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EXTRINSIC HOMOGENEOUS ALMOST HERMITIAN 6-DIMENSIONAL SUBMANIFOLDS IN THE OCTONIONS

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1. Introduction

In ([Br1]), R. L. Bryant showed that any oriented 6-dimensional submanifold $\varphi : M^6 \to \mathfrak{C}$ of the octonions $\mathfrak{C}$ admits the almost complex (Hermitian) structure J defined by

$$\varphi_*(JX) = \varphi_*(X)(\eta \times \xi),$$

where $\{\xi, \eta\}$ is a local oriented orthonormal frame field of the normal bundle of φ over a neighborhood of each point of M^6. The induced almost complex (Hermitian) structure is a $Spin(7)$-invariant in the following sense.

Let $\varphi_1, \varphi_2 : M^6 \to \mathfrak{C}$ be two isometric immersions from the same source manifold to the octonions. If there exists an element $g \in Spin(7)$ such that $g \circ \varphi_1 = \varphi_2$ (up to a parallel translation), then the two maps are said to be $Spin(7)$-congruent. If the immersions φ_1 and φ_2 are $Spin(7)$-congruent, then the induced almost complex structures coincide.

We shall give a classification (Theorem 5.1) of 6-dimensional extrinsic homogeneous almost Hermitian submanifolds of the octonions $\mathfrak{C}$ by making use of the classification of the homogeneous isoparametric hypersurfaces of a unit sphere ([HsL], [TT]), and also introduce a list of 6-dimensional submanifolds of $\mathfrak{C}$ which are Riemannian homogeneous but not homogeneous with respect to the induced almost complex structure (§6).

2. Preliminaries

Let $\mathbf{H}$ be the skew field of all quaternions with canonical basis $\{1, i, j, k\}$, which satisfies

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The octonions (or Cayley algebra) $\mathfrak{C}$ over $\mathbf{R}$ can be considered as a direct sum $\mathbf{H} \oplus \mathbf{H} = \mathfrak{C}$ with the following multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac - \bar{d}b + (da + b\bar{c})\varepsilon,$$

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where $\varepsilon = (0,1) \in \mathbf{H} \oplus \mathbf{H}$ and $a, b, c, d \in \mathbf{H}$, where the symbol "$^-$" denotes the conjugation of the quaternion. For any $x, y \in \mathfrak{C}$, we have

$$\langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle,$$

which is called "normed algebra" in ([H-L]). The octonions is a non-commutative, non-associative alternative division algebra. The group of automorphisms of the octonions is the exceptional simple Lie Group

$$G_2 = \{g \in SO(8) \mid g(uv) = g(u)g(v) \text{ for any } u, v \in \mathfrak{C}\}.$$

In this paper, we shall concern the Lie group $Spin(7)$ which is defined by

$$Spin(7) = \{g \in SO(8) \mid g(uv) = g(u)\chi_g(v) \text{ for any } u, v \in \mathfrak{C}\},$$

where $\chi_g(v) = g(g^{-1}(1)v)$. Note that G_2 is a Lie subgroup of $Spin(7)$:

$$G_2 = \{g \in Spin(7) \mid g(1) = 1\}.$$

The map χ defines a double covering map from $Spin(7)$ onto $SO(7)$, which satisfies the following equivariance

$$g(u) \times g(v) = \chi_g(u \times v),$$

for any $u, v \in \mathfrak{C}$, where $u \times v = (1/2)(\bar{v}u - \bar{u}v)$ (which is called the "exterior product") where $\bar{v} = 2\langle v, 1 \rangle - v$ is the conjugation of $v \in \mathfrak{C}$. We note that $u \times v$ is pure-imaginary for any $u, v \in \mathfrak{C}$.

The Lie algebra $\mathfrak{g}_2$ is the subalgebra of $\mathfrak{so}(7)$ whose basis is given by

$$\tag{2.1} \begin{cases} aG_{23} + bG_{45} + cG_{76}, \\ aG_{31} + bG_{46} + cG_{57}, \\ aG_{12} + bG_{47} + cG_{65}, \\ aG_{51} + bG_{73} + cG_{62}, \\ aG_{14} + bG_{72} + cG_{36}, \\ aG_{17} + bG_{24} + cG_{53}, \\ aG_{61} + bG_{34} + cG_{25}, \end{cases}$$

where $a, b, c \in \mathbf{R}$ with $a + b + c = 0$, and $G_{ij}(e_k) = \delta_{jk}e_i - \delta_{ik}e_j$. The Lie algebra $\mathfrak{spin}(7)$ is the subalgebra of $\mathfrak{so}(8)$ whose basis is given by

$$\tag{2.2} \begin{cases} dG_{10} + aG_{23} + bG_{45} + cG_{76}, \\ dG_{20} + aG_{31} + bG_{46} + cG_{57}, \\ dG_{30} + aG_{12} + bG_{47} + cG_{65}, \\ dG_{40} + aG_{51} + bG_{73} + cG_{62}, \\ dG_{50} + aG_{14} + bG_{72} + cG_{36}, \\ dG_{60} + aG_{17} + bG_{24} + cG_{53}, \\ dG_{70} + aG_{61} + bG_{34} + cG_{25}, \end{cases}$$

where $a, b, c, d \in \mathbf{R}$ with $a + b + c + d = 0$.

2.1. *Spin*(7)-structure equations

In this section, we shall recall the structure equation of *Spin*(7) which was established by R. Bryant ([Br1]). To do this, we fix a basis of the complexification of the octonions $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$ over $\mathbf{C}$ given by

$$N = (1/2)(1 - \sqrt{-1}\varepsilon),\ \overline{N} = (1/2)(1 + \sqrt{-1}\varepsilon),$$
$$E_1 = iN,\ E_2 = jN,\ E_3 = -kN,\ \overline{E}_1 = i\overline{N},\ \overline{E}_2 = j\overline{N},\ \overline{E}_3 = -k\overline{N}.$$

We extend the multiplication of the octonions complex linearly on $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$. Then we have the following multiplication table;

$A\backslash B$	N	E_1	E_2	E_3	$\overline{N}$	$\overline{E}_1$	$\overline{E}_2$	$\overline{E}_3$
N	N	0	0	0	0	$\overline{E}_1$	$\overline{E}_2$	$\overline{E}_3$
E_1	E_1	0	$-\overline{E}_3$	$\overline{E}_2$	0	$-\overline{N}$	0	0
E_2	E_2	$\overline{E}_3$	0	$-\overline{E}_1$	0	0	$-\overline{N}$	0
E_3	E_3	$-\overline{E}_2$	$\overline{E}_1$	0	0	0	0	$-\overline{N}$
$\overline{N}$	0	E_1	E_2	E_3	$\overline{N}$	0	0	0
$\overline{E}_1$	0	$-N$	0	0	$\overline{E}_1$	0	$-E_3$	E_2
$\overline{E}_2$	0	0	$-N$	0	$\overline{E}_2$	E_3	0	$-E_1$
$\overline{E}_3$	0	0	0	$-N$	$\overline{E}_3$	$-E_2$	E_1	0

We define a $\mathfrak{C} \rtimes Spin(7)$ admissible frame field as follows. Let o be the origin of the octonions. The Lie group $\mathfrak{C} \rtimes Spin(7)$ acts on $\mathfrak{C} \oplus End(\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C})$ as follows

$$\begin{aligned}(x,g)(o;N,E,\overline{N},\overline{E}) &= (g \cdot o + x, g(N), g(E), g(\overline{N}), g(\overline{E})) \\ &= (x, g(N), g(E), g(\overline{N}), g(\overline{E})) \\ &= (o;N,E,\overline{N},\overline{E})\begin{pmatrix} 1 & 0_{1\times 8} \\ \rho(x) & \rho(g) \end{pmatrix},\end{aligned}$$

where $(x,g) \in \mathfrak{C} \rtimes Spin(7)$ and $\begin{pmatrix} 1 & 0_{1\times 8} \\ \rho(x) & \rho(g) \end{pmatrix}$ is its matrix representation. A frame $(x;n,f,\bar{n},\bar{f})$ is said to be a $\mathfrak{C} \rtimes Spin(7)$ admissible frame if there exists an element $(x,g) \in \mathfrak{C} \rtimes Spin(7)$ such that

$$(x;n,f,\bar{n},\bar{f}) = (x,g)(o;N,E,\overline{N},\overline{E}).$$

PROPOSITION 2.1 ([Br1]). *The Maurer-Cartan form of* $\mathfrak{C} \rtimes Spin(7)$ *is given by*

$$d(x;n,f,\bar{n},\bar{f}) = (x;n,f,\bar{n},\bar{f})\left(\begin{array}{c|cc|cc} 0 & 0 & 0_{1\times 3} & 0 & 0_{1\times 3} \\ \hline v & \sqrt{-1}\rho & -{}^t\bar{\mathfrak{h}} & 0 & -{}^t\theta \\ \omega & \mathfrak{h} & \kappa & \theta & [\bar{\theta}] \\ \hline \bar{v} & 0 & -{}^t\bar{\theta} & -\sqrt{-1}\rho & -{}^t\mathfrak{h} \\ \bar{\omega} & \bar{\theta} & [\theta] & \bar{\mathfrak{h}} & \bar{\kappa} \end{array}\right)$$

$$= (x;n,f,\bar{n},\bar{f})\psi,$$

where ψ *is a* $\mathfrak{spin}(7) \oplus \mathfrak{C}(\subset M_{9\times 9}(\boldsymbol{C}))$*-valued* 1*-form,* ρ *is a real-valued* 1*-form,* v *is a complex valued* 1*-form,* ω, $\mathfrak{h}$, θ *are* $M_{3\times 1}$*-valued* 1*-forms,* κ *is a* $\mathfrak{u}(3)$*-valued* 1*-form which satisfy* $\sqrt{-1}\rho + \operatorname{tr}\kappa = 0$, *and*

