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MINIMUM MODULI OF WEIGHTED COMPOSITION OPERATORS ON ALGEBRAS OF ANALYTIC FUNCTIONS

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Abstract

We study the minimum moduli of weighted composition operators on the disk algebra and the space of bounded analytic functions.

1. Introduction

Let $\mathbf{D}$ be the open unit disk, $\overline{\mathbf{D}}$ its closure and $\mathbf{T}$ the unit circle. Let $H^\infty = H^\infty(\mathbf{D})$ be the set of all bounded analytic functions on $\mathbf{D}$ and A be the set of all analytic functions bounded on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$, called the disc algebra. Then H^∞ and A are Banach algebras with the supremum norm

$$\|f\|_\infty = \sup_{z \in \mathbf{D}} |f(z)|.$$

In this paper, we will deal with the minimum modulus of analytic functions on $\mathbf{D}$ and $\mathbf{T}$. For $f \in H^\infty$, the radial limit f^* of f is defined almost everywhere on $\mathbf{T}$. We denote that

$$\|f\|_{-\infty,\mathbf{D}} = \inf_{z \in \mathbf{D}} |f(z)|$$

and

$$\|f\|_{-\infty,\mathbf{T}} = \operatorname*{essinf}_{\omega \in \mathbf{T}} |f^*(\omega)|.$$

Let $S(\mathbf{D})$ be the set of all analytic self-map of $\mathbf{D}$. For $\varphi \in S(\mathbf{D})$, we can define the composition operator C_φ on H^∞ as $C_\varphi f = f \circ \varphi$. Moreover, for $u \in H^\infty$, we can define the multiplication operator M_u on H^∞ as $M_u f = uf$. Hence the weighted composition operator uC_φ is the product of M_u and C_φ, that is, $uC_\varphi f = M_u C_\varphi f = uf \circ \varphi$.

To define the weighted composition operators on A, it is necessary that $\varphi, u \in A$. Denote by $S(\overline{\mathbf{D}})$ the closed unit ball of A. If $\varphi \equiv \omega \in \mathbf{T}$, φ is not in

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$S(\mathbf{D})$ but in $S(\overline{\mathbf{D}})$, and C_φ is the point evaluation at ω which acts on A. We can identify the set of all point evaluations at boundary points with $\mathbf{T}$. By the maximum modulus principle, it is shown that $S(\overline{\mathbf{D}})\backslash\mathbf{T} \subsetneq S(\mathbf{D})$.

As well known, $\|uC_\varphi\| = \|u\|_\infty$ both on H^∞ and on A. Putting $u \equiv 1$, we have that $\|C_\varphi\| = 1$.

Let X and Y be Banach spaces and T be a bounded linear operator from X to Y. The operator norm $\|T\|$ of T is the maximum modulus of its image of the closed unit ball $U_X = \{x \in X : \|x\|_X \le 1\}$. In [2], Müller introduced two quantities as the minimum moduli of $T(U_X)$. We can regard $j(T)$ as the minimum modulus of $T(U_X)$ estimating from the outside and $k(T)$ as the minimum modulus estimating from the inside.

Definition 1.1. *Let T be a bounded linear operator from X to Y.*
(i) *The injectivity modulus $j(T)$ of T is defined by*

$$j(T) = \inf\{\|Tx\|_Y : \|x\|_X = 1\}.$$

(ii) *The surjectivity modulus $k(T)$ of T is defined by*

$$k(T) = \sup\{r \ge 0 : T(U_X) \supset rU_Y\}.$$

Though the operator norm holds the triangular inequality, neither $j(T)$ nor $k(T)$ hold it. Some properties of $j(T)$ and $k(T)$ are studied in [2].

Proposition 1.2 [2]. *Let T be a bounded linear operator from X to Y.*
(i) *Clearly $0 \le j(T) \le \|T\|$ and $0 \le k(T) \le \|T\|$.*
(ii) *If T is invertible, then $j(T) = k(T) = \|T^{-1}\|^{-1}$.*
(iii) *$j(T) > 0$ (this is said that T is bounded below) if and only if T is one-to-one and* Ran T *is closed.*
(iv) *$k(T) > 0$ if and only if T is onto.*
(v) *$j(T) = k(T^*)$ and $k(T) = j(T^*)$.*

Example 1.3. Let $l^2(\mathbf{N})$ be the Hilbert space of square summable one-sided complex sequences.
(i) Let F be the forward shift operator on $l^2(\mathbf{N})$. Then $\|F\| = j(F) = 1$ but $k(F) = 0$.
(ii) Let B be the backward shift operator on $l^2(\mathbf{N})$. Then $\|B\| = k(B) = 1$ but $j(B) = 0$.

2. Minimum moduli of weighted composition operators on H^∞

In this section we estimate $j(uC_\varphi)$ and $k(uC_\varphi)$ on H^∞. First, we concern with the trivial cases. If $u \equiv 0$ or $\varphi \equiv p \in \mathbf{D}$, then Ran uC_φ is a zero or one dimensional subspace spanned by u. Hence we have the following.

Proposition 2.1. *If $u \equiv 0$ or $\varphi \equiv p \in \mathbf{D}$, then $j(uC_\varphi) = k(uC_\varphi) = 0$.*

In the sequel, to exclude these cases, we assume that $u \in H^\infty$ is not identically zero and $\varphi \in S(\mathbf{D})$ is not constant. Under this assumption, we call uC_φ non-trivial. We remark that uC_φ is injective on H^∞ if uC_φ is non-trivial. This fact and (iv) of Proposition 1.2 imply that $k(uC_\varphi) > 0$ if and only if $(uC_\varphi)^{-1}$ is bounded on H^∞. Then φ is an automorphism of $\mathbf{D}$ and $1/u$ is in H^∞, that is, $\|u\|_{-\infty,\mathbf{D}} > 0$. Since $(uC_\varphi)^{-1} = M_{1/v}C_{\varphi^{-1}}$ where $v = u \circ \varphi^{-1}$, we have the following theorem.

