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L^2 CONTINUITY OF THE CALDERÓN TYPE COMMUTATOR FOR THE LITTLEWOOD-PALEY OPERATOR WITH ROUGH VARIABLE KERNEL

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Abstract

For $b \in Lip(\mathbf{R}^n)$, the Calderón type commutator for the Littlewood-Paley operator with variable kernel is defined by

$$\mu_{\Omega,1;b}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t}\right)^{1/2}.$$

By giving a method based on Littlewood-Paley theory, Fourier transform and the spherical harmonic development, we prove the L^2 norm inequalities for the rough operators $\mu_{\Omega,1;b}$ with $\Omega(x,z') \in L^\infty(\mathbf{R}^n) \times L^q(S^{n-1})$ $\left(q > \dfrac{2(n-1)}{n}\right)$ satisfying certain cancellation conditions.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbf{R}^n$ $(n \ge 2)$ with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. A function $\Omega(x,z)$ defined on $\mathbf{R}^n \times \mathbf{R}^n$ is said to belong to $L^\infty(\mathbf{R}^n) \times L^q(S^{n-1})$, $q \ge 1$, if it satisfies the following conditions:

(i) $\Omega(x,\lambda z) = \Omega(x,z)$ for any $x, z \in \mathbf{R}^n$ and $\lambda > 0$;

(ii) $\|\Omega\|_{L^\infty(\mathbf{R}^n)\times L^q(S^{n-1})} := \sup_{x\in\mathbf{R}^n}\left(\int_{S^{n-1}} |\Omega(x,z')|^q\,d\sigma(z')\right)^{1/q} < \infty$,

where $z' = \dfrac{z}{|z|}$, for any $z \in \mathbf{R}^n\backslash\{0\}$.

For $\alpha \ge 0$, the singular integral operator $T_{\Omega,\alpha}$ with variable kernel is defined by

$$T_{\Omega,\alpha}f(x) = \text{p.v.}\int_{\mathbf{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n+\alpha}} f(y)\,dy,$$

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where $f \in \mathscr{S}(\mathbf{R}^n)$ and $\Omega(x,z') \in L^\infty(\mathbf{R}^n) \times L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(x,z')\, d\sigma(z') = 0. \tag{1.1}$$

When $\alpha = 0$, we denote $T_{\Omega,0}$ by T for simplicity. It is easy to check that, by (1.1), $Tf(x)$ exists for almost every $x \in \mathbf{R}^n$. As is well known, L^2 continuity of T was initially studied by Calderón and Zygmund [2]. Furthermore, they showed that the operator T is closely related to the problem about the second-order linear elliptic equations with variable coefficients. They obtained

Theorem A (see [2]). *If* $\Omega(x,z') \in L^\infty(\mathbf{R}^n) \times L^q(S^{n-1})$, $q > \dfrac{2(n-1)}{n}$, *satisfies* (1.1), *then for all* $f \in \mathscr{S}(\mathbf{R}^n)$, *there is a constant* $C > 0$ *such that*

$$\|Tf\|_{L^2(\mathbf{R}^n)} \le C\|f\|_{L^2(\mathbf{R}^n)}.$$

Afterwards, the continuity properties of the singular operator $T_{\Omega,\alpha}f$ have been intensively investigated (see [4, 10, 18, 21, 31] for example). Nevertheless, we would like to mention the recent paper [9], where the decomposition techniques via spherical harmonics were used to extend the $(\dot{L}^2_\alpha, L^2)$-boundedness of $T_{\Omega,\alpha}$ to all $\alpha \ge 0$. We recall the following result of [9].

Theorem B (see [9]). *Let* $\alpha \ge 0$. *If* $\Omega(x,z') \in L^\infty(\mathbf{R}^n) \times L^q(S^{n-1})$ *with* $q > \max\left\{1, \dfrac{2(n-1)}{n+2\alpha}\right\}$, *satisfies*

$$\int_{S^{n-1}} \Omega(x,z') Y_m(z')\, d\sigma(z') = 0, \tag{1.2}$$

for all spherical harmonic polynomials Y_m *with degree* $\le [\alpha]$. *Then there is a constant* $C > 0$ *such that*

$$\|T_{\Omega,\alpha} f\|_{L^2(\mathbf{R}^n)} \le C\|f\|_{\dot{L}^2_\alpha(\mathbf{R}^n)},$$

where $\dot{L}^2_\alpha(\mathbf{R}^n)$ *is the homogeneous* L^2 *Sobolev space with the order* α.

An analogous operator of the singular integral $T_{\Omega,\alpha}$ is the Marcinkiewicz integral operator μ^ρ_Ω, which is defined by

$$\mu^\rho_\Omega(f)(x) = \left(\int_0^\infty |F^\rho_{\Omega,t}(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $0 < \rho < n$, and

$$F^\rho_{\Omega,t}(x) = \frac{1}{t^\rho} \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-\rho}} f(y)\, dy.$$

If $\Omega(x, z') = \Omega(z')$, μ_Ω^1 is a Littlewood-Paley function, that was studied by many authors (see [1, 23, 32, 33, 37, 36, 39]). For a general $\Omega(x, z')$, Ding, Lin and Shao [25] gave the L^2 boundedness of μ_Ω^1 with variable kernel. Recently, many new advances have been made in the study of the L^p boundedness of μ_Ω^ρ, see [8, 12, 24] for further details.