$$[\theta] = \begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix},$$

for $\theta = {}^t(\theta^1, \theta^2, \theta^3)$. *The* 1*-form* ψ *satisfies the following integrability condition* $d\psi + \psi \wedge \psi = 0$. *More precisely*

$$dx = (n,f,\bar{n},\bar{f})\begin{pmatrix} v \\ \omega \\ \bar{v} \\ \bar{\omega} \end{pmatrix},$$

$$dn = n\sqrt{-1}\rho + f\mathfrak{h} + \bar{f}\bar{\theta},$$

$$df = n(-{}^t\bar{\mathfrak{h}}) + f\kappa + \bar{n}(-{}^t\bar{\theta}) + \bar{f}[\theta],$$

and the integrability conditions are given by

$$dv = \sqrt{-1}\rho \wedge v + {}^t\bar{\mathfrak{h}} \wedge \omega + {}^t\bar{\theta} \wedge \bar{\omega},$$

$$d\omega = -\mathfrak{h} \wedge v - \kappa \wedge \omega - \theta \wedge \bar{v} - [\theta] \wedge \bar{\omega},$$

$$d(\sqrt{-1}\rho) = {}^t\bar{\mathfrak{h}} \wedge \mathfrak{h} + {}^t\theta \wedge \bar{\theta},$$

$$d\mathfrak{h} = -\mathfrak{h} \wedge \sqrt{-1}\rho - \kappa \wedge \mathfrak{h} - [\bar{\theta}] \wedge \bar{\theta},$$

$$d\theta = \theta \wedge \sqrt{-1}\rho - \kappa \wedge \theta - [\bar{\theta}] \wedge \bar{\mathfrak{h}},$$

$$d\kappa = \mathfrak{h} \wedge {}^t\bar{\mathfrak{h}} - \kappa \wedge \kappa + \theta \wedge {}^t\bar{\theta} - [\bar{\theta}] \wedge [\theta].$$

3. Gram-Schmidt construction of *Spin*(7)-frame fields

In order to construct a *Spin*(7)-frame field, we first recall the Gram-Schmidt construction of a G_2-frame.

LEMMA 3.1. *For a pair of mutually orthogonal unit vectors e_1, e_4 in $\mathfrak{C}_0$ put $e_5 = e_1 e_4$. Take a unit vector e_2, which is perpendicular to e_1, e_4 and e_5. If we put $e_3 = e_1 e_2$, $e_6 = e_2 e_4$ and $e_7 = e_3 e_4$ then the matrix*

$$g = [e_1, e_2, e_3, e_4, e_5, e_6, e_7] \in SO(7),$$

is an element of G_2.

From Lemma 3.1, by taking $e_4 = \eta \times \xi$, we obtain a G_2-frame field given by

$$\begin{aligned} N^* &= (1/2)(1 - \sqrt{-1}e_4), & \bar{N}^* &= (1/2)(1 + \sqrt{-1}e_4), \\ E_1^* &= (1/2)(e_1 - \sqrt{-1}e_5), & \bar{E}_1^* &= (1/2)(e_1 + \sqrt{-1}e_5), \\ E_2^* &= (1/2)(e_2 - \sqrt{-1}e_6), & \bar{E}_2^* &= (1/2)(e_2 + \sqrt{-1}e_6), \\ E_3^* &= -(1/2)(e_3 - \sqrt{-1}e_7), & \bar{E}_3^* &= -(1/2)(e_3 + \sqrt{-1}e_7). \end{aligned}$$

Then we see that $span_{\mathbf{C}}\{N^*, E_1^*, E_2^*, E_3^*\}$ is a $\sqrt{-1}$-eigenspace $T_p^{(1,0)}\mathfrak{C}(\subset \mathfrak{C} \otimes \mathbf{C})$ with respect to the almost complex structure $J = R_{\eta\times\xi}$ at $p \in \mathfrak{C}$. On the other hand, $n = (1/2)(\xi - \sqrt{-1}\eta)$ is a local orthonormal frame field of the complexified normal bundle $T^{\perp(1,0)}M$. Since $T_{\varphi(m)}^{\perp(1,0)}M \subset T_{\varphi(m)}^{(1,0)}\mathfrak{C}$, there exists a $M_{4\times 1}(\mathbf{C})$-valued function $a_1 = {}^t(a_{11}, a_{21}, a_{31}, a_{41})$, such that

$$n = (1/2)(\xi - \sqrt{-1}\eta) = (N^*, E_1^*, E_2^*, E_3^*)a_1.$$

By applying the Gram-Schmidt orthonormalization to the pair $\{n, a_1\}$ with respect to the Hermitian inner product of $T_{\varphi(m)}^{(1,0)}\mathfrak{C}$, we may obtain three $M_{4\times 1}(\mathbf{C})$-valued functions $\{a_2, a_3, a_4\}$ such that $\{a_1, a_2, a_3, a_4\}$ is a special unitary frame. We set

$$f_i = (N^*, E_1^*, E_2^*, E_3^*)a_{i+1},$$

for $i = 1, 2, 3$, then

$$(n, f, \bar{n}, \bar{f}) = (n, f_1, f_2, f_3, \bar{n}, \bar{f}_1, \bar{f}_2, \bar{f}_3),$$

is a (local) $Spin(7)$-frame field on M.

Remark 3.1. The above procedure comes from the following relation

$$Spin(7)/Spin(6) = Spin(7)/SU(4) = S^6 \cong G_2/SU(3).$$

4. *Spin*(7) invariants

We shall recall the invariants of $Spin(7)$-congruence classes for 6-dimensional submanifolds (M, φ) in $\mathfrak{C}$. By Proposition 2.1, we have

PROPOSITION 4.1 ([Br1]). *Let $\varphi : M \to \mathfrak{C}$ be an isometric immersion from an oriented 6-dimensional manifold to the octonions. Then*

$$d\varphi = f\omega + \bar{f}\bar{\omega}, \tag{4.1}$$

$$\nu = 0, \tag{4.2}$$

$$dn = n\sqrt{-1}\rho + f\mathfrak{h} + \bar{f}\bar{\theta}, \tag{4.3}$$

$$df = n(-{}^t\bar{\mathfrak{h}}) + f\kappa + n(-{}^t\bar{\theta}) + \bar{f}[\theta], \tag{4.4}$$

and the integrability conditions imply that

$$d\omega = -\kappa \wedge \omega - [\theta] \wedge \bar{\omega}, \tag{4.5}$$

$$d(\sqrt{-1}\rho) = {}^t\bar{\mathfrak{h}} \wedge \mathfrak{h} + {}^t\theta \wedge \bar{\theta}, \tag{4.6}$$

$$d\mathfrak{h} = -\mathfrak{h} \wedge \sqrt{-1}\rho - \kappa \wedge \mathfrak{h} - [\bar{\theta}] \wedge \bar{\theta}, \tag{4.7}$$

$$d\theta = \theta \wedge \sqrt{-1}\rho - \kappa \wedge \theta - [\bar{\theta}] \wedge \bar{\mathfrak{h}}, \tag{4.8}$$

$$d\kappa = \mathfrak{h} \wedge {}^t\bar{\mathfrak{h}} - \kappa \wedge \kappa + \theta \wedge {}^t\bar{\theta} - [\bar{\theta}] \wedge [\theta]. \tag{4.9}$$

The second fundamental form II is given by

$$\mathrm{II} = -2\,\mathrm{Re}\{({}^t\bar{\mathfrak{h}} \circ \omega + {}^t\bar{\theta} \circ \bar{\omega}) \otimes n\},$$

where the symbol "$\circ$" is the symmetric tensor product. By Cartan's Lemma (since $\nu = 0$), there exist $M_{3\times 3}$-valued matrices A, B, C such that

$$\begin{pmatrix} \mathfrak{h} \\ \theta \end{pmatrix} = \begin{pmatrix} \bar{B} & \bar{A} \\ {}^tB & \bar{C} \end{pmatrix} \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}, \tag{4.10}$$

where ${}^tA = A$ and ${}^tC = C$. We have the following decomposition

$$\mathrm{II}^{(2,0)} = (-{}^t\omega \circ A\omega) \otimes n,$$

$$\mathrm{II}^{(1,1)} = (-{}^t\bar{\omega} \circ {}^tB\omega - {}^t\omega \circ B\bar{\omega}) \otimes n,$$

$$\mathrm{II}^{(0,2)} = (-{}^t\bar{\omega} \circ \bar{C}\bar{\omega}) \otimes n.$$

We shall write each elements more explicitly. There exists a unitary frame $\{e_i, Je_i\}$ for $i = 1, 2, 3$, such that

$$n = (1/2)(\xi - \sqrt{-1}\eta), \quad f_i = (1/2)(e_i - \sqrt{-1}Je_i).$$

Thus the elements of the second fundamental form are given by

$$A_{ij} = -2\langle \mathrm{II}(f_i, f_j), \bar{n}\rangle,$$

$$B_{ij} = -2\langle \mathrm{II}(f_i, \bar{f}_j), \bar{n}\rangle,$$

$$C_{ij} = -2\langle \mathrm{II}(\bar{f}_i, \bar{f}_j), \bar{n}\rangle.$$

We shall recall the relation of Ricci $*$-tensor ρ^* and $*$-scalar curvature τ^* which are fundamental invariants of almost Hermitian geometry. The Ricci $*$-tensor and $*$-scalar curvature of an almost Hermitian manifold $M = (M, J, \langle\, ,\rangle)$ of dimension $2n$, are defined by

$$\rho^*(x, y) = \frac{1}{2}\sum_{i=1}^{2n} \langle R(e_i, Je_i)Jy, x\rangle$$

and

$$\tau^* = \sum_{i=1}^{2n} \rho^*(e_i, e_i),$$

respectively. Note that Ricci $*$-tensor is neither symmetric nor skew-symmetric, in general.

