holds. By Proposition 3.2 and the homogeneity of the double Hurwitz zeta function

$$\zeta_2(4, cx, c\boldsymbol{\omega}) = \frac{1}{c^4}\zeta_2(4, x, \boldsymbol{\omega}) \quad (c \neq 0), \tag{4.1}$$

we may assume $\omega_1 = 1$, $\omega_2 = \tau$ with $\mathrm{Im}(\tau) > 0$. We put $\boldsymbol{\tau} = (1, \tau)$. Setting $l_1 = 4$, $l_2 = 3$ in Lemma 2.2 yields

$$\frac{1}{p^4 q^3} = \sum_{k=0}^{3}\binom{k+2}{k}\frac{1}{p^{4-k} r^{3+k}} + \sum_{l=0}^{2}\binom{l+3}{l}\frac{1}{q^{3-l} r^{4+l}}. \tag{4.2}$$

We add together the two identities obtained by putting

$$(p, q) = (x + \boldsymbol{m} \cdot \boldsymbol{\tau}, q = y + (\boldsymbol{n} - \boldsymbol{m}) \cdot \tau), (x - \boldsymbol{m} \cdot \boldsymbol{\tau}, y + (\boldsymbol{n} + \boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau})$$

in (4.2) and apply $\sum_{m_1, m_2 \geq 0}$ to this. By observing that

$$\begin{aligned}
&\mathrm{Cot}_2(x, \boldsymbol{\tau}) - \mathrm{Cot}_2(y + (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau}, \boldsymbol{\tau}) \\
&\quad = \sum_{m_1, m_2 \geq 0}\left(\frac{1}{x + \boldsymbol{m} \cdot \boldsymbol{\tau}} - \frac{1}{x - (\boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau}} - \frac{1}{y + (\boldsymbol{n} + \boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau}} + \frac{1}{y + (\boldsymbol{n} - \boldsymbol{m}) \cdot \boldsymbol{\tau}}\right),
\end{aligned}$$

we have

$$\sum_{m_1, m_2 \geq 0} \left(\frac{1}{(x + \boldsymbol{m} \cdot \boldsymbol{\tau})^4 (y + (\boldsymbol{n} - \boldsymbol{m}) \cdot \boldsymbol{\tau})^3} - \frac{1}{(x - (\boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau}))^4 (y + (\boldsymbol{n} + \boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau})^3} \right)$$
$$= \sum_{k=0}^{3} \binom{k+2}{k} \frac{(-1)^{3-k} \operatorname{Cot}_2^{(3-k)}(x, \boldsymbol{\tau})}{(3-k)!(z + \boldsymbol{n} \cdot \boldsymbol{\tau})^{3+k}} - \sum_{l=0}^{2} \binom{l+3}{l} \frac{(-1)^{2-l} \operatorname{Cot}_2^{(2-l)}(y + (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(2-l)!(z + \boldsymbol{n} \cdot \boldsymbol{\tau})^{4+l}}. \tag{4.3}$$

Then we subtract the identity which is obtained by replacing $\boldsymbol{n} \cdot \boldsymbol{\tau}$ with $-(\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau}$ in (4.3) from (4.3), and apply $\sum_{n_1, n_2 \geq 0}$ to this. Since $\operatorname{Im}(\tau) > 0$,

$$\sum_{n_1, n_2 \geq 0} \sum_{m_1, m_2 \geq 0} \frac{1}{(x + \boldsymbol{m} \cdot \boldsymbol{\tau})^4 (y + (\boldsymbol{n} - \boldsymbol{m}) \cdot \boldsymbol{\tau})^3},$$
$$\sum_{n_1, n_2 \geq 0} \sum_{m_1, m_2 \geq 0} \frac{1}{(x - (\boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau})^4 (y + (\boldsymbol{n} + \boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau})^3}$$

are both absolutely convergent. Thus, we have

$$\frac{1}{2} \sum_{m_1, m_2 \geq 0} \left(\frac{\operatorname{Cot}_2^{(2)}(y - \boldsymbol{m} \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(x + \boldsymbol{m} \cdot \boldsymbol{\tau})^4} - \frac{\operatorname{Cot}_2^{(2)}(y + (\boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(x - (\boldsymbol{m} + \mathbf{1}) \cdot \boldsymbol{\tau})^4} \right)$$
$$= \sum_{k=0}^{3} \binom{k+2}{k} \frac{(-1)^{3-k}}{(3-k)!} \frac{(-1)^{2+k}}{(2+k)!} \operatorname{Cot}_2^{(3-k)}(x, \boldsymbol{\tau}) \operatorname{Cot}_2^{(2+k)}(z, \boldsymbol{\tau})$$
$$- \sum_{l=0}^{2} \binom{l+3}{l} \frac{(-1)^{2-l}}{(2-l)!}$$
$$\times \sum_{n_1, n_2 \geq 0} \left(\frac{\operatorname{Cot}_2^{(2-l)}(y + (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(z + \boldsymbol{n} \cdot \boldsymbol{\tau})^{4+l}} - \frac{\operatorname{Cot}_2^{(2-l)}(y - \boldsymbol{n} \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(z - (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau})^{4+l}} \right) \tag{4.4}$$

if the series

$$\sum_{n_1, n_2 \geq 0} \left(\frac{\operatorname{Cot}_2^{(2-l)}(y + (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(z + \boldsymbol{n} \cdot \boldsymbol{\tau})^{4+l}} - \frac{\operatorname{Cot}_2^{(2-l)}(y - \boldsymbol{n} \cdot \boldsymbol{\tau}, \boldsymbol{\tau})}{(z - (\boldsymbol{n} + \mathbf{1}) \cdot \boldsymbol{\tau})^{4+l}} \right) \tag{4.5}$$

is convergent.

We prove the convergence of (4.5). By (3.1), we have

$$\sum_{n_1,n_2\geq 0}\left(\frac{\operatorname{Cot}_2^{(2-l)}(y+(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z+\boldsymbol{n}\cdot\boldsymbol{\tau})^{4+l}}-\frac{\operatorname{Cot}_2^{(2-l)}(y-\boldsymbol{n}\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau})^{4+l}}\right)$$

$$=\sum_{n_1,n_2\geq 0}\left(\frac{\operatorname{Cot}_2^{(2-l)}(y,\boldsymbol{\tau})-\sum_{k=0}^{n_2}c_1^{(2-l)}(y+k\tau)-\frac{1}{\tau^{3-l}}\sum_{k=0}^{n_1}c_1^{(2-l)}\left(\frac{y+k}{\tau}\right)}{(z+\boldsymbol{n}\cdot\boldsymbol{\tau})^{4+l}}\right.$$
$$\left.-\frac{\operatorname{Cot}_2^{(2-l)}(y,\boldsymbol{\tau})+\sum_{k=1}^{n_2}c_1^{(2-l)}(y-k\tau)+\frac{1}{\tau^{3-l}}\sum_{k=1}^{n_1}c_1^{(2-l)}\left(\frac{y-k}{\tau}\right)}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau})^{4+l}}\right).$$