On the other hand, it is well known that commutators have played a crucial role in harmonic analysis and partial differential equations (see [3, 5, 19, 27, 29, 30] for example). For $b \in Lip(\mathbf{R}^n)$, Calderón [5] introduced the following commutator on $\mathbf{R}$:

$$[b, dH/dx]f(x) = \text{p.v.}(-1)\int_{-\infty}^{+\infty} \frac{b(x) - b(y)}{x - y} \frac{f(y)}{x - y}\, dy,$$

which is called first Calderón commutator. Obviously, if $b(x) = -x$, then $[b, dH/dx]$ reduces to the Hilbert transform. Thus, its role in the theory of partial differential equations becomes apparent. Some known results show that the commutator $[b, dH/dx]$ is also of fundamental importance in the study of Cauchy integral along Lipschitz curves and the Kato square root problem, see [6, 7, 34, 35] for its history and significance.

In addition, there are large classes of commutators of singular integrals, which are of interest in the theory of nondivergent elliptic equations with discontinuous coefficients, see [16, 17, 22]. Moreover, there is also an interesting connection between the nonlinear commutator, considered by Coifman, Rochberg and Weiss in [20], and the Jacobian mapping of vector functions. They have been applied in the study of nonlinear partial differential equations, see [28] and the references therein.

In this paper, for $b \in Lip(\mathbf{R}^n)$, we will study the Calderón commutator for the Littlewood-Paley operator with variable kernel defined by

$$\mu_{\Omega,1;b}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}}(b(x) - b(y))f(y)\, dy\right|^2 \frac{dt}{t}\right)^{1/2},$$

which is a new kind of commutator of Littlewood-Paley operator. When $\Omega(x, z')$ is independent of x, Chen and Ding [14] showed that if $\Omega \in L(log^+ L)^{1/2}(S^{n-1})$, then $\mu_{\Omega,1;b}$ is of type $(2, 2)$. Motivated by [5] and [14], it is an interesting problem that if $\mu_{\Omega,1;b}$ is still bounded on $L^2(\mathbf{R}^n)$ when $\Omega(x, z')$ does depend on x. We will give an affirmative answer to this question. Our main result can be stated as follows.

THEOREM 1.1. *If $\Omega(x, z') \in L^\infty(\mathbf{R}^n) \times L^q(S^{n-1})$, $q > \dfrac{2(n-1)}{n}$, satisfies (1.2) for all spherical harmonic polynomials Y_m with degree ≤ 1, then for $b \in Lip(\mathbf{R}^n)$, there is a constant $C > 0$ such that*

$$\|\mu_{\Omega,1;b}f\|_{L^2(\mathbf{R}^n)} \le C\|b\|_{Lip(\mathbf{R}^n)}\|f\|_{L^2(\mathbf{R}^n)}.$$

Remark 1.2. In fact, since the integral kernel in Theorem 1.1 has no any smoothness, the usual manner such as the method of rotation, which is effective in [5] fails utterly to treat the operator $\mu_{\Omega,1;b}$. Therefore, we are forced to give here a new approach which is fundamentally different from the one in [5]. The proof of Theorem 1.1 involves careful decompose arguments using the properties of Littlewood-Paley functions and Fourier transform, and the well known techniques for treating "variable kernel" operators via spherical harmonics.

2. Preliminaries and Lemmas

As usual, the notations "$\wedge$" and "$\vee$" denote the Fourier transform and the inverse Fourier transform, respectively. Denote by $\mathscr{S}(\mathbf{R}^n)$ the Schwartz class and $\mathscr{S}'(\mathbf{R}^n)$ the space of tempered distributions. If $E \subset \mathbf{R}^n$ is a measurable set, then $|E|$ stands for the Lebesgue measure of E. f_E represents the mean value of f on E, namely, $f_E = |E|^{-1}\int_E f(x)\,dx$. For brevity, we write $Lip_1(\mathbf{R}^n) = Lip(\mathbf{R}^n)$. Throughout this paper, the letter C indicates a positive constant whose value may change from appearance to appearance. We also denote $f(x) \simeq g(x)$ if there exist positive constants A and B independent of x such that $Af(x) \le g(x) \le Bf(x)$.

For $x \in \mathbf{R}^n$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we set $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. $\partial^\alpha f$ denotes the derivative $\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denotes its size.

LEMMA 2.1 (see [38]). *Let $n \ge 2$, and $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree k. Then the Fourier transform of f has the form $\hat{f} = F_0(|x|)P(x)$, where*

$$F_0(r) = 2\pi i^{-k} r^{-[(n+2k-2)/2]}\int_0^\infty f_0(s)J_{(n+2k-2)/2}(2\pi rs)s^{(n+2k)/2}\,ds,$$

$r = |x|$, and J_υ is the Bessel function.