Proposition 4.2 ([H2]). *The Ricci $*$-tensor and $*$-scalar curvature of an oriented 6-dimensional submanifold in $\mathfrak{C}$ are given by*

$$\begin{aligned}\rho^*(x, y) &= {}^t\alpha(A\bar{B} - BC - {}^t(A\bar{B} - BC))\beta \\ &\qquad - {}^t\alpha(A\bar{A} - B\,{}^t\bar{B} - {}^t\bar{B}B + C\bar{C})\bar{\beta} + \textit{its conjugation} \\ \tau^* &= -4(\mathrm{tr}\, A\bar{A} - 2\,\mathrm{tr}\,{}^t\bar{B}B + \mathrm{tr}\, C\bar{C}),\end{aligned}$$

where $x = f\alpha + \bar{f}\bar{\alpha}$, $y = f\beta + \bar{f}\bar{\beta}$ and $\alpha, \beta \in M_{3\times 1}(\boldsymbol{C})$.

4.2. *Spin*(7)-congruence theorem

In this section, we shall give an equivalent condition for two isometric immersions to be $Spin(7)$-congruent. Namely, we have the following.

Proposition 4.3. *Let M^6 be a connected, oriented 6-dimensional manifold and $\varphi_1, \varphi_2 : M^6 \to \mathfrak{C}$ be two isometric immersions with same induced metrics and almost complex structures. Let $\mathrm{II}^{(2,0)}_{\varphi_1}$, $\mathrm{II}^{(2,0)}_{\varphi_2}$ be the corresponding $(2,0)$ part of the 2nd fundamental forms. Then there exists an element $g \in Spin(7)$ such that $g \circ \varphi_1 = \varphi_2$ if and only if $\mathrm{II}^{(2,0)}_{\varphi_1} \cong \mathrm{II}^{(2,0)}_{\varphi_2}$.*

Proof. By (4.1) in Proposition 4.1, ω, $\bar{\omega}$ are determined by the induced almost Hermitian structure. We may check that ρ, $\mathfrak{h}$, and θ depend on ω, $\bar{\omega}$ and $\mathrm{II}^{(2,0)}$. By (4.4), κ and θ depend only on the unitary frame f, $\bar{f}$, df and $d\bar{f}$. Hence they depend only on the induced almost Hermitian structure. By (4.10), B and C are also. If we fix $\mathrm{II}^{(2,0)}$, we get the desired complete information of the immersion. q.e.d

5. Orbits of the isotropy representations

In this paper, we shall determine the 6-dimensional *extrinsic* homogeneous almost Hermitian submanifolds of the octonions. The terminology of *extrinsic* *homogeneous* for submanifolds of $\mathfrak{C}$ means that a submanifold is obtained as the orbit (at some point) of a Lie subgroup of $\mathbf{R}^8 \rtimes Spin(7)$. In this case, we see that the index of relative nullity is constant on such a homogeneous submanifold. We note that the notion of the index of relative nullity is not intrinsic one (see [KN]).

Let (M^m, φ) be a homogeneous submanifold in an n-dimensional Euclidean space $\mathbf{R}^n$, and let k be the index of relative nullity of (M^m, φ). Taking account of completeness of M^m, we see that M^m admits the splitting $\varphi(M^m) = \mathbf{R}^k \times \phi(M^{m-k})$ and $M^m = \mathbf{R}^k \times M^{m-k}$, where ϕ is an isometric immersion $\phi : M^{m-k} \to \mathbf{R}^{n-k}$. In this splitting, we may assume that the image of ϕ is included in a sphere S^{n-k-1}, (and hence, $\varphi(M^m)$ is included in the generalized cylinder $\mathbf{R}^k \times S^{n-k-1} \subset \mathbf{R}^k \times \mathbf{R}^{n-k}$.

From the classification of ([HsL], [TT]) of homogeneous isoparametric hypersurfaces of a sphere, we see that the following 6-dimensional manifolds $M^6 = (M^6, \varphi)$ are extrinsic homogeneous almost Hermitian submanifolds of the octonions $\mathfrak{C}$ with respect to the almost Hermitian structure induced by the isometric immersion φ.

1. φ is totally geodesic. ($\varphi : \mathbf{R}^6 \to \mathfrak{C} \simeq \mathbf{R}^8$, flat Kähler manifold).
2. φ is totally umbilic in $\operatorname{Im}\mathfrak{C}$. ($\varphi : S^6 \to \operatorname{Im}\mathfrak{C} \simeq \mathbf{R}^7$, nearly Kähler 6-sphere).
3. $\varphi : S^1 \times \mathbf{R}^5 \to \operatorname{Im}\mathfrak{C} \simeq \mathbf{R}^7$ is defined by

$$
\begin{aligned}
&\varphi(e^{\sqrt{-1}\theta}, x_1, z_2, z_3) \\
&= (o; \varepsilon, E, \overline{E}) \left(\begin{array}{c|c|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
z_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
z_2 & 0 & 0 & e^{-\sqrt{-1}\theta} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\sqrt{-1}\theta} & 0 & 0 & 0 \\ \hline
\overline{z_1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\overline{z_2} & 0 & 0 & 0 & 0 & 0 & e^{\sqrt{-1}\theta} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\sqrt{-1}\theta}
\end{array}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \varepsilon x_1 + E_1 z_1 + E_2 z_2 + E_3 e^{\sqrt{-1}\theta} + \overline{E_1 z_1 + E_2 z_2 + E_3\, e^{\sqrt{-1}\theta}},
\end{aligned}
$$

for $(e^{i\theta}, x_1, z_1, z_2) \in S^1 \times \mathbf{R} \times \mathbf{C} \times \mathbf{C}$ $(\simeq S^1 \times \mathbf{R}^5)$

4. $\varphi : \mathbf{R}^1 \times \mathbf{R}^5 \to \mathfrak{C}$ is define by ($\gamma : \mathbf{R} \to \mathbf{R}^3$ is a helix)

$$
\begin{aligned}
&\varphi(t, x_0, x_4, x_5, x_6, x_7) \\
&\quad = 1x_0 + i\cos(at) + j\sin(at) + kbt + \varepsilon x_4 + i\varepsilon x_5 + j\varepsilon x_6 + k\varepsilon x_7 \\
&\quad = (o; N, E, \overline{N}, \overline{E}) A v_0,
\end{aligned}
$$

where $v_0 = {}^t(1\quad 0\quad 1\quad 0\quad 0\quad 0\quad 1\quad 0\quad 0)$ and

$$A = \left(\begin{array}{c|cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x_0 + \sqrt{-1}x_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{-1}x_5 & 0 & \cos(at) & -\sin(at) & 0 & 0 & 0 & 0 & 0 \\ \sqrt{-1}x_6 & 0 & \sin(at) & \cos(at) & 0 & 0 & 0 & 0 & 0 \\ -bt - \sqrt{-1}x_7 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline x_0 - \sqrt{-1}x_4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\sqrt{-1}x_5 & 0 & 0 & 0 & 0 & 0 & \cos(at) & -\sin(at) & 0 \\ -\sqrt{-1}x_6 & 0 & 0 & 0 & 0 & 0 & \sin(at) & \cos(at) & 0 \\ -bt + \sqrt{-1}x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

for $(t, x_0, x_4, x_5, x_6, x_7) \in \mathbf{R}^1 \times \mathbf{R}^5$.

5. $\varphi : S^2 \times \mathbf{R}^4 \to \operatorname{Im}\mathfrak{C}$ (quasi-Kähler manifold)

$$\varphi(q, x_1, x_2, x_3, x_4) = qi\bar{q} + \varepsilon x_1 + i\varepsilon x_2 + j\varepsilon x_3 + k\varepsilon x_4,$$

where $q \in Sp(1) \cong S^3 \subset \mathbf{H}$ and $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$.

6. $\varphi : S^3 \times \mathbf{R}^3 \to \operatorname{Im}\mathfrak{C}$ is defined by

$$\varphi(q, x_1, x_2, x_3) = ix_1 + jx_2 + kx_3 + q\varepsilon,$$

for $(q, x_1, x_2, x_3) \in S^3 \times \mathbf{R}^3$.

7. $\varphi : S^5 \times \mathbf{R}^1 \to \operatorname{Im}\mathfrak{C}$, is defined by

$$\begin{aligned}\varphi(x, z_1, z_2, z_3) &= (o; \varepsilon, E, \overline{E}) \begin{pmatrix} 1 & 0 & 0_{1\times 3} & 0_{1\times 3} \\ x & 1 & 0_{1\times 3} & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & U & 0_{3\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 3} & \overline{U} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \varepsilon x + \sum_{i=1}^{3} E_i z_i + \sum_{i=1}^{3} \overline{E_i z_i},\end{aligned}$$

where $p_0 = (\varepsilon, E, \overline{E})\,{}^t(0\quad 1\quad 0\quad 0\quad 1\quad 0\quad 0)$, $U \in SU(3)$ and ${}^t(z_1, z_2, z_3) = U\,{}^t(1, 0, 0)$.