By interchanging the order of summation, we obtain

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\sum_{k=0}^{n_2}\frac{c_1^{(2-l)}(y+k\tau)}{(z+\boldsymbol{n}\cdot\boldsymbol{\tau})^{4+l}}=\sum_{n_1\geq 0}\sum_{k\geq 0}\sum_{n_2\geq k}\frac{c_1^{(2-l)}(y+k\tau)}{(z+n_1+n_2\tau)^{4+l}}$$
$$=\sum_{n_1\geq 0}\sum_{k\geq 0}\sum_{n_2\geq 0}\frac{c_1^{(2-l)}(y+k\tau)}{(z+n_1+n_2\tau+k\tau)^{4+l}},$$

but since $\operatorname{Im}(\tau)>0$ and thereby $|c_1^{(2-l)}(y+k\tau)|$ is bounded, the last series is absolutely convergent. Since the other terms can be calculated similarly, we see that (4.5) is convergent.

By integrating the both sides of (4.4) with respect to y twice, we have

$$\sum_{k=0}^{3}\binom{3}{k}\operatorname{Cot}_2^{(3-k)}(x,\boldsymbol{\tau})\operatorname{Cot}_2^{(k)}(z,\boldsymbol{\tau})$$
$$=-6\sum_{m_1,m_2\geq 0}\left(\frac{\operatorname{Cot}_2(y-\boldsymbol{m}\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(x+\boldsymbol{m}\cdot\boldsymbol{\tau})^4}-\frac{\operatorname{Cot}_2(y+(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(x-(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\tau})^4}\right)$$
$$-6\sum_{n_1,n_2\geq 0}\left(\frac{\operatorname{Cot}_2(y+(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z+\boldsymbol{n}\cdot\boldsymbol{\tau})^4}-\frac{\operatorname{Cot}_2(y-\boldsymbol{n}\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau})^4}\right)+f(x)y+g(x),$$

where $f(x)$ and $g(x)$ are functions depending only on x. Similar calculations to that in the proof of the convergence of (4.5) show that

$$-6\sum_{m_1,m_2\geq 0}\left(\frac{\operatorname{Cot}_2(y-\boldsymbol{m}\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(x+\boldsymbol{m}\cdot\boldsymbol{\tau})^4}-\frac{\operatorname{Cot}_2(y+(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(x-(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\tau})^4}\right)$$
$$=\operatorname{Cot}_2^{(3)}(x,\boldsymbol{\tau})\operatorname{Cot}_2(y,\boldsymbol{\tau})+6R(y,1+\tau-x,\boldsymbol{\tau})$$

and

$$-6\sum_{n_1,n_2\geq 0}\left(\frac{\mathrm{Cot}_2(y+(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z+\boldsymbol{n}\cdot\boldsymbol{\tau})^4}-\frac{\mathrm{Cot}_2(y-\boldsymbol{n}\cdot\boldsymbol{\tau},\boldsymbol{\tau})}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\tau})^4}\right)$$
$$=\mathrm{Cot}_2(y,\boldsymbol{\tau})\,\mathrm{Cot}_2^{(3)}(z,\boldsymbol{\tau})-6R(y,z,\boldsymbol{\tau}).$$

We substitute $y_n=(n+1/2)(\tau-1)$ into y and take the limit as $n\to\infty$. Then, by Corollary 3.6 we have

$$\lim_{n\to\infty} f(x)y_n+g(x)=\frac{\pi i}{\tau}x\,\mathrm{Cot}_2^{(3)}(x,\boldsymbol{\tau})+\frac{\pi i}{\tau}\,\mathrm{Cot}_2^{(2)}(x,\boldsymbol{\tau})$$
$$-6\pi i\left(\sum_{k=0}^{\infty}\zeta_2(4,1+\tau-x+k\tau,\boldsymbol{\tau})+\sum_{k=1}^{\infty}\zeta_2(4,x+k\tau,\boldsymbol{\tau})\right)$$
$$-6\frac{\pi i}{\tau}\left(\sum_{k=0}^{\infty}\zeta_2(4,1+\tau-x+k,\boldsymbol{\tau})+\sum_{k=1}^{\infty}\zeta_2(4,x+k,\boldsymbol{\tau})\right).$$

We consider the quantity in the right hand side. We see that

$$\sum_{k=1}^{\infty}\zeta_2(4,x+k\tau,\boldsymbol{\tau})+\frac{1}{\tau}\sum_{k=1}^{\infty}\zeta_2(4,x+k,\boldsymbol{\tau})$$
$$=\sum_{n_1\geq 0,n_2\geq 1}\frac{n_2}{(x+n_1+n_2\tau)^4}+\frac{1}{\tau}\sum_{n_1\geq 1,n_2\geq 0}\frac{n_1}{(x+n_1+n_2\tau)^4}$$
$$=\frac{1}{\tau}\sum_{n_1,n_2\geq 0}\frac{n_1+n_2\tau}{(x+n_1+n_2\tau)^4}$$
$$=-\frac{x}{\tau}\zeta_2(4,x,\boldsymbol{\tau})+\frac{1}{\tau}\zeta_2(3,x,\boldsymbol{\tau})$$

and by the similar calculation

$$\sum_{k=0}^{\infty}\zeta_2(4,1+\tau-x+k\tau,\boldsymbol{\tau})+\frac{1}{\tau}\sum_{k=0}^{\infty}\zeta_2(4,1+\tau-x+k,\boldsymbol{\tau})$$
$$=\frac{x}{\tau}\zeta_2(4,1+\tau-x,\boldsymbol{\tau})+\frac{1}{\tau}\zeta_2(3,1+\tau-x,\boldsymbol{\tau}).$$

Proposition 3.3 shows that

$$\zeta_2(3,x,\boldsymbol{\tau})+\zeta_2(3,1+\tau-x,\boldsymbol{\tau})=\frac{1}{2}\,\mathrm{Cot}_2^{(2)}(x,\boldsymbol{\tau}),$$
$$\zeta_2(4,x,\boldsymbol{\tau})-\zeta_2(4,1+\tau-x,\boldsymbol{\tau})=-\frac{1}{6}\,\mathrm{Cot}_2^{(3)}(x,\boldsymbol{\tau}).$$

Thus we have

$$\lim_{n\to\infty} f(x)y_n+g(x)=0$$

and thereby $f(x) = g(x) = 0$, which completes the proof when the condition (i) holds.