LEMMA 2.2. *Suppose that $n \ge 2$, $0 < \beta < 1$, $m \in \mathbf{N}$ and $k \in \mathbf{Z}$. Denote by $\mathcal{H}_m$ the space of surface spherical harmonics of degree m on S^{n-1} with its dimension d_m. $\{Y_{m,j}\}_{j=1}^{d_m}$ denotes the normalized complete system in $\mathcal{H}_m$. For $t > 0$, let*

$$\sigma_{k,t,m,j}(x) = (2^k t)^{-2}\frac{Y_{m,j}(x')}{|x|^{n-1}}\chi_{\{x:0<|x|\le 2^k t\}}(x).$$

Then for $m \ge 2$

$$|\widehat{\sigma_{k,t,m,j}}(\xi)| \le C(2^k t)^{-1}m^{-\lambda-1+\beta/2}\min\{|2^k t\xi|^2, |2^k t\xi|^{-\beta/2}\}|Y_{m,j}(\xi')|, \tag{2.1}$$

where $\lambda = (n-2)/2$ and $\xi' = \dfrac{\xi}{|\xi|}$. For any fixed multi-index α with $m > |\alpha|$, we have

$$|\partial^\alpha\widehat{\sigma_{k,t,m,j}}(\xi)| \le C(2^k t)^{(-1+|\alpha|)}\min\{|2^k t\xi|^{m-|\alpha|}, 1\}. \tag{2.2}$$

Proof. To estimate (2.1), we set $P_{m,j}(x) = Y_{m,j}(x')|x|^m$, then $P_{m,j}$ is a solid spherical harmonic of degree m and

$$\sigma_{k,t,m,j}(x) = (2^k t)^{-2}|x|^{-n-m+1} P_{m,j}(x)\chi_{\{x:|x|\le 2^k t\}}(x).$$

Obviously, $\psi_0(|x|) := (2^k t)^{-2}|x|^{-n-m+1}\chi_{\{x:|x|\le 2^k t\}}(x)$ is a radial function in x for fixed $t > 0$, by Lemma 2.1, we have

$$\widehat{\sigma_{k,t,m,j}}(\xi) = \Psi_0(|\xi|)P_{m,j}(\xi) = Y_{m,j}(\xi')|\xi|^m\Psi_0(|\xi|), \tag{2.3}$$

where

$$\begin{aligned}\Psi_0(r) &= 2\pi i^{-m} r^{-[(n+2m-2)/2]} \int_0^\infty \psi_0(s) J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2}\, ds \\ &= 2\pi i^{-m}(2^k t)^{-2} r^{-[(n+2m-2)/2]} \int_0^{2^k t} s^{-n+1-m} J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2}\, ds \\ &= (2\pi)^{n/2+1} i^{-m} r^{-m+1}(2\pi 2^k t r)^{-2} \int_0^{2\pi 2^k t r} \frac{J_{(n+2m-2)/2}(s)}{s^{(n-2)/2}}\, ds.\end{aligned}$$

From this and (2.3), it follows that

$$\widehat{\sigma_{k,t,m,j}}(\xi) = (2\pi)^{n/2+1} i^{-m}(2\pi 2^k t|\xi|)^{-2}\, Y_{m,j}(\xi')|\xi| \int_0^{2\pi 2^k t|\xi|} \frac{J_{m+\lambda}(s)}{s^\lambda}\, ds, \tag{2.4}$$

where $\lambda = (n-2)/2$. Now we can distinguish three cases as follows:

Case 1°: $2^k t|\xi| \le 1$. The classical formula of the Bessel function yields (see [40, p. 48])

$$|J_{m+\lambda}(s)| \le C\frac{s^{m+\lambda}}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}.$$

For $x > 1$, using Stirling's formula, we get

$$\sqrt{2\pi}x^{x-1/2}e^{-x} \le \Gamma(x) \le 2\sqrt{2\pi}x^{x-1/2}e^{-x}.$$

Then, in view of $m \ge 2$, we obtain

$$\begin{aligned}&(2\pi 2^k t|\xi|)^{-2}|\xi| \left|\int_0^{2\pi 2^k t|\xi|} \frac{J_{m+\lambda}(s)}{s^\lambda}\, ds\right| \\ &\le C\frac{(2\pi 2^k t|\xi|)^{-2}}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}|\xi| \int_0^{2\pi 2^k t|\xi|} s^m\, ds \\ &\le C(2^k t)^{-1}\frac{1}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}\cdot\frac{(2\pi 2^k t|\xi|)^m}{m} \\ &\le C(2^k t)^{-1}\frac{e^{m+\lambda}(2\pi 2^k t|\xi|)^{m-2}}{2^{m+\lambda}(m+\lambda+1/2)^{m+\lambda}}\cdot\frac{(2^k t|\xi|)^2}{m} \\ &\le C(2^k t)^{-1}(2^k t|\xi|)^2\frac{e^{m+\lambda}}{2^{m+\lambda}}\cdot\frac{(2\pi)^m}{m(m+\lambda+1/2)^{m+\lambda}}\end{aligned}$$

$$\leq C(2^k t)^{-1} m^{-\lambda-1} (2^k t|\xi|)^2 \left(\frac{e}{2}\right)^{\lambda} \cdot \frac{(2e\pi)^m}{(2m+2\lambda)^m}$$
$$\leq C(2^k t)^{-1} m^{-\lambda-1} (2^k t|\xi|)^2.$$
$$\leq C(2^k t)^{-1} m^{-\lambda-1+\beta/2} (2^k t|\xi|)^2.$$