8. $\varphi : S^1 \times S^5 \to S^7 \subset \mathbf{R}^8 \cong \mathfrak{C}$ is defined by

$$\varphi(e^{\sqrt{-1}\theta}, z_1, z_2, z_3)$$

$$= (N, E, \overline{N}, \overline{E}) \begin{pmatrix} e^{\sqrt{-1}\theta} & 0_{1\times 3} & 0 & 0_{1\times 3} \\ 0_{3\times 1} & e^{-(\sqrt{-1}\theta)/3} U & 0 & 0_{3\times 3} \\ 0 & 0_{1\times 3} & e^{-\sqrt{-1}\theta} & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 3} & 0_{3\times 1} & e^{(\sqrt{-1}\theta)/3}\overline{U} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \hline 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= Ne^{\sqrt{-1}\theta} + e^{(\sqrt{-1}\theta)/3} \sum_{i=1}^{3} E_i z_i + \overline{N} e^{-\sqrt{-1}\theta} + e^{-(\sqrt{-1}\theta)/3} \sum_{i=1}^{3} \overline{E_i z_i},$$

where $p_0 = (N, E, \overline{N}, \overline{E})\, {}^t(1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0)$, $U \in SU(3)$ and ${}^t(z_1, z_2, z_3) = U\, {}^t(1, 0, 0)$.

9. $\varphi : S^3 \times S^3 \to S^7 \subset \mathfrak{C}$ is defined by

$$\varphi(q_1, q_2) = q_1 + q_2 \varepsilon,$$

for $(q_1, q_2) \in S^3 \times S^3$.

10. $\varphi : T^2 \times \mathbf{R}^4 \to \mathfrak{C}$ is defined by

$$\begin{aligned} \varphi(e^{\sqrt{-1}\theta_1}, e^{\sqrt{-1}\theta_1}, z_1, z_2) &= (o; N, E, \overline{N}, \overline{E}) A v_0 \\ &= Ne^{\sqrt{-1}\theta_1} + E_1 e^{\sqrt{-1}\theta_1} + E_2 z_1 + E_3 z_2 \\ &\quad + \overline{Ne^{\sqrt{-1}\theta_1} + E_1 e^{\sqrt{-1}\theta_1} + E_2 z_1 + E_3 z_2}, \end{aligned}$$

where

$$v_0 = {}^t(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0),$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\sqrt{-1}\theta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\sqrt{-1}\theta_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ z_1 & 0 & 0 & e^{-\sqrt{-1}\theta_1} & 0 & 0 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 & e^{-\sqrt{-1}\theta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\sqrt{-1}\theta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\sqrt{-1}\theta_2} & 0 & 0 \\ \overline{z_1} & 0 & 0 & 0 & 0 & 0 & 0 & e^{\sqrt{-1}\theta_1} & 0 \\ \overline{z_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\sqrt{-1}\theta_2} \end{pmatrix}.$$

11. $\varphi : \mathbf{R}^2 \times S^1 \times S^3 \to \mathfrak{C}$ is defined by

$$\begin{aligned}\varphi(x_1, x_2, e^{i\theta}, q) &= x_1 1 + x_2 i + (\cos\theta + i\sin\theta) j(\cos\theta - i\sin\theta)\\ &\quad + (q(\cos\theta - i\sin\theta))\varepsilon\\ &= x_1 1 + x_2 i + \cos(2\theta) j + \sin(2\theta) k + (q(\cos\theta - i\sin\theta))\varepsilon,\end{aligned}$$

where $(x_1, x_2, e^{i\theta}, q) \in \mathbf{R}^2 \times S^1 \times S^3$.

12. $\tilde{\varphi} : \mathbf{R}^1 \times S^2 \times S^3 \to \mathfrak{C}$ is defined by

$$\varphi(x_1, q_1, q_2) = x_1 1 + q_1 i \bar{q}_1 + (q_2 \bar{q}_1)\varepsilon,$$

for $(x_1, q_1, q_2) \in \mathbf{R}^1 \times S^3 \times S^3$. The image of $\varphi(\mathbf{R}^1 \times S^3 \times S^3)$ is $\mathbf{R}^1 \times S^2 \times S^3 \subset \mathfrak{C}$.

13. $\varphi_{t_0} : SO(2) \times SO(3) \times \mathbf{R}^2 \to S^5 \times \mathbf{R}^2 \subset \mathbf{C}^3 \oplus \mathbf{R}^2 = \mathbf{R}^8$ is defined by

$$\begin{aligned}\varphi_{t_0}(\theta, q, x_1, x_2) &= x_1 1 + x_2 \varepsilon + \cos t_0(\cos\theta q i \bar{q} + \sin\theta(q i \bar{q})\varepsilon)\\ &\quad + \sin t_0(-\sin\theta q j \bar{q} + \cos\theta(q j \bar{q})\varepsilon),\end{aligned}$$

for $(\theta, q) \in S^1 \times Sp(1)$, $(x_1, x_2) \in R^2$ and $0 < t_0 < \pi/4$ (constant).

Remark 5.1. We note that the above immersion $\varphi \times id : SO(2) \times SO(3) \times \mathbf{R}^2 \to S^5 \times \mathbf{R}^2 \subset \mathbf{R}^6 \oplus \mathbf{R}^2 = \mathbf{R}^8$ is a product of the immersion $\varphi : SO(2) \times SO(3) \to S^5$, which is the isoparametric hypersurface with four distinct principal curvatures and the image $\varphi(SO(2) \times SO(3)) = SO(2) \times SO(3)/Z_2$. The induced almost complex structure is homogeneous, since this manifold is a principal orbit of the adjoint action of $SO(2) \times SO(3)$ which is included in $U(3) \subset SO(6)$ as a Lie subgroup. We note that it is also included in $SU(4)$. The action of $SO(2) \times SO(3)$ is an isotropy representation of the symmetric space $SO(5)/SO(2) \times SO(3)$. We define the action ρ of the Lie subgroup $SO(2) \times SO(3)$ of $Spin(7)$ by

$$\begin{aligned}&\rho(\theta, q)(a_0 \cdot 1 + a_1 + (b_0 + b_1)\varepsilon)\\ &\quad = (a_0 \cos(3\theta) + b_0 \sin(3\theta)) \cdot 1 + (\cos(\theta) q a_1 \bar{q} - \sin(\theta) q b_1 \bar{q})\\ &\qquad + \{(-a_0 \sin(3\theta) + b_0 \cos(3\theta)) \cdot 1 + (\sin(\theta) q a_1 \bar{q} + \cos(\theta) q b_1 \bar{q})\}\varepsilon,\end{aligned}$$

where $a_0, b_0 \in \mathbf{R}$ and $a_1, b_1 \in \mathrm{Im}\,\mathbf{H}$. Then the immersion is given by

$$\varphi_{t_0}(\theta, q) = \rho(\theta, q)(\cos(t_0) i + \sin(t_0) j\varepsilon),$$

where $0 < t_0 < \pi/4$.

Further, we have the following result.

THEOREM 5.1. *Let $M^6 = (M^6, \varphi)$ be an extrinsic homogeneous almost Hermitian submanifold of the octonions $\mathfrak{C}$ with respect to the induced almost Hermitian structure induced by the isometric immersion φ. Then (M^6, φ) is one of the above submanifolds in* (1)–(13).

6. 6-dimensional Riemannian homogeneous, but not homogeneous almost Hermitian submanifolds of $\mathfrak{C}$

The following 6-dimensional submanifolds (M, φ) of 8-dimensional Euclidean space are Riemannian homogeneous, but not homogeneous with respect to the almost complex structure induced by φ.

1. $(S^4 \times \mathbf{R}^2, \varphi)$
2. $(S^4 \times S^1 \times \mathbf{R}, \varphi)$
3. $(S^4 \times S^2, \varphi)$
4. $(S^2 \times S^2 \times \mathbf{R}^2, \varphi)$
5. $(S^1 \times S^2 \times \mathbf{R}^3, \varphi)$

   (In the above (1)–(5) the isometric immersion φ represents the standard one, respectively.)

   Further, the submanifolds (M^6, φ) of $\mathfrak{C}$ in (6)–(10) are given respectively as the orbits of adjoint action at the origin of some Riemannian symmetric spaces G/K and they are all isoparametric hypersurfaces in S^4 or S^7 which are Riemannian homogeneous. Denoting by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the standard decomposition of the Lie algebra $\mathfrak{g}$ corresponding to the Cartan involution on G/K. Then the subspace $\mathfrak{m}$ of $\mathfrak{g}$ is identified with the tangent space of G/K at the origin eK.
6. $(M^3 \times \mathbf{R}^3, \varphi \times id)$, where $\varphi \times id$ is a product immersion of the identity map $id : \mathbf{R}^3 \to \mathbf{R}^3$ and $\varphi : M^3 \to S^4 \subset \mathbf{R}^5$ is the isometric immersion from M^3 into $S^4 (\subset \mathbf{R}^5)$ such that (M^3, φ) is the Cartan hypersurface of the unit 4-sphere S^4 with three distinct constant principal curvatures. It is well-known that (M^3, φ) is obtained as a principal orbit of the adjoint action defined by the isotropy representation of the symmetric space $SU(3)/SO(3)$, namely
$$SO(3)/Z_2 \oplus Z_2 = Ad(SO(3))(p_0),$$
   for some nonzero tangent vector p_0 at the origin $o = e(SO(3))$.
7. The isoparametric hypersurface
$$SU(3)/T^2 = Ad(SU(3))(p_0)$$
   of S^7 with three distinct principal curvatures is given by the orbit of the isotropy representation (an orbit of the adjoint action) of the symmetric space $(SU(3) \times SU(3))/SU(3)$ for some nonzero tangent vector p_0 at the origin $o = e(SU(3))$.
8. The isoparametric hypersurface
$$SO(2) \times SO(4)/(Z_2 \times SO(2)) = Ad(SO(2) \times SO(4))(p_0)$$
   of S^7 with four distinct principal curvatures is an orbit of the isotropy representation (an orbit of the adjoint action) of the symmetric space $SO(6)/SO(2) \times SO(4)$ at the origin $o = e(SO(2) \times SO(4))$ where p_0 is a nonzero tangent vector at o.