Next we assume that the condition (iii) holds. To prove the theorem, we use the signed double Poisson summation formula, which was established in [9]:

Proposition 4.1 (Signed double Poisson summation formula [9]). *Let $H(t)$ be an odd function in $L^1(\mathbf{R})$ with $H(t) = O(t^{-2})$ as $|t| \to \infty$. We put*

$$\tilde{H}(u) = \int_{-\infty}^{\infty} H(t)e^{itu}\,dt.$$

Assume that a/b and b/a are both generic and that the test function $H(t)$ satisfies

$$\tilde{H}(x) = O(\mu^x)$$

as $x \to \infty$ for some $0 < \mu < 1$. Then we have

$$\sum_{k,n>0} H\left(2\pi\left(\frac{k}{a}+\frac{n}{b}\right)\right) + \frac{1}{2}\left(\sum_{k>0} H\left(2\pi\frac{k}{a}\right) + \sum_{n>0} H\left(2\pi\frac{n}{b}\right)\right)$$
$$= -\frac{ia}{4\pi}\sum_{k>0} \cot\left(\pi\frac{ka}{b}\right)\tilde{H}(ka) - \frac{ib}{4\pi}\sum_{n>0} \cot\left(\pi\frac{nb}{a}\right)\tilde{H}(nb) - \frac{iab}{8\pi^2}\tilde{H}'(0).$$

In the proof, by Proposition 3.2 and (4.1), we may assume that $\omega_1, \omega_2 > 0$. Since both sides of (1.2) are meromorphic functions of x and $\mathrm{Cot}_2(x, \boldsymbol{\omega})$ satisfies the reflection formula

$$\mathrm{Cot}_2(\omega_1 + \omega_2 - x, \boldsymbol{\omega}) = \mathrm{Cot}_2(x, \boldsymbol{\omega}),$$

it is enough to prove the theorem when $\mathrm{Im}(x) > 0$ and $\mathrm{Im}(y) > 0$.

Now put

$$H(u) = \frac{\mathrm{Cot}_2(y-u, \boldsymbol{\omega})}{(x+u)^4} - \frac{\mathrm{Cot}_2(y+u, \boldsymbol{\omega})}{(x-u)^4} \quad (u > 0),$$
$$I_r(u) = \frac{\mathrm{Cot}_2(y-u\omega_r, \boldsymbol{\omega})}{(x+u\omega_r)^4} \quad (r = 1, 2).$$

Then we have

$$\sum_{m_1,m_2\geq 0}\left(\frac{\mathrm{Cot}_2(y - \boldsymbol{m}\cdot\boldsymbol{\omega}, \boldsymbol{\omega})}{(x+\boldsymbol{m}\cdot\boldsymbol{\omega})^4} - \frac{\mathrm{Cot}_2(y+(\boldsymbol{m}+\mathbf{1})\cdot\boldsymbol{\omega}, \boldsymbol{\omega})}{(x-(\boldsymbol{m}+\mathbf{1})\cdot\boldsymbol{\omega})^4}\right)$$
$$= \sum_{m_1,m_2\geq 1} H(\boldsymbol{m}\cdot\boldsymbol{\omega}) + \frac{1}{2}\sum_{m_1\geq 1} H(m_1\omega_1) + \frac{1}{2}\sum_{m_2\geq 1} H(m_2\omega_2)$$
$$+ \frac{1}{2}\sum_{m_1\in\mathbf{Z}} I_1(m_1) + \frac{1}{2}\sum_{m_2\in\mathbf{Z}} I_2(m_2).$$

Cauchy's integral theorem gives

$$\tilde{H}(u) = 2\pi i\left\{\operatorname*{Res}_{t=x} H(t)e^{iut} + \sum_{n_1,n_2\geq 0}\left(\operatorname*{Res}_{t=y+\boldsymbol{n}\cdot\boldsymbol{\omega}} H(t)e^{iut} + \operatorname*{Res}_{t=y-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega}} H(t)e^{iut}\right)\right\}$$

and

$$\widehat{I_r}(u) = \begin{cases} -2\pi i \operatorname*{Res}_{t=-x/\omega_r} I(t)e^{-2\pi itu} & (u\geq 0)\\ 2\pi i\displaystyle\sum_{n_1,n_2\geq 0}\left(\operatorname*{Res}_{t=(y+\boldsymbol{n}\cdot\boldsymbol{\omega})/\omega_r} I(t)e^{-2\pi itu} + \operatorname*{Res}_{t=(y-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega})/\omega_r} I(t)e^{-2\pi itu}\right) & (u<0). \end{cases}$$

Recall that we put $z = x + y$. Simple residue calculations show that

$$\begin{aligned}\tilde{H}(u) = -2\pi i\Bigg(& e^{iux}\sum_{k=0}^{3}\frac{(iu)^{3-k}}{(3-k)!k!}(-1)^{k+1}\operatorname{Cot}_2^{(k)}(z,\boldsymbol{\omega})\\ &+\sum_{n_1,n_2\geq 0}\left(\frac{e^{iu(y+\boldsymbol{n}\cdot\boldsymbol{\omega})}}{(z+\boldsymbol{n}\cdot\boldsymbol{\omega})^4}-\frac{e^{iu(y-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega})}}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega})^4}\right)\Bigg)\end{aligned}$$

and

$$\widehat{I_r}(u) = \begin{cases} \dfrac{-2\pi i}{\omega_r}\displaystyle\sum_{k=0}^{3}\frac{(-1)^k}{(3-k)!k!}\left(-\frac{2\pi iu}{\omega_r}\right)^{3-k}\operatorname{Cot}_2^{(k)}(z,\boldsymbol{\omega})e^{2\pi iux/\omega_r} & (u\geq 0)\\ \dfrac{-2\pi i}{\omega_r}\displaystyle\sum_{n_1,n_2\geq 0}\left\{\frac{e^{-2\pi iu(y+\boldsymbol{n}\cdot\boldsymbol{\omega})/\omega_r}}{(z+\boldsymbol{n}\cdot\boldsymbol{\omega})^4}-\frac{e^{-2\pi iu(y-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega})/\omega_r}}{(z-(\boldsymbol{n}+\boldsymbol{1})\cdot\boldsymbol{\omega})^4}\right\} & (u<0). \end{cases}$$

Furthermore, by the above expression of $\tilde{H}(u)$, we have

$$\tilde{H}'(0) = 2\pi\operatorname{Cot}_2^{(2)}(z,\boldsymbol{\omega}) + \frac{2}{3}\pi x\operatorname{Cot}_2^{(3)}(z,\boldsymbol{\omega}).$$