CASE 2°: $1 < 2^k t|\xi| < \dfrac{m+\lambda}{4\pi}$. In this Case, we have

$$(2\pi 2^k t|\xi|)^{-2} |\xi| \left| \int_0^{2\pi 2^k t|\xi|} \frac{J_{m+\lambda}(s)}{s^{\lambda}}\, ds \right|$$
$$\leq C(2^k t)^{-1} \frac{1}{2^{m+\lambda}\Gamma(m+\lambda+1/2)} \cdot \frac{(2\pi 2^k t|\xi|)^m}{m}$$
$$\leq C(2^k t)^{-1} \frac{e^{m+\lambda}}{2^{m+\lambda}} \cdot \frac{(2\pi 2^k t|\xi|)^m}{m(m+\lambda+1/2)^{m+\lambda}}$$
$$\leq C(2^k t)^{-1} \left(\frac{e}{4}\right)^m \cdot \left(\frac{e}{2}\right)^{\lambda} \cdot \frac{(m+\lambda)^m}{m(m+\lambda+1/2)^{m+\lambda}}$$
$$\leq C(2^k t)^{-1} (m+\lambda)^{-\lambda} m^{-1}$$
$$\leq C(2^k t)^{-1} (m+\lambda)^{-\lambda+\beta/2} m^{-1} (2^k t|\xi|)^{-\beta/2}$$
$$\leq C(2^k t)^{-1} m^{-1-\lambda+\beta/2} (2^k t|\xi|)^{-\beta/2}.$$

CASE 3°: $2^k t|\xi| \geq \dfrac{m+\lambda}{4\pi}$. By the second integral mean value theorem, arguing as in [11, p. 195], we have

$$(2\pi 2^k t|\xi|)^{-2} |\xi| \left| \int_0^{2\pi 2^k t|\xi|} \frac{J_{m+\lambda}(s)}{s^{\lambda}}\, ds \right|$$
$$= (2\pi 2^k t|\xi|)^{-2} |\xi| \left| \sum_{j=-\infty}^{0} \int_{2\pi 2^{k+j-1} t|\xi|}^{2\pi 2^{k+j} t|\xi|} \frac{J_{m+\lambda}(s)}{s^{\lambda}}\, ds \right|$$
$$\leq C(2\pi 2^k t|\xi|)^{-2} |\xi| \left(\left| \sum_{j=-\infty}^{0} (2\pi 2^{k+j-1} t|\xi|) \int_h^{2\pi 2^{k+j} t|\xi|} \frac{J_{m+\lambda}(s)}{s^{\lambda+1}}\, ds \right| \right.$$
$$\left. + \left| \sum_{j=-\infty}^{0} (2\pi 2^{k+j} t|\xi|) \int_{2\pi 2^{k+j-1} t|\xi|}^{h} \frac{J_{m+\lambda}(s)}{s^{\lambda+1}}\, ds \right| \right)$$
$$\leq C(2\pi 2^k t|\xi|)^{-2} |\xi| \cdot (2^k t|\xi|) \cdot (2^k t|\xi|)^{-\lambda-1}$$
$$\leq C(2^k t)^{-1} m^{-1-\lambda+\beta/2} (2^k t|\xi|)^{-\beta/2},$$

where $2\pi 2^{k+j-1}t|\xi| \le h \le 2\pi 2^{k+j}t|\xi|$. Hence, combining (2.4) and the estimates above, we arrive at

$$|\widehat{\sigma_{k,t,m,j}}(\xi)| \le C(2^k t)^{-1} m^{-\lambda-1+\beta/2} \min\{|2^k t\xi|^2, |2^k t\xi|^{-\beta/2}\}|Y_{m,j}(\xi')|.$$

This gives the proof of (2.1). We omit the proof of (2.2) since it is essentially similar to the proof of Lemma 3.1(3.3) in [13, p. 88–89].

LEMMA 2.3 (see [15]). *For* $0 < \delta < \infty$, $m \in \mathbf{N}$ *and* $j = 1, 2, \dots, d_m$, *take* $\Gamma_{\delta,m,j} \in C_c^\infty(\mathbf{R}^n)$ *such that* $\operatorname{supp}(\Gamma_{\delta,m,j}) \subset \{\xi : \delta/2 \le |\xi| \le 2\delta\}$. *Let* $T_{\delta,m,j}$ *be the multiplier operators defined by*

$$\widehat{T_{\delta,m,j}f}(\xi) = \Gamma_{\delta,m,j}\hat{f}(\xi), \quad j = 1, 2, \dots, d_m.$$

Moreover, for $b \in Lip(\mathbf{R}^n)$, *denote by* $[b, T_{\delta,m,j}]$ *the commutator of* $T_{\delta,m,j}$ *and b. Define* $T_{\delta,m;b}$ *by*

$$T_{\delta,m;b}f(x) = \left(\sum_{j=1}^{d_m} |[b, T_{\delta,m,j}]f(x)|^2\right)^{1/2}.$$

If for some constant $0 < \beta < 1$, $\Gamma_{\delta,m,j}$ *satisfies*

$$|\Gamma_{\delta,m,j}(\xi)| \le C2^{-k} m^{-\lambda-1+\beta/2} \min\{\delta, \delta^{-\beta/2}\}|Y_{m,j}(\xi')|,$$

where $\lambda = (n-2)/2$, *and for any multi-index* α *with* $|\alpha| > \dfrac{2}{\beta}(\lambda + 1 - \beta/2)$ *and* τ *with* $|\tau| = 2$,

$$\begin{cases} \|\partial^\alpha \Gamma_{\delta,m,j}\|_{L^\infty} \le C2^{-k}, & m > |\alpha|, \\ \|\partial^\tau \Gamma_{\delta,m,j}\|_{L^\infty} \le C2^{-k}, & m \le |\alpha|, \end{cases}$$

then for some $0 < v_1, v_2 < 1$, *there exists a positive constant* $C = C(|\alpha|)$ *such that*

$$\|T_{\delta,m;b}f\|_{L^2} \le \begin{cases} C2^{-k}m^{-1+\beta} \min\{\delta^{v_1}, \delta^{-\beta v_1/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m > |\alpha|, \\ C2^{-k} \min\{\delta^{v_2}, \delta^{-\beta v_2/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m \le |\alpha|. \end{cases}$$