9. The isoparametric hypersurface
$$S(U(2) \times U(2))/S^1 = Ad(S(U(2) \times U(2)))(p_0)$$
of S^7 with four distinct principal curvatures is an orbit of the isotropy representation (an orbit of the adjoint action) of the symmetric space $SU(4)/S(U(2) \times U(2))$ at the origin $o = e(S(U(2) \times U(2)))$ where p_0 is a nonzero tangent vector at o.
10. The isoparametric hypersurface
$$Ad(SO(4))(p_0)$$
of S^7 with six distinct principal curvatures is an orbit of the isotropy representation (an orbit of the adjoint action) of the symmetric space $G_2/SO(4)$ at the origin $o = e(SO(4))$ where p_0 is a nonzero tangent vector at o.

6.1. A brief proof for the assertion in §6

We shall give a brief proof for the assertion stated in §6, namely that the ten 6-dimensional submanifolds of $\mathfrak{C}$ introduced in a previous section are all Riemannian homogeneous, but not homogeneous with respect to the induced almost complex structure defined in §1. To do this, we shall show that the Lie subalgebra of $\mathfrak{so}(8)$ corresponding to the subgroups of $SO(8)$ acting transitively on each manifolds of the 10 examples can not be included in $Spin(7)$.

1. $\mathbf{R}^2 \times S^4$. By (2.2), we can easily see that a Lie group $SO(5)$ which acts on $\mathbf{R}^5$ (canonically) can not be realized as the Lie subgroup of $Spin(7)$.
2. $\mathbf{R}^1 \times S^1 \times S^4$. A Lie group $SO(2) \times SO(5)$ which acts on $\mathbf{R}^7$ can not be realized as the Lie subgroup of $Spin(7)$.
3. $S^2 \times S^4$. A Lie group $SO(3) \times SO(5)$ which acts on $\mathbf{R}^8$ can not be realized as the Lie subgroup of $Spin(7)$.
4. $S^2 \times S^2 \times \mathbf{R}^2$. By (2.2), a Lie group $SO(3) \times SO(3)$ which acts on $\mathbf{R}^6$ canonically, can not be realized as the Lie subgroup of $Spin(7)$.
5. $S^1 \times S^2 \times \mathbf{R}^3$. By (2.2), we see that the canonical action of $SO(2) \times SO(3)$ on $\mathbf{R}^5 \subset \mathbf{R}^7$, can not be realized as the one of subgroup of $Spin(7)$. Now, we show that the submanifold is not homogeneous with respect to the induced almost Hermitian structure. Since the universal covering group of the isometry of S^2 is isomrphic to $SU(2)$, so we have to examine the possibility of representation from the Lie group $S^1 \times SU(2)$ to $Spin(7)$ such that the image through some point in $\mathfrak{C}$ is diffeomorphic to $S^1 \times S^2$. We shall consider the following two cases. First, we assume that $g(1) = 1$ for any $g \in SU(2)$, then $SU(2)$ is a subgroup of G_2. Taking account of the classification of the representation of such $SU(2)$ as the subgroups of G_2, we may easily see that its orbit is (i) $S^3 \subset \mathbf{R}^4$ or (ii) a hypersurface in $S^2 \times S^2$ or (iii) a 3-dimensional submanifold of $S^2 \times S^3$, which does not coincide with S^2, or (iv) the one of an irreducible representation of $SU(2)$ whose representation space is $\mathbf{R}^7$. In any cases, we can not represent $S^1 \times S^2$ as an orbit of a

subgroup of $Spin(7)$ in $\mathrm{Im}\,\mathfrak{C}$. Secondary, we assume that $g(1) \neq 1$ for some $g \in SU(2)$. Then $\{g(1) \mid g \in SU(2)\} = S^2 \subset span_{\mathbf{R}}\{1, e_1, e_2\}$ where $e_1, e_2 \in \mathrm{Im}\,\mathfrak{C}$. Then, we may deduce a contradiction by using the equivariance of the exterior product of $\mathfrak{C}$.

6. $\varphi \times id : M^3 \times \mathbf{R}^3 \to S^4 \times \mathbf{R}^3 \subset \mathbf{R}^5 \oplus \mathbf{R}^3 = \mathbf{R}^8$, where $\varphi : M^3 \to S^4$ is the Cartan hypersurface of a 4-dimensional sphere (the isoparametric hypersurface with three distinct principal curvatures). This submanifold is a Riemannian homogeneous but the induced almost complex structure is not homogeneous. M^3 is obtained as an orbit of the the irreducible representation of $SO(3) \subset SO(5)$ on $\mathbf{R}^5$. The action of $SO(3)$ on $\mathbf{R}^5$ is given by an isotropy representation of the symmetric space $SU(3)/SO(3)$. More explicitely, M^3 is an adjoint orbit $Ad(g)(p_0)$ at some point $p_0 \in \mathfrak{m}$ where $\mathfrak{m}$ is the subspace (corresponding to the tangent space at the origin of the symmetric space $SU(3)/SO(3)$) defined by $\mathfrak{su}(3) = \mathfrak{m} \oplus \mathfrak{so}(3)$. We note that the Cartan involution σ is given by

$$\sigma(u) = \bar{u},$$

for any $u \in SU(3)$, and $\mathfrak{m}$ is the (-1)-eigenspace of σ_*. The Lie algebra $\mathfrak{su}(3)$ is spanned by

$$e_0 = \mathrm{diag}(0, \sqrt{-1}, -\sqrt{-1}) = \sqrt{-1}(E_{22} - E_{33}),$$

$$e_1 = (1/\sqrt{3})\,\mathrm{diag}(2\sqrt{-1}, -\sqrt{-1}, -\sqrt{-1}) = (\sqrt{-1}/\sqrt{3})(2E_{11} - E_{22} - E_{33})$$

$$e_2 = G_{12} = E_{12} - E_{21}, \quad e_3 = \sqrt{-1}(E_{12} + E_{21}),$$

$$e_4 = G_{13} = E_{13} - E_{31}, \quad e_5 = \sqrt{-1}(E_{13} + E_{31}),$$

$$e_6 = G_{23} = E_{23} - E_{32}, \quad e_7 = \sqrt{-1}(E_{23} + E_{32}),$$

where E_{ij} denote the elementary matrix. $\{e_0, e_1, e_3, e_5, e_7\}$ is a basis of subspace $\mathfrak{m}$. For any $X \in \mathfrak{so}(3)$, X can be represented by

$$X = \alpha_1 e_2 + \alpha_2 e_4 + \alpha_3 e_6,$$

for $\alpha_i \in \mathbf{R}$. Let $A(X)$ be the representation matrix with respect to the above orthonormal frame, that is

$$ad(X)(e_0, e_1, e_3, e_5, e_7) = (e_0, e_1, e_3, e_5, e_7)A(X).$$

By direct calculation, we have

$$\begin{aligned} A(X) = {} & \alpha_1\{-G_{02} + \sqrt{3}G_{12} + G_{34}\} \\ & + \alpha_2\{G_{03} - \sqrt{3}G_{13} - G_{24}\} \\ & + \alpha_3\{2G_{04} + G_{23}\}. \end{aligned}$$

From this and (2.2), we get a contradiction.

7. Let M^6 be a 6-dimensional isoparametric hypersurfaces of S^7 with three distinct principal curvatures. It is known that this submanifold M^6 is obtained as an principal orbit of the isotropy representation of the Riemannian symmetric space $SU(3) \times SU(3)/SU(3)$ as follows. Let

$$\mathfrak{m} = \{(X, -X) \mid X \in \mathfrak{su}(3)\},$$

be the subspace of the $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$. Then we have the decomposition $\mathfrak{su}(3) \oplus \mathfrak{su}(3) = \mathfrak{m} \oplus \ker \sigma_*$, (Cartan decomposition) where

$$\sigma(g, h) = (h, g),$$

for $(g, h) \in SU(3) \times SU(3)$ is the Cartan involution. We may easily check that

$$\ker \sigma_* = \{(X, X) \mid X \in \mathfrak{su}(3)\}.$$

We may identify $\mathfrak{m}$ with $\mathfrak{su}(3)$. Then the isotropy representation is given by

$$Ad(g)u = gug^{-1}.$$

for $u \in \mathfrak{m}$. Hence the tangent space of the corresponding orbit is given by

$$ad(X)u,$$

for $X \in \mathfrak{su}(3) \cong \ker \sigma_*$. If the hypersurface M admits the homogeneous almost complex structure, then

$$\mathfrak{l} = \{ad(X) \mid X \in \mathfrak{su}(3)\},$$

is a subalgebra of $\mathfrak{spin}(7)$. We may derive the contradiction as follows. To do this, we shall consider the linear representation $ad(X)$ on $\mathfrak{m}$. Let $\{e_i\}_{i=0}^7$ be the orthonormal basis of $\mathfrak{m}$ with respect to the canonical inner product $\langle X, Y \rangle = -\mathrm{tr}\, XY$ on $\mathfrak{spin}(7)$ given by