Therefore Proposition 4.1 and the (usual) Poisson summation formula yield

$$\begin{aligned}(4.6)\quad &\sum_{m_1,m_2\geq 0}\left(\frac{\operatorname{Cot}_2(y-\boldsymbol{m}\cdot\boldsymbol{\omega})}{(x+\boldsymbol{m}\cdot\boldsymbol{\omega})^4}-\frac{\operatorname{Cot}_2(y+(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\omega})}{(x-(\boldsymbol{m}+\boldsymbol{1})\cdot\boldsymbol{\omega})^4}\right)\\ &= -\frac{1}{6}\sum_{k=0}^{3}\binom{3}{k}\operatorname{Cot}_2^{(k)}(z,\boldsymbol{\omega})\\ &\quad\times\left\{\frac{1}{\omega_1}\sum_{m_1>0}\left(c_1\left(m_1\frac{\omega_2}{\omega_1}\right)-\pi i\right)\left(\frac{2\pi im_1}{\omega_1}\right)^{3-k}e^{2\pi im_1x/\omega_1}\right.\\ &\quad\left.+\frac{1}{\omega_2}\sum_{m_2>0}\left(c_1\left(m_2\frac{\omega_1}{\omega_2}\right)-\pi i\right)\left(\frac{2\pi im_2}{\omega_2}\right)^{3-k}e^{2\pi im_2x/\omega_2}-\left(\frac{\pi i}{\omega_1}+\frac{\pi i}{\omega_2}\right)\delta_{3k}\right\}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{n_1, n_2 \geq 0} \left(\frac{\sum_{(i,j)=(1,2),(2,1)} \frac{1}{\omega_i} \sum_{m>0} \left(c_1 \left(m \frac{\omega_j}{\omega_i} \right) + \pi i \right) e^{2\pi i m (y + \boldsymbol{n} \cdot \boldsymbol{\omega})/\omega_i}}{(z + \boldsymbol{n} \cdot \boldsymbol{\omega})^{-4}} \right. \\
&\qquad \left. - \frac{\sum_{(i,j)=(1,2),(2,1)} \frac{1}{\omega_i} \sum_{m>0} \left(c_1 \left(m \frac{\omega_j}{\omega_i} \right) + \pi i \right) e^{2\pi i m (y - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega})/\omega_i}}{(z - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega})^{-4}} \right) \\
&- \frac{i}{2\omega_1\omega_2} \left(2\pi \operatorname{Cot}_2^{(2)}(z, \boldsymbol{\omega}) + \frac{2}{3} \pi x \operatorname{Cot}_2^{(3)}(z, \boldsymbol{\omega}) \right),
\end{aligned}
$$

where δ_{ij} denotes the Kronecker delta. Now, by Proposition 3.5, we have

$$
\begin{aligned}
&-\frac{1}{6} \sum_{k=0}^{3} \binom{3}{k} \operatorname{Cot}_2^{(k)}(z, \boldsymbol{\omega}) \left\{ \frac{1}{\omega_1} \sum_{m_1>0} \left(c_1 \left(m_1 \frac{\omega_2}{\omega_1} \right) - \pi i \right) \left(\frac{2\pi i m_1}{\omega_1} \right)^{3-k} e^{2\pi i m_1 x/\omega_1} \right. \\
&\qquad \left. + \frac{1}{\omega_2} \sum_{m_2>0} \left(c_1 \left(m_2 \frac{\omega_1}{\omega_2} \right) - \pi i \right) \left(\frac{2\pi i m_2}{\omega_2} \right)^{3-k} e^{2\pi i m_2 x/\omega_2} - \left(\frac{\pi i}{\omega_1} + \frac{\pi i}{\omega_2} \right) \delta_{3k} \right\} \\
&= -\frac{1}{6} \sum_{k=0}^{3} \binom{3}{k} \operatorname{Cot}_2^{(k)}(x, \boldsymbol{\omega}) \operatorname{Cot}_2^{(3-k)}(z, \boldsymbol{\omega}) + \frac{\pi i}{2\omega_1\omega_2} \operatorname{Cot}_2^{(2)}(z, \boldsymbol{\omega}) \\
&\qquad + \frac{\pi i}{6} \left(\frac{x}{\omega_1\omega_2} + \frac{1}{2\omega_1} + \frac{1}{2\omega_2} \right) \operatorname{Cot}_2^{(3)}(z, \boldsymbol{\omega})
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n_1, n_2 \geq 0} \left(\frac{\sum_{(i,j)=(1,2),(2,1)} \frac{1}{\omega_i} \sum_{m>0} \left(c_1 \left(m \frac{\omega_j}{\omega_i} \right) + \pi i \right) e^{2\pi i m (y + \boldsymbol{n} \cdot \boldsymbol{\omega})/\omega_i}}{(z + \boldsymbol{n} \cdot \boldsymbol{\omega})^{-4}} \right. \\
&\qquad \left. - \frac{\sum_{(i,j)=(1,2),(2,1)} \frac{1}{\omega_i} \sum_{m>0} \left(c_1 \left(m \frac{\omega_j}{\omega_i} \right) + \pi i \right) e^{2\pi i m (y - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega})/\omega_i}}{(z - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega})^{-4}} \right) \\
&= \sum_{n_1, n_2} \left(\frac{\operatorname{Cot}_2(y + (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega}, \boldsymbol{\omega}) - \frac{\pi i}{\omega_1\omega_2}(y + \boldsymbol{n} \cdot \boldsymbol{\omega}) - \frac{\pi i}{2} \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} \right)}{(z + \boldsymbol{n} \cdot \boldsymbol{\omega})^4} \right. \\
&\qquad \left. - \frac{\operatorname{Cot}_2(y - \boldsymbol{n} \cdot \boldsymbol{\omega}, \boldsymbol{\omega}) - \frac{\pi i}{\omega_1\omega_2}(y - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega}) - \frac{\pi i}{2} \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} \right)}{(z - (\boldsymbol{n}+\boldsymbol{1}) \cdot \boldsymbol{\omega})^4} \right)
\end{aligned}
$$

$$= \sum_{n_1, n_2 \geq 0} \left(\frac{\mathrm{Cot}_2(y + (\boldsymbol{n} + \boldsymbol{1}) \cdot \boldsymbol{\omega}, \boldsymbol{\omega})}{(z + \boldsymbol{n} \cdot \boldsymbol{\omega})^4} - \frac{\mathrm{Cot}_2(y - \boldsymbol{n} \cdot \boldsymbol{\omega}, \boldsymbol{\omega})}{(z - (\boldsymbol{n} + \boldsymbol{1}) \cdot \boldsymbol{\omega})^4} \right)$$

$$+ \frac{\pi i}{12} \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} - \frac{2x}{\omega_1 \omega_2} \right) \mathrm{Cot}_2^{(3)}(z, \boldsymbol{\omega}) - \frac{\pi i}{2\omega_1 \omega_2} \mathrm{Cot}_2^{(2)}(z, \boldsymbol{\omega}).$$

Thus the right hand side of (4.6) becomes

$$-\frac{1}{6} \sum_{k=0}^{3} \binom{3}{k} \mathrm{Cot}_2^{(k)}(x, \boldsymbol{\omega}) \mathrm{Cot}_2^{(3-k)}(z, \boldsymbol{\omega})$$

$$- \sum_{n_1, n_2 \geq 0} \left(\frac{\mathrm{Cot}_2(y + (\boldsymbol{n} + \boldsymbol{1}) \cdot \boldsymbol{\omega}, \boldsymbol{\omega})}{(z + \boldsymbol{n} \cdot \boldsymbol{\omega})^4} - \frac{\mathrm{Cot}_2(y - \boldsymbol{n} \cdot \boldsymbol{\omega}, \boldsymbol{\omega})}{(z - (\boldsymbol{n} + \boldsymbol{1}) \cdot \boldsymbol{\omega})^4} \right).$$

The remaining proof is the same as (i).