LEMMA 2.4 (see [14]). *Let* $\phi \in \mathscr{S}(\mathbf{R}^n)$ *be a radial function such that* $\operatorname{supp}\phi \subset \{\xi : 1/2 \le |\xi| \le 2\}$. *Define the multiplier operator* S_j *by* $\widehat{S_j f}(\xi) = \phi(2^{-j}\xi)\hat{f}(\xi)$ *for* $j \in \mathbf{Z}$. *Let* $b \in Lip(\mathbf{R}^n)$. *Then for* $f \in L^2(\mathbf{R}^n)$, *we have*

$$\left(\sum_{j \in \mathbf{Z}} 2^{2j}\|[b, S_j]f\|_{L^2}^2\right)^{1/2} \le C\|b\|_{Lip}\|f\|_{L^2}.$$

3. Proof of Theorem 1.1

Using the spherical harmonic development [4] and (1.2), we get

$$\Omega(x, z') = \sum_{m=2}^{\infty}\sum_{j=1}^{d_m} a_{m,j}(x) Y_{m,j}(z'),$$

where

$$a_{m,j}(x) = \int_{S^{n-1}} \Omega(x,z')\overline{Y_{m,j}(z')}\,d\sigma(z')$$

and $d_m \simeq m^{n-2}$ (see [4]). Denote

$$a_m(x) = \left(\sum_{j=1}^{d_m} |a_{m,j}(x)|^2\right)^{1/2} \quad \text{and} \quad b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}.$$

Then

$$\sum_{j=1}^{d_m} b_{m,j}^2(x) = 1 \tag{3.1}$$

and

$$\Omega(x,z') = \sum_{m\ge 2} a_m(x) \sum_{j=1}^{d_m} b_{m,j}(x)\,Y_{m,j}(z').$$

If we write

$$\mu_{m,j;b}f(x) = \left(\int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t}\right)^{1/2},$$

then by Hölder's inequality and (3.1), we have

$$\begin{aligned}
&(\mu_{\Omega,1;b}f(x))^2 \\
&\quad= \int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t} \\
&\quad= \int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \sum_{m\ge 2} a_m(x)\sum_{j=1}^{d_m} b_{m,j}(x)\frac{Y_{m,j}(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t} \\
&\quad\le \left(\sum_{m\ge 2} a_m^2(x)m^{-\varepsilon}\right) \\
&\qquad\times \sum_{m\ge 2} m^{\varepsilon}\int_0^\infty \left|\frac{1}{t^2}\int_{|x-y|\le t} \sum_{j=1}^{d_m} b_{m,j}(x)\frac{Y_{m,j}(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2 \frac{dt}{t}
\end{aligned}$$

$$\le \left(\sum_{m\ge 2} a_m^2(x)m^{-\varepsilon}\right)\sum_{m\ge 2} m^{\varepsilon}\int_0^{\infty}\left(\sum_{j=1}^{d_m} b_{m,j}^2(x)\right)$$
$$\times \sum_{j=1}^{d_m}\left|\frac{1}{t^2}\int_{|x-y|\le t}\frac{Y_{m,j}(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)\,dy\right|^2\frac{dt}{t}$$
$$= \left(\sum_{m\ge 2} a_m^2(x)m^{-\varepsilon}\right)\left(\sum_{m\ge 2} m^{\varepsilon}\sum_{j=1}^{d_m}(\mu_{m,j;b}f(x))^2\right),$$

where $0<\varepsilon<1$. By [4, p. 230], if we take ε sufficiently close to 1, then

$$\left(\sum_{m\ge 2} a_m^2(x)m^{-\varepsilon}\right)^{1/2} \le C\left(\int_{S^{n-1}}|\Omega(x,z')|^q\,d\sigma(z')\right)^{1/q} \le C\|\Omega\|_{L^\infty(\mathbf{R}^n)\times L^q(S^{n-1})}. \tag{3.2}$$

for $q>2(n-1)/n$. Let

$$\mu_{m;b}f(x) = \left(\sum_{j=1}^{d_m}|\mu_{m,j;b}f(x)|^2\right)^{1/2}.$$

Minkowski's inequality and (3.2) imply that

$$\|\mu_{\Omega,1;b}f\|_{L^2}^2 \le C\|\Omega\|^2_{L^\infty(\mathbf{R}^n)\times L^q(S^{n-1})}\sum_{m\ge 2} m^{\varepsilon}\|\mu_{m;b}f\|_{L^2}^2. \tag{3.3}$$

If we can show that for some $0<\beta<(1-\varepsilon)/2$, such that

$$\|\mu_{m;b}f\|_{L^2}^2 \le Cm^{-2+2\beta}\|b\|_{Lip}^2\|f\|_{L^2}^2, \tag{3.4}$$

then from (3.3) and (3.4), we get immediately the conclusion of Theorem 1.1. Hence, it remains to show (3.4) to prove Theorem 1.1.