$$e_0 = \mathrm{diag}(0, \sqrt{-1}, -\sqrt{-1}) = \sqrt{-1}(E_{22} - E_{33}),$$

$$e_1 = (1/\sqrt{3})\,\mathrm{diag}(2\sqrt{-1}, -\sqrt{-1}, -\sqrt{-1}) = (\sqrt{-1}/\sqrt{3})(2E_{11} - E_{22} - E_{33}),$$

$$e_2 = G_{12} = E_{12} - E_{21}, \quad e_3 = \sqrt{-1}(E_{12} + E_{21}),$$

$$e_4 = G_{13} = E_{13} - E_{31}, \quad e_5 = \sqrt{-1}(E_{13} + E_{31}),$$

$$e_6 = G_{23} = E_{23} - E_{32}, \quad e_7 = \sqrt{-1}(E_{23} + E_{32}),$$

where E_{ij} denote the elementary matrix. For any $X \in \mathfrak{su}(3)$, it can be represented by

$$\begin{aligned} X = {} & \alpha_1\sqrt{-1}E_{11} + \alpha_2\sqrt{-1}E_{22} + \alpha_3\sqrt{-1}E_{33} \\ & + p_1 G_{12} + p_2\sqrt{-1}(E_{12} + E_{21}) + q_1 G_{13} \\ & + q_2\sqrt{-1}(E_{13} + E_{31}) + r_1 G_{23} + r_2\sqrt{-1}(E_{23} + E_{32}), \end{aligned}$$

for $\alpha_i, p_i, q_i, r_i \in \mathbf{R}$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $A(X)$ be the representation matrix corresponding to the above orthonormal frame $\{e_i\}_{i=0}^{7}$, that is

$$ad(X)(e_0, e_1, e_2, \dots, e_7) = (e_0, e_1, e_2, \dots, e_7)A(X).$$

By direct calculation, we get

$$\begin{aligned} A(X) = {} & \alpha_1\{-G_{23} - 2G_{45} - G_{67}\} \\ & + \alpha_2\{-G_{23} - G_{45} - 2G_{67}\} \\ & + p_1\{-G_{03} + \sqrt{3}G_{13} + G_{46} + G_{57}\} \\ & + p_2\{G_{02} + \sqrt{3}G_{12} - G_{47} + G_{56}\} \\ & + q_1\{G_{05} + \sqrt{3}G_{15} - G_{26} + G_{37}\} \\ & + q_2\{-G_{04} - \sqrt{3}G_{14} - G_{27} - G_{36}\} \\ & + r_1\{2G_{07} + G_{24} + G_{35}\} \\ & + r_2\{-2G_{06} + G_{25} - G_{34}\}. \end{aligned}$$

We can see that the subspace of $\mathfrak{so}(8)$ generated by the above basis can not be contained in $\mathfrak{spin}(7)$, by (2.2), this is a contradiction. q.e.d

8. The isoparametric hypersurface M^6 of S^7 with four distinct principal curvatures is the isotropy representation (an principal orbit of the adjoint action as follows) of the symmetric space $G_2(\mathbf{R}^6) = SO(6)/SO(2) \times SO(4)$ (the Grassmann manifold of oriented 2-planes of 6-dimensional Euclidean space). It is the principal orbit of the tangent space at the origin $o = e(SO(2) \times SO(4))$ where e is the unit element of $SO(6)$. Then $M^6 = SO(2) \times SO(4)/\mathrm{Z}_2 \times SO(2) = Ad(SO(2) \times SO(4))(p_0)$ where p_0 is a nonzero tangent vector at o. The corresponding Cartan involution σ is given by

$$\sigma(u) = sus^{-1},$$

where $s = \mathrm{diag}(1, 1, -1, -1, -1, -1)$. Let $\mathfrak{m}$ be the subspace of $\mathfrak{so}(6)$ which is corresponding to the tangent space at the origin ((-1)-eigenspace with respect to the Cartan involution), that is, $\mathfrak{so}(6) = \mathfrak{m} \oplus (\mathfrak{so}(2) \oplus \mathfrak{so}(4))$. The action of $Ad(SO(2) \times SO(4)) = SO(2) \otimes SO(4)$ on $\mathfrak{m}$ is given by

$$Ad(R(\theta), A)\begin{pmatrix} 0 & u \\ -{}^t u & 0 \end{pmatrix} = \begin{pmatrix} 0 & R(\theta)u^t A \\ -A^t u^t R(\theta) & 0 \end{pmatrix},$$

where $(R(\theta), A) \in SO(2) \times SO(4)$, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & u \\ -{}^t u & 0 \end{pmatrix} \middle| u \in M_{2\times 4}(\mathbf{R}) \right\}.$$

The tangent space of this action on $\mathfrak{m}$ is spanned by

$$G_{04} + G_{15} + G_{26} + G_{37},$$
$$G_{01} + G_{45}, \quad G_{12} + G_{56}, \quad G_{02} + G_{46},$$
$$G_{13} + G_{57}, \quad G_{03} + G_{47}, \quad G_{23} + G_{67}.$$

We can show that this subspace of $\mathfrak{so}(8)$ which is generated by the above basis, by (2.2), can not be included in $\mathfrak{spin}(7)$. q.e.d

9. It is well-known that an isoparametric hypersurface of unit sphere S^7 with four distinct constant principal curvatures is obtained as a principal orbit of the the adjoint action defined by the isotropy representation of the symmetric space $SU(4)/S(U(2) \times U(2))$, namely

$$M^6 = S(U(2) \times U(2))/S^1 = Ad(S(U(2) \times U(2)))(p_0)$$

for some nonzero tangent vector p_0 at the origin $o = e(S(U(2) \times U(2)))$. The corresponding Cartan involution σ is given by

$$\sigma(u) = sus^{-1},$$

where $s = \mathrm{diag}(1, 1, -1, -1)$. Let $\mathfrak{m}$ is the subspace of $\mathfrak{su}(4)$ which is corresponding to the tangent space at the origin, $((-1)$-eigenspace with respect to the Cartan involution), that is, $\mathfrak{su}(4) = \mathfrak{m} \oplus (\mathfrak{su}(2) \oplus \mathfrak{su}(2))$. The action of $Ad(S(U(2) \times U(2)))$ on $\mathfrak{m}$ is given by

$$Ad\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ -{}^t\bar{\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A\alpha{}^t\bar{B} \\ -B{}^t\bar{\alpha}{}^t\bar{A} & 0 \end{pmatrix},$$

where $(A, B) \in S(U(2) \times U(2))$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & \alpha \\ -{}^t\bar{\alpha} & 0 \end{pmatrix} \middle| \alpha \in M_{2\times 2}(\mathbf{C}) \right\}.$$

The tangent space of this action on $\mathfrak{m}$ is spanned by

$$G_{01} + G_{23}, \quad G_{45} + G_{67}, \quad G_{01} + G_{45}, \quad G_{23} + G_{67},$$
$$G_{02} + G_{13} + G_{46} + G_{57}, \quad G_{03} - G_{12} + G_{47} - G_{56},$$
$$G_{04} + G_{15} + G_{26} + G_{37}, \quad -G_{05} + G_{14} - G_{27} + G_{36}.$$

By (2.2), we can show that this subspace of $\mathfrak{so}(8)$ generated by the above basis, can not be included in $\mathfrak{spin}(7)$. q.e.d

10. It is well known that an isoparametric hypersurface M^6 of S^7 with six distinct principal curvatures is obtained by the isotropy representation of the symmetric space $G_2/SO(4)$. We may observe that it is given as an principal orbit of an adjoint action of $SO(4)$ on the tangent space at the origin $o = e(SO(4))$ of $G_2/SO(4)$, say, $M^6 = Ad(SO(4))(p_0)$ for some nonzero tangent vector p_0 at the origin o. We note that the action of $SO(4)$ in G_2 is given by

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a \overline{q_1} + (q_2 b \overline{q_1})\varepsilon,$$

where $(q_1, q_2) \in Sp(1) \times Sp(1)$ and $a + b\varepsilon \in \operatorname{Im} \mathfrak{C}$. The Cartan involution is given by

$$\sigma(u) = sus^{-1},$$

where $s = \operatorname{diag}(1, 1, 1, -1, -1, -1, -1)$. Then $\mathfrak{m}$ is the subspace of $\mathfrak{g}_2$ which is corresponding to the tangent space at the origin, ((-1)-eigenspace with respect to the Cartan involution), which is spanned by following basis.