The proof when (ii) holds is similar to that of (iii), by using the following proposition (the proof of the proposition is analogous to that of Proposition 4.1 and we omit it):

PROPOSITION 4.2. *Let a, b be coprime positive integers. Assume that $H(t)$ and $\tilde{H}(u)$ satisfy the same conditions as Proposition* 4.1. *Then we have*

$$\sum_{k, n > 0} H\left(2\pi \left(\frac{k}{a} + \frac{n}{b} \right) \right) + \frac{1}{2} \left(\sum_{k > 0} H\left(2\pi \frac{k}{a} \right) + \sum_{n > 0} H\left(2\pi \frac{n}{b} \right) \right)$$

$$= -\frac{ia}{4\pi} \sum_{\substack{m > 0 \\ b \nmid m}} \cot\left(\pi \frac{ma}{b} \right) \tilde{H}(ma) - \frac{ib}{4\pi} \sum_{\substack{n > 0 \\ a \nmid n}} \cot\left(\pi \frac{nb}{a} \right) \tilde{H}(nb)$$

$$- \frac{iab}{8\pi^2} \tilde{H}'(0) - \frac{iab}{4\pi^2} \sum_{k > 0} \tilde{H}'(kab).$$

We finish the proof of the main theorem.

Remark 4.3. The differential equation of the double sine function ([11, Theorem 2.15]) shows

$$\mathrm{Cot}_2(x, (1, 1)) = (1 - x) c_1(x). \tag{4.7}$$

We investigate what Theorem 1.1 means when $\omega_1 = \omega_2 = 1$.

When $\omega_1 = \omega_2 = 1$, $R(x_1, x_2, \boldsymbol{\omega})$ is expressed as follows:

$$\begin{aligned} (4.8) \quad & R(x_1, x_2, (1, 1)) \\ &= -2c_1(x_1) \left(\sum_{k=0}^{\infty} \zeta_2(4, x_2 + k, (1, 1)) + \sum_{k=1}^{\infty} \zeta_2(4, x_2 - 2 - k, (-1, -1)) \right) \\ &= -c_1(x_1) \left(\sum_{n=0}^{\infty} \frac{(n + 2)(n + 1)}{(x_2 + n)^4} + \sum_{n=3}^{\infty} \frac{(-n + 2)(-n + 1)}{(x_2 - n)^4} \right) \end{aligned}$$

$$= -c_1(x_1)\sum_{n\in\mathbf{Z}}\frac{(n+1)(n+2)}{(x_2+n)^4}$$

$$= -c_1(x_1)\sum_{n\in\mathbf{Z}}\frac{(x_2+n)^2+(3-2x_2)(x_2+n)+x_2^2-3x_2+2}{(x_2+n)^4}$$

$$= c_1(x_1)\left(c_1^{(1)}(x_2)+\left(x_2-\frac{3}{2}\right)c_1^{(2)}(x_2)+\frac{x_2^2-3x_2+2}{6}c_1^{(3)}(x_2)\right).$$

Here we used (2.3). By (4.7) and (4.8), (1.2) becomes as follows:

$$\begin{aligned}(4.9)\quad &-6(c_1^{(1)}(x)c_1(y)-c_1(y)c_1^{(1)}(x+y)-c_1(x)c_1^{(1)}(x+y)-c_1^{(1)}(x)c_1(x+y))\\ &+3(2-2x-y)\left(c_1^{(2)}(x)c_1(y)-c_1(y)c_1^{(2)}(x+y)\right.\\ &\qquad\left.-\sum_{i=0}^{2}\binom{2}{i}c_1^{(2-i)}(x)c_1^{(i)}(x+y)\right)\\ &-(1-x)(1-x-y)\left(c_1^{(3)}(x)c_1(y)-c_1(y)c_1^{(3)}(x+y)\right.\\ &\qquad\left.-\sum_{i=0}^{3}\binom{3}{i}c_1^{(3-i)}(x)c_1^{(i)}(x+y)\right)=0.\end{aligned}$$

The expression in the first bracket is the left hand side of Theorem 2.1. The expressions in the second and third brackets are obtained by differentiating the left hand side of Theorem 2.1 with respect to x once and twice, respectively. Therefore (4.9) follows from Theorem 2.1.

5. Application of the main theorem

In this section, we consider some applications of the main theorem. For the applications, we use the following theorem, which is obtained from the main theorem:

THEOREM 5.1. *Assume that one of the conditions in Theorem* 1.1 *holds. Then we have*

$$\frac{1}{\omega_1^2\omega_2}c_1'\left(\frac{x}{\omega_1}\right)c_1\left(\frac{x+y}{\omega_2}\right)+\frac{1}{\omega_1\omega_2^2}c_1\left(\frac{x}{\omega_1}\right)c_1'\left(\frac{x+y}{\omega_2}\right)$$
$$=-\frac{1}{\omega_2}\sum_{m\in\mathbf{Z}}\frac{c_1\left(\dfrac{y-m\omega_1}{\omega_2}\right)}{(x+m\omega_1)^2}+\frac{1}{\omega_1}\sum_{n\in\mathbf{Z}}\frac{c_1\left(\dfrac{y+n\omega_2}{\omega_1}\right)}{(x+y+n\omega_2)^2}.$$

Proof. First we observe that the double Hurwitz zeta function satisfies the following periodicity:

$$\zeta_2(s, x+\omega_i, \boldsymbol{\omega}) = \zeta_2(s, x, \boldsymbol{\omega}) - \zeta_1(s, x, \omega_j), \tag{5.1}$$

where we put

$$\zeta_1(s, x, \omega_j) = \sum_{n=0}^{\infty} (x + n\omega_j)^{-s}.$$

Now, in Theorem 1.1, fix y and replace $x + \omega_2$ with x. Then, by (5.1) and Proposition 3.1, the theorem becomes

$$\begin{aligned}
\sum_{m=0}^{3} \binom{3}{m} &\left(\mathrm{Cot}_2^{(m)}(x+y, \boldsymbol{\omega}) - \frac{c_1^{(m)}((x+y)/\omega_1)}{\omega_1^{m+1}} \right) \\
&\times \left(\mathrm{Cot}_2^{(3-m)}(x, \boldsymbol{\omega}) - \frac{c_1^{(3-m)}(x/\omega_1)}{\omega_1^{3-m}} \right) \\
= &\left(\mathrm{Cot}_2^{(3)}(x, \boldsymbol{\omega}) - \frac{c_1^{(3)}(x/\omega_1)}{\omega_1^{3}} \right) \mathrm{Cot}_2(y, \boldsymbol{\omega}) \\
&+ \mathrm{Cot}_2(y, \boldsymbol{\omega}) \left(\mathrm{Cot}_2^{(3)}(x+y, \boldsymbol{\omega}) - \frac{c_1^{(3)}((x+y)/\omega_1)}{\omega_1^{4}} \right) \\
&+ 6(R(y, \omega_1 + \omega_2 - x, \boldsymbol{\omega}) - r(y, \omega_1 - x)) \\
&- 6(R(y, x+y, \boldsymbol{\omega}) + r(y, x+y)),
\end{aligned} \tag{5.2}$$