Let $\psi\in C_c^\infty(\mathbf{R}^n)$ be a radial function such that $0\le\psi\le 1$, $\operatorname{supp}\psi\subset\{\xi: 1/2\le\xi\le 2\}$ and $\sum_{l\in\mathbf{Z}}\psi^2(2^{-l}\xi)=1$ for $|\xi|\ne 0$. Define the multiplier S_l by $\widehat{S_l f}(\xi)=\psi(2^{-l}\xi)\hat{f}(\xi)$. Let

$$\sigma_{k,t,m,j}(x) = (2^k t)^{-2}\frac{Y_{m,j}(x')}{|x|^{n-1}}\chi_{\{x:0<|x|\le 2^k t\}}(x)$$

for $k\in\mathbf{Z}$, $m=1,2,\dots,$ and $j=1,\dots,d_m$. Set

$$\Gamma_{k,t,m,j}(\xi)=\widehat{\sigma_{k,t,m,j}}(\xi),\quad \Gamma^l_{k,t,m,j}(\xi)=\Gamma_{k,t,m,j}(\xi)\psi(2^{k-l}\xi).$$

Denote by $F_{k,t,m,j}$ the convolution operator whose kernel is $\Gamma_{k,t,m,j}$ and $[b,F_{k,t,m,j}]$ is the commutator of $F_{k,t,m,j}$. Define the operator $F^l_{k,t,m,j}$ by

$$\widehat{F^l_{k,t,m,j}f}(\xi)=\Gamma^l_{k,t,m,j}(\xi)\hat{f}(\xi).$$

Let $[b,F^l_{k,t,m,j}]$ denote the commutator of $F^l_{k,t,m,j}$. Then

$$\begin{aligned}
\mu_{m,j;b}f(x) &= \left(\sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \frac{1}{t^2} \int_{|x-y|\le t} \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) f(y)\, dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \left(\sum_{k\in\mathbf{Z}} \int_{1}^{2} \left| \frac{1}{(2^k t)^2} \int_{|x-y|\le 2^k t} \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) f(y)\, dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \left(\int_1^2 \sum_{k\in\mathbf{Z}} |[b, F_{k,t,m,j}]f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
&= \left(\int_1^2 \sum_{k\in\mathbf{Z}} \left| \sum_{l\in\mathbf{Z}} [b, F_{k,t,m,j} S_{l-k}^2] f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.
\end{aligned}$$

So we get

$$\begin{aligned}
\|\mu_{m;b}f\|_{L^2}^2 &= \sum_{j=1}^{d_m} \int_{\mathbf{R}^n} |\mu_{m,j;b}f(x)|^2\, dx \\
&= \sum_{j=1}^{d_m} \int_{\mathbf{R}^n} \int_1^2 \sum_{k\in\mathbf{Z}} \left| \sum_{l\in\mathbf{Z}} [b, F_{k,t,m,j} S_{l-k}^2] f(x) \right|^2 \frac{dt}{t}\, dx.
\end{aligned}$$

By Minkowski's inequality, we have

$$\|\mu_{m;b}f\|_{L^2} \le \sum_{l\in\mathbf{Z}} \left(\int_1^2 \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} |[b, F_{k,t,m,j} S_{l-k}^2] f(x)|^2\, dx \frac{dt}{t} \right)^{1/2}. \tag{3.5}$$

It is easy to see that

$$[b, F_{k,t,m,j} S_{l-k}^2] f(x) = [b, F_{k,t,m,j}^l] S_{l-k} f(x) + F_{k,t,m,j}^l [b, S_{l-k}] f(x),$$

then

$$\begin{aligned}
&\int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} |[b, F_{k,t,m,j} S_{l-k}^2] f(x)|^2\, dx \\
&\quad\le \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} \left[\left(\int_{\mathbf{R}^n} |[b, F_{k,t,m,j}^l] S_{l-k} f(x)|^2\, dx \right)^{1/2} \right. \\
&\qquad\qquad \left. + \left(\int_{\mathbf{R}^n} |F_{k,t,m,j}^l [b, S_{l-k}] f(x)|^2\, dx \right)^{1/2} \right]^2 \\
&\quad= \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} (\|[b, F_{k,t,m,j}^l] S_{l-k} f\|_{L^2} + \|F_{k,t,m,j}^l [b, S_{l-k}] f\|_{L^2})^2.
\end{aligned}$$

Thus

$$\left(\int_1^2 \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} |[b, F_{k,t,m,j} S_{l-k}^2] f(x)|^2 \, dx \frac{dt}{t}\right)^{1/2}$$
$$\leq \left(\int_1^2 \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} \|[b, F_{k,t,m,j}^l] S_{l-k} f\|_{L^2}^2 \frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_1^2 \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} \|F_{k,t,m,j}^l [b, S_{l-k}] f\|_{L^2}^2 \frac{dt}{t}\right)^{1/2}$$
$$:= I + II.$$

If we can show that for $0 < \beta < (1-\varepsilon)/2$, there exists a constant $0 < \gamma < 1$ such that

$$\max\{I, II\} \leq C m^{-1+\beta} 2^{-\gamma|l|} \|b\|_{Lip} \|f\|_{L^2}, \tag{3.6}$$

then (3.4) follows. So it remains to show (3.6) to prove Theorem 1.1. Let

$$F_{k,t,m}^l f(x) = \left(\sum_{j=1}^{d_m} |F_{k,t,m,j}^l f(x)|^2\right)^{1/2}$$

and

$$F_{k,t,m;b}^l f(x) = \left(\sum_{j=1}^{d_m} |[b, F_{k,t,m,j}^l] f(x)|^2\right)^{1/2}.$$