$$e_0 = (1/\sqrt{3})(-2G_{51} + G_{73} + G_{62}), \quad e_1 = G_{73} - G_{62},$$

$$e_2 = (1/\sqrt{3})(-2G_{14} + G_{72} + G_{36}), \quad e_3 = G_{72} - G_{36},$$

$$e_4 = (1/\sqrt{3})(-2G_{17} + G_{24} + G_{53}), \quad e_5 = G_{24} - G_{53},$$

$$e_6 = (1/\sqrt{3})(-2G_{61} + G_{34} + G_{25}), \quad e_7 = G_{34} - G_{25},$$

where $G_{ij} = E_{ij} - E_{ji}$ is a basis of $\mathfrak{so}(7)$. The subspace $\mathfrak{m}$ is invariant under the adjoint action of $SO(4)$. We may identified with $\mathfrak{m}$ to the tangent space at the origin o of $G_2/SO(4)$. For any $X \in \mathfrak{so}(4)$, we denote the representation matrix $A(X)$ corresponding to the above orthonormal frame, that is

$$ad(X)(e_0, e_1, e_2, \ldots, e_7) = (e_0, e_1, e_2, \ldots, e_7)A(X).$$

By direct calculation, we get

$$\begin{aligned} A(X) = {} & \alpha_1\{G_{01} - G_{23} + 3(G_{45} - G_{67})\} \\ & + \alpha_2\{-2(G_{02} + G_{13}) + \sqrt{3}(G_{06} - G_{17} + G_{24} - G_{35})\} \\ & + \alpha_3\{-2(G_{03} + G_{12}) + \sqrt{3}(G_{07} + G_{16} - G_{25} - G_{34})\} \\ & + \beta_1\{-G_{01} - G_{23} + G_{45} + G_{76}\} \\ & + \beta_2\{-G_{02} + G_{13} + G_{46} - G_{57}\} \\ & + \beta_3\{-G_{03} - G_{12} - G_{47} - G_{56}\}, \end{aligned}$$

where $X = \sum_{i=1}^{3} \alpha_i u_i + \beta_i v_i$, $(\alpha_i, \beta_i \in \mathbf{R})$ and

$$u_1 = -2G_{23} + G_{45} + G_{76}, \quad u_2 = -2G_{31} + G_{46} + G_{57},$$

$$u_3 = -2G_{12} + G_{47} + G_{65},$$

$$v_1 = -G_{45} + G_{76}, \quad v_2 = -G_{46} + G_{57}, \quad v_3 = -G_{47} + G_{65},$$

which span the Lie algebra $\mathfrak{so}(4)$. We can see that the subspace of $\mathfrak{so}(8)$ which is generated by the above basis related to $A(X)$ can not be included in $\mathfrak{spin}(7)$, we get the desired result by (2.2). q.e.d

7. Some product immersions

In this section, we shall consider the almost Hermitian structure derived from the specified product immersion $\varphi \times id : M^2 \times \mathbf{R}^4 \to \mathbf{H} \oplus \mathbf{H} \cong \mathfrak{C}$. Now, we introduce several $Spin(7)$-invariants for such submanifolds. In the next section §7.1 we calculate them for some concrete examples of such submanifolds.

First we shall give the surface theory in the quaternions $\mathbf{H}$. Let $\varphi : M^2 \to \mathbf{H}$ be the surface in $\mathbf{H}$ and let η, ξ a local orthonormal frame field of the normal bundle. We can define the induced almost complex structure J in the same way of the octonions case, as follows

$$\varphi_*(JX) = \varphi_*(X)(\eta \times \xi),$$

where $\eta \times \xi$ denote the exterior product of the quaternions (which coincide with that of the octonions.) Let $\{e_1, e_2\}$ be an orthonormal frame (local) field of the tangent bundle. we may take $e_2 = Je_1$ and that $\xi(\eta \times \xi) = \eta$ and $\eta(\eta \times \xi) = \xi$. Since $\eta \times \xi \in S^2 \subset \operatorname{Im} \mathbf{H}$, there exists a $q \in S^3 \subset \mathbf{H}$ such that $\eta \times \xi = qi\bar{q}$, by taking account of the Hopf fibration $S^3 \to S^2$. We define the $G_2 (\subset Spin(7))$-frame field along the immersion $\varphi \times id : M^2 \times \mathbf{R}^4 \to \mathfrak{C}$ as follows

$$\begin{aligned} n^* &= (1/2)(1 - \sqrt{-1}qi\bar{q}), \\ f_1^* &= (1/2)(qj\bar{q} - \sqrt{-1}(-qk\bar{q})), \\ f_2^* &= (1/2)(\bar{q}\varepsilon - \sqrt{-1}(-i\bar{q})\varepsilon), \\ f_3^* &= -(1/2)((j\bar{q})\varepsilon - \sqrt{-1}(k\bar{q})\varepsilon). \end{aligned}$$

We may choose $\mathbf{C}$-valued functions a_{11}, a_{12} satisfying the following equality

$$(1/2)(\xi - \sqrt{-1}\eta) = n^* a_{11} + f_1^* a_{12}. \tag{7.1}$$

We set

$$\xi = \xi_0 1 + \xi_1 qi\bar{q} + \xi_2 qj\bar{q} + \xi_3 qk\bar{q}, \tag{7.2}$$

$$\eta = \eta_0 1 + \eta_1 qi\bar{q} + \eta_2 qj\bar{q} + \eta_3 qk\bar{q}. \tag{7.3}$$

Then, since $\eta \times \xi = qi\bar{q}$ and $\xi(\eta \times \xi) = \eta$ imply that

$$(\eta_0, \eta_1, \eta_2, \eta_3) = (-\xi_1, \xi_0, \xi_3, -\xi_2). \tag{7.4}$$

By (7.1), (7.2), (7.3) and (7.4), we get $a_{11} = \xi_0 + \sqrt{-1}\xi_1$, $a_{12} = \xi_2 - \sqrt{-1}\xi_3$. Therefore, if we set

$$(a_1 \quad a_2 \quad a_3 \quad a_4) = \begin{pmatrix} \xi_0 + \sqrt{-1}\xi_1 & -(\xi_2 + \sqrt{-1}\xi_3) & 0 & 0 \\ \xi_2 + \sqrt{-1}\xi_3 & \xi_0 - \sqrt{-1}\xi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then, we see that $(a_1 \quad a_2 \quad a_3 \quad a_4) \in SU(4)$. The (local) $Spin(7)$-frame field along the immersion $\varphi \times id$ is given by

$$\begin{aligned} n &= (1/2)(\xi - \sqrt{-1}\eta), \\ f_1 &= (1/2)(1 - \sqrt{-1}(qi\bar{q}))(-(\xi_2 + \sqrt{-1}\xi_3)), \\ &\quad + (1/2)(qj\bar{q} - \sqrt{-1}(-qk\bar{q}))(\xi_0 - \sqrt{-1}\xi_1), \\ f_2 &= (1/2)(\bar{q}\varepsilon - \sqrt{-1}(-i\bar{q})\varepsilon), \\ f_3 &= -(1/2)((j\bar{q})\varepsilon - \sqrt{-1}(k\bar{q})\varepsilon). \end{aligned}$$

We may remark that $q \in S^3$ is determined up to the action of S^1, therefore locally, we take q as the local section of the Hopf fibration $\pi : S^3 \to S^2$ which is defined by $\pi(q) = qi\bar{q}$. Let $x = \varphi \times id : M^2 \times \mathbf{R}^4 \to \mathfrak{C}$ be a product immersion. Then we have

$$d\varphi = e_1 \otimes \mu^1 + e_2 \otimes \mu^2,$$

where $\{\mu^1, \mu^2\}$ are dual 1-forms with respect to the orthonormal basis $\{e_1, e_2\}$ of M^2. Then the dual 1-forms $\{\omega_i\}$ of the product immersion x are given by

$$\begin{aligned} \omega_1 &= 2\langle dx, \bar{f}_1\rangle = \mu^1 + \sqrt{-1}\mu^2, \\ \omega_2 &= 2\langle dx, \bar{f}_2\rangle = \langle dy, \bar{q}\rangle - \sqrt{-1}\langle dy, i\bar{q}\rangle, \\ \omega_3 &= 2\langle dx, \bar{f}_3\rangle = \langle dy, j\bar{q}\rangle + \sqrt{-1}\langle dy, k\bar{q}\rangle, \end{aligned}$$

where $x(m, y) = \varphi(m) + y\varepsilon$ for any $(m, y) \in M^2 \times \mathbf{R}^4$. On the other hand, we have

$$dn = n \otimes \sqrt{-1}\rho + f \otimes \mathfrak{h} + \bar{f} \otimes \bar{\theta},$$

therefore, the 1-forms $\mathfrak{h}$, θ are given by

$$\mathfrak{h}^i = 2\langle dn, \bar{f}_i\rangle \quad \text{and} \quad \bar{\theta}^i = 2\langle dn, f_i\rangle,$$

respectively. If necessary, exchanging the special unitary frame from (f_1, f_2, f_3) into the unitary frame $(f_1 e^{i\theta}, f_2 e^{-i\theta}, f_3)$, we obtain the following *Spin*(7)-frame field given by (denoting it again by using the same symbol)

$$\begin{aligned} n &= (1/2)(\xi - \sqrt{-1}\eta), \\ f_1 &= (1/2)(\varphi_*(e_1) - \sqrt{-1}\varphi_*(e_1)), \\ f_2 &= (1/2)e^{-i\theta}(\bar{q}\varepsilon - \sqrt{-1}(-i\bar{q})\varepsilon), \\ f_3 &= -(1/2)((j\bar{q})\varepsilon - \sqrt{-1}(k\bar{q})\varepsilon). \end{aligned}$$

Then $\mathfrak{h}^1$ is given by

$$\begin{aligned} \mathfrak{h}^1 = (-1/4)\{&(\mathrm{tr}\, A_\xi - \sqrt{-1}\ \mathrm{tr}\, A_\eta)\omega^1 \\ &+ [(h^1_{11} - h^1_{22} + 2h^2_{12}) - \sqrt{-1}(h^2_{11} - h^2_{22} - 2h^1_{12})]\overline{\omega^1}\}. \end{aligned}$$