where we put

$$\begin{aligned}
r(x_1, x_2) = \frac{1}{\omega_1} &\left\{ \sum_{k=0}^{\infty} c_1\left(\frac{x_1 + k\omega_2}{\omega_1} \right) \zeta_1(4, x_2 + k\omega_2, \omega_1) \right. \\
&\left. - \sum_{k=1}^{\infty} c_1\left(\frac{x_1 - k\omega_2}{\omega_1} \right) \zeta_1(4, x_2 - \omega_1 - k\omega_2, -\omega_1) \right\} \\
+ \frac{1}{\omega_2} &\left\{ \sum_{k=0}^{\infty} c_1\left(\frac{x_1 + k\omega_1}{\omega_2} \right) \zeta_1(4, x_2 + k\omega_1, \omega_1) \right. \\
&\left. - \sum_{k=1}^{\infty} c_1\left(\frac{x_1 - k\omega_1}{\omega_2} \right) \zeta_1(4, x_2 - \omega_1 - k\omega_1, -\omega_1) \right\}.
\end{aligned}$$

Taking difference between (1.2) and (5.2) yields

$$
\begin{aligned}
(5.3)\quad & \sum_{m=0}^{3}\binom{3}{m}\left(\frac{c_1^{(m)}\left(\dfrac{x+y}{\omega_1}\right)}{\omega_1^{m+1}}\operatorname{Cot}_2^{(3-m)}(x,\boldsymbol{\omega})+\frac{c_1^{(3-m)}\left(\dfrac{x}{\omega_1}\right)}{\omega_1^{4-m}}\operatorname{Cot}_2^{(m)}(x+y,\boldsymbol{\omega})\right.\\
&\qquad\left.-\frac{c_1^{(m)}\left(\dfrac{x+y}{\omega_1}\right)c_1^{(3-m)}\left(\dfrac{x}{\omega_1}\right)}{\omega_1^{5}}\right)\\
&=\frac{c_1^{(3)}\left(\dfrac{x}{\omega_1}\right)}{\omega_1^4}\operatorname{Cot}_2(y,\boldsymbol{\omega})+\frac{c_1^{(3)}\left(\dfrac{x+y}{\omega_1}\right)}{\omega_1^4}\operatorname{Cot}_2(y,\boldsymbol{\omega})\\
&\quad+6r(y,\omega_1-x)+6r(y,x+y).
\end{aligned}
$$

Next, in (5.3), fix x and replace y with $y+\omega_1$. Then, by Proposition 3.1 and the periodicity of $\zeta_1(s,x,\omega_j)$ given by

$$\zeta_1(s,x+\omega_j,\omega_j)=\zeta_1(s,x,\omega_j)-\frac{1}{x^s},$$

we have

$$
\begin{aligned}
(5.4)\quad & -\frac{6}{\omega_2}\sum_{k\in\mathbf{Z}}\frac{c_1\left(\dfrac{y-k\omega_1}{\omega_2}\right)}{(x+k\omega_1)^4}+\frac{6}{\omega_1}\sum_{k\in\mathbf{Z}}\frac{c_1\left(\dfrac{y+k\omega_2}{\omega_1}\right)}{(x+y+k\omega_2)^4}\\
&=\sum_{m=0}^{3}\binom{3}{m}\frac{c_1^{(3-m)}\left(\dfrac{x}{\omega_1}\right)c_1^{(m)}\left(\dfrac{x+y}{\omega_2}\right)}{\omega_1^{4-m}\omega_2^{m+1}}.
\end{aligned}
$$

Integrating both sides of (5.4) with respect to x twice yields

$$
\begin{aligned}
& -\frac{1}{\omega_2}\sum_{m\in\mathbf{Z}}\frac{c_1\left(\dfrac{y-m\omega_1}{\omega_2}\right)}{(x+m\omega_1)^2}+\frac{1}{\omega_1}\sum_{n\in\mathbf{Z}}\frac{c_1\left(\dfrac{y+n\omega_2}{\omega_1}\right)}{(x+y+n\omega_2)^2}\\
&\quad=\frac{1}{\omega_1^2\omega_2}c_1^{(1)}\left(\frac{x}{\omega_1}\right)c_1\left(\frac{x+y}{\omega_2}\right)+\frac{1}{\omega_1\omega_2^2}c_1\left(\frac{x}{\omega_1}\right)c_1^{(1)}\left(\frac{x+y}{\omega_2}\right)+h(y)x+k(y),
\end{aligned}
$$

where $h(y)$ and $k(y)$ are functions depending only on y. When the condition (i) of Theorem 1.1 holds, we substitute $(k+1/2)(\omega_1+\omega_2)$ $(k\in\mathbf{Z}_{>0})$ into x and take the limit as $k\to\infty$. When the condition (ii) or (iii) holds, we take the limit as $\operatorname{Im}(x)\to\infty$. In both cases, we have $h(y)=k(y)=0$ and thus obtain the theorem. □

When $\omega_1=\omega_2=1$ in Theorem 5.1, the right hand side becomes $c_1'(x)c_1(y)-c_1(y)c_1'(x+y)$ because of the periodicity of $c_1(x)$. Thus we recover Theorem

2.1. We show that, in addition to Theorem 2.1, several formulas can be deduced from Theorem 5.1.

First we show the reciprocity law for the Dedekind sums. The Dedekind sum $s(h,k)$ is defined as follows. Let h, k be coprime positive integers and we put

$$s(h,k) = \sum_{\mu=1}^{k}\left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right),$$

where we set

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & (x \notin \mathbf{Z}) \\ 0 & (x \in \mathbf{Z}). \end{cases}$$

and $\lfloor x \rfloor$ is the largest integer less than or equal to x. By [12, p. 18 (26)], $s(h,k)$ has the following expression:

$$s(h,k) = \frac{1}{4\pi^2 k}\sum_{m=1}^{k-1} c_1\left(\frac{m}{k}\right)c_1\left(\frac{hm}{k}\right). \tag{5.5}$$

Corollary 5.2 (Reciprocity law for Dedekind sums).