The proof of (3.6) needs the following fact: for $t \in [1,2]$, $0 < \beta < (1-\varepsilon)/2$, there exists a constant $0 < \theta < 1$ such that

$$\|F_{k,t,m}^l f\|_{L^2} \leq C 2^{-k} m^{-1+\beta} \min\{2^{2l}, 2^{-\beta l/2}\} \|f\|_{L^2}, \tag{3.7}$$

and

$$\|F_{k,t,m;b}^l f\|_{L^2} \leq C m^{-1+\beta} 2^{-\theta|l|} \|b\|_{Lip} \|f\|_{L^2}. \tag{3.8}$$

First we prove (3.7). In fact, for any fixed constant $0 < \beta < 1$, by Lemma 2.2(2.1), we have for $t \in [1,2]$,

$$\begin{aligned} |\Gamma_{k,t,m,j}^l(\xi)| &= |\Gamma_{k,t,m,j}(\xi) \psi(2^{k-l}\xi)| \\ &\leq C 2^{-k} m^{-\lambda-1+\beta/2} \min\{2^{2l}, 2^{-\beta l/2}\} |Y_{m,j}(\xi')|. \end{aligned} \tag{3.9}$$

From this, the Plancherel theorem and the fact $\sum_{j=1}^{d_m}|Y_{m,j}(x')|^2 \sim m^{2\lambda}$, (see [4, p. 225]), we get

$$\begin{aligned}
\|F^l_{k,t,m}f\|_{L^2} &= \left(\sum_{j=1}^{d_m}\|F^l_{k,t,m,j}f\|^2_{L^2}\right)^{1/2}\\
&= \left(\sum_{j=1}^{d_m}\int_{\mathbf{R}^n}|\Gamma^l_{k,t,m,j}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi\right)^{1/2}\\
&\le C2^{-k}m^{-1+\beta/2}\min\{2^{2l},2^{-\beta l/2}\}\|\hat{f}\|_{L^2}\\
&= C2^{-k}m^{-1+\beta}\min\{2^{2l},2^{-\beta l/2}\}\|f\|_{L^2}.
\end{aligned}$$

This gives the proof of (3.7). To show (3.8), we define operator $\tilde{F}^l_{k,t,m,j}$ by $\widehat{\tilde{F}^l_{k,t,m,j}f}(\xi) = \Gamma^l_{k,t,m,j}(2^{-k}\xi)\hat{f}(\xi)$, and denote by

$$\tilde{F}^l_{k,t,m;b}f(x) = \left(\sum_{j=1}^{d_m}|[b,\tilde{F}^l_{k,t,m,j}]f(x)|^2\right)^{1/2}.$$

Then we have supp $\Gamma^l_{k,t,m,j}(2^{-k}\cdot) \subset \{\xi : 2^{l-1} \le |\xi| \le 2^{l+1}\}$, and by (3.9), we obtain

$$\begin{aligned}
|\Gamma^l_{k,t,m,j}(2^{-k}\xi)| &\le C2^{-k}m^{-\lambda-1+\beta/2}\min\{2^{2l},2^{-\beta l/2}\}|Y_{m,j}(\xi')|\\
&\le C2^{-k}m^{-\lambda-1+\beta/2}\min\{2^{l},2^{-\beta l/2}\}|Y_{m,j}(\xi')|.
\end{aligned}\tag{3.10}$$

On the other hand, for any multi-index α, we have

$$\begin{aligned}
\partial^\alpha\Gamma^l_{k,t,m,j}(\xi) &= \partial^\alpha(\Gamma_{k,t,m,j}(\xi)\psi(2^{k-l}\xi))\\
&= \sum_\eta C^{\alpha_1}_{\eta_1}\cdots C^{\alpha_n}_{\eta_n}(\partial^\eta\Gamma_{k,t,m,j}(\xi))(\partial^{\alpha-\eta}\psi(2^{k-l}\xi)),
\end{aligned}$$

where the sum is taken over all multiindexes η with $0 \le \eta_j \le \alpha_j$ for $1 \le j \le n$. We take α with $|\alpha| > \frac{2\gamma}{\beta}(\lambda+1-\beta/2)$. If $m > |\alpha|$, by Lemma 2.2(2.2), we have for $t \in [1,2]$

$$\begin{aligned}
|\partial^\alpha\Gamma^l_{k,t,m,j}(\xi)| &\le C\sum_{0\le|\eta|\le|\alpha|}2^{(k-l)(|\alpha|-|\eta|)}|\partial^\eta\Gamma_{k,t,m,j}(\xi)|\\
&\le C\sum_{0\le|\eta|\le|\alpha|}2^{(k-l)(|\alpha|-|\eta|)}(2^kt)^{-1}(2^kt)^{|\eta|}\min\{|2^kt\xi|^{m-|\eta|},1\}\\
&\le C\sum_{0\le|\eta|\le|\alpha|}2^{(k-l)(|\alpha|-|\eta|)}2^{-k}2^{k|\eta|}\min\{|2^k\xi|^{m-|\eta|},1\}\\
&\le C\sum_{0\le|\eta|\le|\alpha|}2^{-k}2^{|\alpha|k}2^{-l(|\alpha|-|\eta|)}\min\{|2^k\xi|^{m-|\eta|},1\}.
\end{aligned}$$