Also we have $\mathfrak{h}^2 = \mathfrak{h}^3 = 0$. Therefore $\mathfrak{h}$ is given by

$$\mathfrak{h} = \begin{pmatrix} \mathfrak{h}^1 \\ \mathfrak{h}^2 \\ \mathfrak{h}^3 \end{pmatrix} = \begin{pmatrix} \bar{B}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} \bar{A}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\omega^1} \\ \overline{\omega^2} \\ \overline{\omega^3} \end{pmatrix},$$

where $\bar{B}_{11} = (-1/4)(\mathrm{tr}\, A_\xi - \sqrt{-1}\ \mathrm{tr}\, A_\eta)$ and

$$\bar{A}_{11} = (-1/4)((h^1_{11} - h^1_{22} + 2h^2_{12}) - \sqrt{-1}(h^2_{11} - h^2_{22} - 2h^1_{12})),$$

where $h^1_{ij} = \langle A_\xi(e_i), e_j\rangle$, $h^2_{ij} = \langle A_\eta(e_i), e_j\rangle$ for $i, j = 1, 2$. In the same way, $\theta^1 = (1/2)\langle d\xi - \sqrt{-1}\, d\eta, \varphi_*(e_1) + \sqrt{-1}J(\varphi_*(e_1))\rangle$ is given by

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} \bar{C}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\omega^1} \\ \overline{\omega^2} \\ \overline{\omega^3} \end{pmatrix},$$

where $B_{11} = (-1/4)(\mathrm{tr}\, A_\xi + \sqrt{-1}\ \mathrm{tr}\, A_\eta)$ and

$$\bar{C}_{11} = (-1/4)((h^1_{11} - h^1_{22} - 2h^2_{12}) - \sqrt{-1}(h^2_{11} - h^2_{22} + 2h^1_{12})).$$

We may remark that Bryant ([Br1]) proved that M^6 is a complex manifold with respect to the induced almost complex structure, if and only if B is identically zero. In this case, the condition is equivalent that M^2 is a minimal surface of $\mathbf{R}^4$. By a simple calculation, we get

$$|\bar{A}_{11}|^2 = (1/16)(|\sigma|^2 - 2K + 4K^\perp), \quad |\bar{C}_{11}|^2 = (1/16)(|\sigma|^2 - 2K - 4K^\perp),$$

where $|\sigma|^2$, K, $K^\perp$ denote the square length of the second fundamental form, the Gauss curvature, and the normal curvature of $\varphi(M^2)$, respectively. We note that the normal curvature is defined by as follows; $K^\perp = \langle [A_\xi, A_\eta](e_1), Je_1\rangle$. We can easily see that the ellipse of curvature

$$\{\sigma(X, X) \in T^\perp M^2 \mid X \in T_m M^2, |X| = 1\},$$

is a circle in the normal bundle, if and only if $|\bar{A}_{11}|^2 = 0$ or $|\bar{C}_{11}|^2 = 0$.

7.1. *Spin*(7)-invariants for some product submanifolds in $\mathfrak{C}$

1. $S^2 \times \mathbf{R}^4$

$$\{x + y\varepsilon \in \mathrm{Im}\,\mathbf{H} \oplus \mathbf{H}\varepsilon \mid x \in \mathrm{Im}\,\mathbf{H}, y \in \mathbf{H}, |x| = r\}.$$

The position vector can be considered as an outward normal vector field. We put $\xi = x/r$ and $\eta = 1$ as an orthonormal frame of the normal bundle. The nontrivial elements of the *Spin*(7)-invariants are given by

$$B_{11} = 2/r.$$

This example is a quasi-Kähler submanifold with vanishing first Chern class ([H1]).

2. (Catenoid $\times \mathbf{R}^4$) we shall consider the map from $\mathbf{R} \times S^1$ to $\mathrm{Im}\,\mathbf{H}$ such that

$$\varphi(t, \theta) = t \cdot i + a \cos\theta \cosh(t/a) \cdot j + a \sin\theta \cosh(t/a) \cdot k,$$

for $(t,\theta) \in \mathbf{R} \times S^1$. The unit normal vector field ξ is given by

$$\xi = (1/\cosh(t/a)) - \sinh(t/a) \cdot i + \cos\theta \cdot j + \sin\theta \cdot k,$$

We can take the another unit normal vector field $\eta = 1$, The nontrivial elements of the $Spin(7)$-invariants are given by

$$A_{11} = C_{11} = -1/(2a\cosh(t/a)^2).$$

The $*$-scalar curvature τ^* is given by $\tau^* = -4/(a\cosh(t/a)^2)^2$. Thus τ^* is a negative non-constant function. The product immersion $\varphi \times id : \mathbf{R} \times S^1 \times \mathbf{R}^4 \to \operatorname{Im} \mathfrak{C}$ is a simplest (non Kähler) complex manifold with vanishing first Chern class.

8. Examples with non-trivial characteristic classes

In this section, we shall introduce an example of product submanifolds in $\mathfrak{C}$ with non-vanishing 1st Chern class. Let $\mathbf{R}P^2$ be a 2-dimensional real projective space, and $\varphi \times id : \mathbf{R}P^2 \times \mathbf{R}^4 \to \mathfrak{C}$ be the product immersion, where $\varphi : \mathbf{R}P^2 \to \mathbf{H}$ is the map defined by

$$\varphi(x,y,z) = (1,i,j,k)\begin{pmatrix} x^2 - y^2 \\ 2xy \\ 2yz \\ 2zx \end{pmatrix},$$

for $x^2 + y^2 + z^2 = 1$. If we set

$$(x,y,z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

for $0 < \phi < \pi$ and $0 < \theta < 2\pi$. Then the above immersion is rewritten as

$$\varphi(\theta,\phi) = (1,i,j,k)\begin{pmatrix} \cos 2\theta \sin^2\phi \\ \sin 2\theta \sin^2\phi \\ \sin\theta \sin 2\phi \\ \cos\theta \sin 2\phi \end{pmatrix}.$$

Then the orthonormal frame of the normal bundle is given by

$$\xi = \frac{1}{\sqrt{1 + 3\cos^2(2\phi)}}(1,i,j,k)\begin{pmatrix} 2\cos 2\theta \cos 2\phi \\ 2\sin 2\theta \sin 2\phi \\ -\sin\theta \sin 2\phi \\ -\cos\theta \sin 2\phi \end{pmatrix},$$

$$\eta = (1,i,j,k)\begin{pmatrix} -\sin 2\theta \cos\phi \\ \cos 2\theta \cos\phi \\ -\cos\theta \sin\phi \\ \sin\theta \sin\phi \end{pmatrix}.$$

The exterior vector cross product $\eta \times \xi$ is given by

$$\eta \times \xi = (i, j, k)(2/\sqrt{1 + 3\cos^2(2\phi)})\begin{pmatrix} \cos^3\phi \\ \cos 3\theta \sin^3\phi \\ -\sin 3\theta \sin^3\phi \end{pmatrix}.$$

By solving the equation

$$\eta \times \xi = qi\bar{q},$$

we may obtain a local $Spin(7)$-frame field in §7.2. With respect to the local orthonormal frame field on $\mathbf{R}P^2$,

$$\{1/(\sin\phi)\partial/\partial\theta, 1/\sqrt{1 + 3\cos^2(2\phi)}\partial/\partial\phi\},$$

the shape operators are given by

$$A_\xi = \begin{pmatrix} (1 - 5\cos 2\phi)/(2\sqrt{1 + 3\cos^2(2\phi)}) & 0 \\ 0 & -4/(\sqrt{1 + 3\cos^2(2\phi)})^3 \end{pmatrix},$$

$$A_\eta = 1/\sqrt{1 + 3\cos^2(2\phi)}\begin{pmatrix} 0 & \cos 2\phi - 1 \\ \cos 2\phi - 1 & 0 \end{pmatrix},$$

from which, we get

$$\begin{aligned} A_{11} = -(1/4)\{&(1 - 5\cos 2\phi)/(2\sqrt{1 + 3\cos^2(2\phi)}) \\ &+ 4/(\sqrt{1 + 3\cos^2(2\phi)})^3 + 2(\cos 2\phi - 1)/(\sqrt{1 + 3\cos^2(2\phi)})\}, \\ B_{11} = -(1/4)\{&(1 - 5\cos 2\phi)/(2\sqrt{1 + 3\cos^2(2\phi)}) - 4/(\sqrt{1 + 3\cos^2(2\phi)})^3\}, \\ C_{11} = -(1/4)\{&(1 - 5\cos 2\phi)/(2\sqrt{1 + 3\cos^2(2\phi)}) \\ &+ 4/(\sqrt{1 + 3\cos^2(2\phi)})^3 - 2(\cos 2\phi - 1)/(\sqrt{1 + 3\cos^2(2\phi)})\}, \end{aligned}$$

and otherwise are zero. By direct calculations, we get

$$\langle c_1(\varphi(S^2) \times \mathbf{R}^4), [\varphi(S^2)]\rangle \neq 0,$$

and hence, the first Chern class does not vanish ([H1]).

Remark 8.1. From the above arguments, we may observe that the product manifold $S^2 \times \mathbf{R}^4$ admits at least three different kinds of almost Hermitian structures from the point of view of almost Hermitian geometry. Namely, the first one is the canonical almost Hermitian structure on $P^1(\mathbf{C}) \times \mathbf{C}^2$ coming from the product Kähler structure. The second one is a quasi-Kähler structure on $S^2 \times \mathbf{R}^4$ which is compatible with the canonical Riemannian metric with vanishing 1-st Chern class. The remaining is the one introduced in the above arguments of the present section.

Remark 8.2. The Gauss curvature K and the normal curvature $K^{\perp}$ of the above example are given by

$$K = \frac{1 - 3\cos 2\phi(2 + 2\cos 2\phi)}{(1 + 3\cos^2(2\phi))^2},$$

$$K^{\perp} = \frac{-8 + (1 + 3\cos^2 2\phi)(1 - 3\cos 2\phi)}{2(1 + 3\cos^2(2\phi))}.$$

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