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right).$$

Proof. Put $z = x + y$. Set $\omega_1 = h$, $\omega_2 = k$ in Theorem 5.1. By the periodicity of $c_1(x)$, Theorem 5.1 becomes

$$\frac{1}{h^2k^3}\sum_{r_1=0}^{k-1} c_1\left(\frac{y + r_1 h}{k}\right)c_1^{(1)}\left(\frac{x - r_1 k}{hk}\right) - \frac{1}{h^3k^2}\sum_{r_2=0}^{h-1} c_1\left(\frac{y + r_2 k}{h}\right)c_1^{(1)}\left(\frac{z + r_2 k}{hk}\right)$$
$$= \frac{1}{h^2k}c_1'\left(\frac{x}{h}\right)c_1\left(\frac{z}{k}\right) + \frac{1}{hk^2}c_1\left(\frac{x}{h}\right)c_1'\left(\frac{z}{k}\right).$$

Integrating the both sides with respect to x yields

$$\begin{aligned} &\frac{1}{k}\sum_{r_1=0}^{k-1} c_1\left(\frac{y + r_1 h}{k}\right)c_1\left(\frac{x - r_1 h}{hk}\right) - \frac{1}{h}\sum_{r_2=0}^{h-1} c_1\left(\frac{y + r_2 k}{h}\right)c_1\left(\frac{z + r_2 k}{hk}\right) \\ &\quad = c_1\left(\frac{x}{h}\right)c_1\left(\frac{z}{k}\right) + f(y), \end{aligned} \tag{5.6}$$

where $f(y)$ is a function depending only on y. Since it is known that

$$\lim_{\mathrm{Im}(x)\to\infty} c_1(x) = -\pi i$$

and for a positive integer N

$$\sum_{m=0}^{N-1} c_1\left(\frac{x+m}{N}\right) = Nc_1(x),$$

as $\mathrm{Im}(x) \to \infty$, the left hand side converges to

$$\begin{aligned}
&-\frac{\pi i}{k}\sum_{r_1=0}^{k-1} c_1\left(\frac{y+r_1 h}{k}\right) + \frac{\pi i}{h}\sum_{r_2=0}^{h-1} c_1\left(\frac{y+r_2 k}{h}\right) \\
&\quad = -\frac{\pi i}{k}\sum_{r_1=0}^{k-1} c_1\left(\frac{y+r_1}{k}\right) + \frac{\pi i}{h}\sum_{r_2=0}^{h-1} c_1\left(\frac{y+r_2}{h}\right) \\
&\quad = -\pi i c_1(y) + \pi i c_1(y) \\
&\quad = 0,
\end{aligned}$$

while the right hand side to

$$(-\pi i)^2 + f(y) = -\pi^2 + f(y).$$

Thus we have $f(y) = \pi^2$. By expanding the both sides of (5.6) into power series of x and y, and comparing the constant terms, we see that

$$\frac{1}{4\pi^2 k}\sum_{r_1=1}^{k-1} c_1\left(\frac{r_1 h}{k}\right) c_1\left(\frac{r_1}{k}\right) + \frac{1}{4\pi^2 h}\sum_{r_2=1}^{h-1} c_1\left(\frac{r_2 k}{h}\right) c_1\left(\frac{r_2}{h}\right) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right).$$

Here we used

$$\cot z = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1} \quad (|z| < \pi), \tag{5.7}$$

where B_j denotes the j-th Bernoulli number. By (5.5), two sums on the left hand side are equal to $s(h,k)$ and $s(k,h)$. Thus we obtain the theorem. □

Next, we show the reciprocity law of Apostol's higher order Dedekind sums. The higher order Dedekind sum $s_{2r-1}(h,k)$ is defined by, for an integer r greater than 1,

$$s_{2r-1}(h,k) = \sum_{n=1}^{k} B_{2r-1}\left(\frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor\right)\left(\left(\frac{n}{k}\right)\right),$$

where $B_j(x)$ donotes the j-th Bernoulli polynomial. By [7, Lemma 4.1], $s_{2r-1}(h,k)$ is represented as

$$s_{2r-1}(h,k) = \frac{(-1)^{r-1}(2r-1)!}{(2\pi)^{2r-1}} \sum_{\substack{n=1 \\ n \not\equiv 0 (\mathrm{mod}\ k)}}^{\infty} \frac{\cot(\pi hn/k)}{n^{2r-1}}. \tag{5.8}$$

Corollary 5.3 (Reciprocity law for the higher order Dedekind sums).

$$2rhk^{2r-1}s_{2r-1}(h,k)+2rkh^{2r-1}s_{2r-1}(k,h) = \sum_{m=0}^{r}\binom{2r}{2m}h^{2m}k^{2(r-m)}B_{2m}B_{2r-2m}+(2r-1)B_{2r}. \tag{5.9}$$

Proof. By setting $\omega_1 = h$, $\omega_2 = k$ and taking the limit as $y \to 0$ in Theorem 5.1, we have

$$\frac{1}{k}\sum_{\substack{n\in\mathbf{Z}\\ n\not\equiv 0 (\mathrm{mod}\ k)}}\frac{c_1(nh/k)}{(x+nh)^2}+\frac{1}{h}\sum_{\substack{n\in\mathbf{Z}\\ n\not\equiv 0 (\mathrm{mod}\ h)}}\frac{c_1(nk/h)}{(x+nk)^2}$$
$$=\frac{1}{h^2k}c_1^{(1)}\left(\frac{x}{h}\right)c_1\left(\frac{x}{k}\right)-\frac{1}{hk^2}c_1\left(\frac{x}{h}\right)c_1^{(1)}\left(\frac{x}{k}\right)+\frac{c_1^{(2)}(x/kh)}{(kh)^3}.$$

Expanding the both sides into power series of x and comparing the coefficients of x^{2r-3} yields

$$2rhk^{2r-1}\frac{(-1)^{r-1}(2r-1)!}{(2\pi)^2}\frac{1}{\pi}\sum_{\substack{n=1\\ n\not\equiv 0 (\mathrm{mod}\ k)}}^{\infty}\frac{c_1(nh/k)}{n^{2r-1}}$$
$$+2rkh^{2r-1}\frac{(-1)^{r-1}(2r-1)!}{(2\pi)^{2r-1}}\frac{1}{\pi}\sum_{\substack{n=1\\ n\not\equiv 0 (\mathrm{mod}\ h)}}^{\infty}\frac{c_1(nk/h)}{n^{2r-1}}$$
$$=\sum_{m=0}^{r}\binom{2r}{2m}h^{2m}k^{2(r-m)}B_{2m}B_{2r-2m}+(2r-1)B_{2r}.$$

Here we used (5.7). By (5.8), two infinite series on the left hand side are equal to $s_{2r-1}(h,k)$ and $s_{2r-1}(k,h)$. This prove the theorem. $\square$

Furthermore we can obtain the following two formulas from Theorem 5.1:

Corollary 5.4 (Lerch's functional equation). *Let r be a integer greater than or equal to 2 and θ be a algebraic irrational number. Then we have*

$$\sum_{m=1}^{\infty}\frac{\cot(\pi m\theta)}{m^{2r-1}}+\theta^{2r-2}\sum_{m=1}^{\infty}\frac{\cot(\pi m/\theta)}{m^{2r-1}}=(-1)^{r-1}(2\pi)^{2r-1}\sum_{k=0}^{r}\theta^{2k-1}\frac{B_{2k}}{(2k)!}\frac{B_{2r-2k}}{(2r-2k)!}.$$

Proof. Thee proof is analogous to that of Corollary 5.3 by setting $(\omega_1,\omega_2) = (1,\theta)$ in Theorem 5.1. $\square$

Remark 5.5. The absolute convergence of the infinite series in the left hand side of Corollary 5.4 follows from [2, Lemma 1].