Then we obtain

$$
\begin{aligned}
(3.11)\qquad |\partial^\alpha(\Gamma^l_{k,t,m,j}(2^{-k}\xi))| &\le C \sum_{0\le|\eta|\le|\alpha|} 2^{-k|\alpha|}2^{-k}2^{|\alpha|k}2^{-l(|\alpha|-|\eta|)}\min\{2^{(m-|\eta|)l},1\} \\
&\le C \sum_{0\le|\eta|\le|\alpha|} 2^{-k}2^{-l(|\alpha|-|\eta|)}\min\{2^{(m-|\eta|)l},1\} \\
&\le C2^{-k}
\end{aligned}
$$

Since

$$\int_{S^{n-1}} Y_{m,j}(x')\,Y_s(x')\,d\sigma(x') = 0,$$

for any spherical harmonic polynomials Y_s with degree $< m$, we have

$$\int_{S^{n-1}} Y_{m,j}(x')x'^{\ell}\,d\sigma(x') = 0,$$

for any multi-index ℓ with $|\ell| < m$. From this, by [26, p. 551], if $2 \le m \le |\alpha|$, we have for any multi-index η with $|\eta| \le 2$,

$$
\begin{aligned}
|\partial^\eta\Gamma_{k,t,m,j}(\xi)| &\le C\left|(2^k t)^{-2}\int_0^{2^k t}\int_{S^{n-1}} Y_{m,j}(y')\,y'^{\eta}e^{-2\pi i\xi\cdot ry'}\,d\sigma(y')r^{|\eta|}\,dr\right| \\
&\le C2^{k(|\eta|-1)}|2^k\xi|^{2-|\eta|}.
\end{aligned}
$$

Then, similar to (3.11), for any multi-index τ with $|\tau| = 2$, we have

$$
\begin{aligned}
(3.12)\qquad |\partial^\tau\Gamma^l_{k,t,m,j}(2^{-k}\xi)| &\le C \sum_{0\le|\eta|\le|\tau|} 2^{-l(|\tau|-|\eta|)}|\partial^\eta\Gamma_{k,m,j}(2^{-k}\xi)| \\
&\le C \sum_{0\le|\eta|\le|\tau|} 2^{-|\eta|k}2^{-l(|\tau|-|\eta|)}2^{k(|\eta|-1)}|\xi|^{2-|\eta|} \le C2^{-k}.
\end{aligned}
$$

From (3.10)–(3.12), using Lemma 2.3 with $\delta = 2^l$, for some $0 < v_1, v_2 < 1$, we get

$$
\|\tilde{F}^l_{k,t,m;b}h\|_{L^2} \le \begin{cases} C2^{-k}m^{-1+\beta}\min\{2^{v_1 l}, 2^{-\beta v_1 l/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m > |\alpha|, \\ C2^{-k}\min\{2^{v_2 l}, 2^{-\beta v_2 l/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m \le |\alpha|. \end{cases}
$$

By dilation and $\|b(2^k\cdot)\|_{Lip} = 2^k\|b\|_{Lip}$, we obtain

$$
\|F^l_{k,t,m;b}h\|_{L^2} \le \begin{cases} Cm^{-1+\beta}\min\{2^{v_1 l}, 2^{-\beta v_1 l/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m > |\alpha|, \\ C\min\{2^{v_2 l}, 2^{-\beta v_2 l/2}\}\|b\|_{Lip}\|f\|_{L^2}, & m \le |\alpha|. \end{cases}
$$

Hence the proof of (3.8) is completed. □

Now we turn our attention to (3.6). Applying (3.8) and Littlewood-Paley theory, we conclude that

$$\begin{aligned}
I^2 &= \int_1^2 \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} \|[b, F^l_{k,t,m,j}]S_{l-k}f\|_{L^2}^2 \frac{dt}{t} \\
&= \int_1^2 \sum_{k\in\mathbf{Z}} \|F^l_{k,t,m;b}S_{l-k}f\|_{L^2}^2 \frac{dt}{t} \\
&\le Cm^{-2+2\beta}2^{-2\theta|l|}\|b\|_{Lip}^2 \sum_{k\in\mathbf{Z}} \|S_{l-k}f\|_{L^2}^2 \\
&\le Cm^{-2+2\beta}2^{-2\theta|l|}\|b\|_{Lip}^2 \|f\|_{L^2}^2.
\end{aligned}$$

From (3.7), Littlewood-Paley theory and Lemma 2.4, we derive

$$\begin{aligned}
II^2 &= \int_1^2 \sum_{k\in\mathbf{Z}} \sum_{j=1}^{d_m} \|F^l_{k,t,m,j}[b, S_{l-k}]f\|_{L^2}^2 \frac{dt}{t} \\
&= \int_1^2 \sum_{k\in\mathbf{Z}} \|F^l_{k,t,m}[b, S_{l-k}]f\|_{L^2}^2 \frac{dt}{t} \\
&\le Cm^{-2+2\beta} \min\{2^{4l}, 2^{-\beta l}\} \sum_{k\in\mathbf{Z}} 2^{-2k}\|[b, S_{l-k}]f\|_{L^2}^2 \\
&= Cm^{-2+2\beta} \min\{2^{2l}, 2^{-(\beta+2)l}\} \sum_{k\in\mathbf{Z}} \|2^{l-k}[b, S_{l-k}]f\|_{L^2}^2 \\
&\le Cm^{-2+2\beta} \min\{2^{2l}, 2^{-(\beta+2)l}\}\|b\|_{Lip}^2 \|f\|_{L^2}^2.
\end{aligned}$$

Combining the estimates of I and II gives (3.6). Hence Theorem 1.1 is proved. □

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