COROLLARY 5.6 (Ramanujan's formula). *Let n be a positive integer. When $\alpha, \beta > 0$, $\alpha\beta = \pi^2$, we have*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\} = (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\} - 2^{2n}\sum_{j=0}^{n+1}(-1)^j\frac{B_{2j}B_{2n+2-2j}}{(2j)!(2n+2-2j)!}\alpha^{n+1-j}\beta^j.$$

Proof. Set $(\omega_1, \omega_2) = (1, -\pi i/\alpha)$ in Theorem 5.1. By taking the limit as $y \to 0$ and comparing the coefficients of x^{2n-1}, we have

$$\frac{\alpha^{-n}}{2i}\sum_{k=1}^{\infty}\frac{\cot(-k\alpha i)}{k^{2n+1}} = \frac{(-\beta)^{-n}}{2i}\sum_{k=1}^{\infty}\frac{\cot(-k\beta i)}{k^{2n+1}} - 2^{2n}\sum_{j=0}^{n+1}(-1)^j\frac{B_{2j}B_{2n+2-2j}}{(2j)!(2n+2-2j)!}\alpha^{n+1-j}\beta^j.$$

Since

$$\frac{\cot(-\theta i)}{2i} = \frac{1}{2} + \frac{1}{e^{2\theta}-1},$$

we have

$$\frac{\alpha^{-n}}{2i}\sum_{k=1}^{\infty}\frac{\cot(-k\alpha i)}{k^{2n+1}} = \alpha^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right)$$

$$\frac{(-\beta)^{-n}}{2i}\sum_{k=1}^{\infty}\frac{\cot(-k\beta i)}{k^{2n+1}} = (-\beta)^{n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right),$$

which completes the proof. $\square$

Appendix A. An alternative proof of Theorem 5.1

Theorem 5.1 may alternatively be obtained directly in terms of the usual Poisson summation formula, as follows. The proof in the case of $\omega_2/\omega_1 \notin \mathbf{R}$ is similar to that of $\omega_2/\omega_1 \in \mathbf{R}$. We prove the theorem when $\omega_2/\omega_1 \in \mathbf{R}$. Since both sides of the identity are meromorphic functions of x and c_1 is a odd function, it is enough to show the theorem when $\mathrm{Im}(x) > 0$ and $\mathrm{Im}(y) > 0$. For simplicity, set $z = x + y$. Put $I(u) = c_1\left(\dfrac{y - u\omega_1}{\omega_2}\right)(x + u\omega_1)^{-2}$, $\hat{I}(u) = \int_{-\infty}^{\infty} I(t)e^{-2\pi i t u}\, dt$. We have $I(u) = O(u^{-2})$ because $\operatorname{Im}(y) > 0$. By Cauchy's integral theorem, we have

$$\hat{I}(u) = \begin{cases} -2\pi i \operatorname*{Res}_{t=-x/\omega_1} I(t)e^{-2\pi i t u} & (u \geq 0) \\ 2\pi i \sum_{n\in\mathbf{Z}} \operatorname*{Res}_{t=(y+n\omega_2)/\omega_1} I(t)e^{-2\pi i t u} & (u < 0) \end{cases}$$

and simple calculations show that

$$\operatorname*{Res}_{t=-x/\omega_1} I(t)e^{-2\pi i t u} = -e^{2\pi i x u/\omega_1}\left(\frac{2\pi i u}{\omega_1^2} c_1\left(\frac{z}{\omega_2}\right) + \frac{1}{\omega_1\omega_2} c_1'\left(\frac{z}{\omega_2}\right)\right)$$

$$\operatorname*{Res}_{t=(y+n\omega_2)/\omega_1} I(t)e^{-2\pi i t u} = -\frac{\omega_2}{\omega_1}\frac{e^{-2\pi i(y+n\omega_2)u/\omega_1}}{(z + n\omega_2)^2}.$$

In particular, for $\varepsilon > 0$, we have $\hat{I}(u) = O(u^{-1-\varepsilon})$. By the Poisson summation formula, it follows that

$$\frac{1}{\omega_2}\sum_{m\in\mathbf{Z}}\frac{c_1\left(\dfrac{y - m\omega_1}{\omega_2}\right)}{(x + m\omega_1)^2} = \frac{(2\pi i)^2}{\omega_1^2\omega_2} c_1\left(\frac{z}{\omega_2}\right)\sum_{m\geq 0} m e^{2\pi i x m/\omega_1} + \frac{2\pi i}{\omega_1\omega_2^2} c_1'\left(\frac{z}{\omega_2}\right)\sum_{m\geq 0} e^{2\pi i x m/\omega_1}$$
$$- \left(\frac{2\pi i}{\omega_1}\right)\sum_{n\in\mathbf{Z}}\sum_{m>0}\frac{e^{2\pi i(y+n\omega_2)m/\omega_1}}{(z + n\omega_2)^2}.$$

Since

$$2\pi i\sum_{n=0}^{\infty} e^{2\pi i n x} = -c_1(x) + \pi i$$

for $\operatorname{Im}(x) > 0$, the right hand side is calculated as follows:

$$-\frac{1}{\omega_1^2\omega_2} c_1'\left(\frac{x}{\omega_1}\right) c_1\left(\frac{z}{\omega_2}\right) - \frac{1}{\omega_1\omega_2^2}\left(c_1\left(\frac{x}{\omega_1}\right) - \pi i\right) c_1'\left(\frac{z}{\omega_2}\right)$$
$$+ \frac{1}{\omega_1}\sum_{n\in\mathbf{Z}}\frac{c_1\left(\dfrac{y + n\omega_2}{\omega_1}\right) + \pi i}{(z + n\omega_2)^2}$$
$$= -\frac{1}{\omega_1^2\omega_2} c_1'\left(\frac{x}{\omega_1}\right) c_1\left(\frac{z}{\omega_2}\right) - \frac{1}{\omega_1\omega_2^2} c_1\left(\frac{x}{\omega_1}\right) c_1'\left(\frac{z}{\omega_2}\right) + \frac{1}{\omega_1}\sum_{n\in\mathbf{Z}}\frac{c_1\left(\dfrac{y + n\omega_2}{\omega_1}\right)}{(z + n\omega_2)^2}.$$

Thus we obtain the theorem.

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