Theorem 2.2. *Let uC_φ be a non-trivial weighted composition operator on H^∞. Then $k(uC_\varphi) > 0$ if and only if $\|u\|_{-\infty,\mathbf{D}} > 0$ and φ is an automorphism of $\mathbf{D}$. Moreover, in such cases, $k(uC_\varphi) = j(uC_\varphi) = \|u\|_{-\infty,\mathbf{D}}$.*

Considering the special cases of $u \equiv 1$ and $\varphi(z) = z$, we have the following corollary.

Corollary 2.3. *Let $u \in H^\infty$ and $\varphi \in S(\mathbf{D})$.*
(i) $k(M_u) = \|u\|_{-\infty,\mathbf{D}}$.
(ii) *If φ is an automorphism of $\mathbf{D}$, $k(C_\varphi) = 1$. Otherwise, $k(C_\varphi) = 0$.*

Next we will consider the estimation of $j(uC_\varphi)$. For convenience, we provide some notation.

Definition 2.4. *Define that $D_\delta(u) = \{z \in \mathbf{D} : |u(z)| \geq \delta\}$.*

In [3], Ohno and Takagi have stated their results in terms of Gelfand transformation and Shilov boundary of H^∞. Our main theorem is expressed in function theoretic terms. We need the following lemma (see [4] and [5]).

Lemma 2.5. *Let G be a subset of $\mathbf{D}$ such that $\bar{G} \supset \mathbf{T}$. Then, for any $f \in H^\infty$,*

$$\sup_{z \in G} |f(z)| = \|f\|_\infty$$

Now we can prove the main theorem.

Theorem 2.6. *Let uC_φ be a non-trivial weighted composition operator on H^∞. Then we have*

$$j(uC_\varphi) = \sup\{\delta : \overline{\varphi(D_\delta(u))} \supset \mathbf{T}\} \tag{1}$$

$$= \inf_{\omega \in \mathbf{T}} \limsup_{\varphi(z_n) \to \omega} |u(z_n)| \tag{2}$$

where we define the supremum in (1) *is equal to* 0 *if such a constant δ does not exist, and we define also the infimum in* (2) *is equal to* 0 *if $\overline{\varphi(\mathbf{D})} \not\supset \mathbf{T}$.*

Proof. Let d be the supremum in (1) and m be the infimum in (2).

First, we will prove that $j(uC_\varphi) \ge d$. We may suppose that $d > 0$. Then for any δ such that $0 < \delta < d$, $\overline{\varphi(D_\delta(u))} \supset \mathbf{T}$. By Lemma 2.5, for any $f \in H^\infty$ such that $\|f\|_\infty = 1$,

$$\begin{aligned} 1 &= \sup\{|f(z)| : z \in \varphi(D_\delta(u))\} \\ &= \sup\{|f(\varphi(z))| : z \in D_\delta(u)\} \\ &\le \delta^{-1} \sup\{|u(z)|\,|f(\varphi(z))| : z \in D_\delta(u)\} \\ &\le \delta^{-1}\|uC_\varphi f\|_\infty. \end{aligned}$$

Hence we have that $j(uC_\varphi) \ge \delta$. Since $\delta \in (0, d)$ is arbitrary, we have that $j(uC_\varphi) \ge d$.

Conversely, suppose that $j(uC_\varphi) > 0$. For any r such that $0 < r < j(uC_\varphi)$, we have that $r < \|uC_\varphi f\|_\infty$ where $\|f\|_\infty = 1$. We will show that $\overline{\varphi(D_r(u))} \supset \mathbf{T}$ by contradiction.

Suppose that there exists $\zeta \in \mathbf{T}\backslash\overline{\varphi(D_r(u))}$. Put

$$f_n(z) = \left(\frac{z+\zeta}{2}\right)^n$$

Clearly, we can see that $f_n \in H^\infty$ and $\|f_n\|_\infty = 1$ for all positive integer n. Since $\zeta \in \mathbf{T}\backslash\overline{\varphi(D_r(u))}$, we have that $|f_1(\varphi(z))| < 1$ for any $z \in D_r(u)$. For enough large n, we can suppose that $|f_n(\varphi(z))| < r\|u\|_\infty^{-1}$ for any $z \in D_r(u)$. Then we have that for any $z \in D_r(u)$,

$$|uC_\varphi f_n(z)| = |u(z) f_n(\varphi(z))| < r\frac{|u(z)|}{\|u\|_\infty} \le r.$$

On the other hand, for any $z \in \mathbf{D}\backslash D_r(u)$,

$$|uC_\varphi f_n(z)| \le r|f_n(\varphi(z))| \le r.$$

Therefore we get $\|uC_\varphi f_n\|_\infty \le r$. Hence we conclude that $j(uC_\varphi) \le r$. This contradicts our assumption. Thus we have that $\overline{\varphi(D_r(u))} \supset \mathbf{T}$ and then $r \le d$. Now we get $j(uC_\varphi) = d$.

Next we prove that $d = m$. Suppose that $m > 0$. Fix $\varepsilon > 0$ such that $m - \varepsilon > 0$. Then for all $\omega \in \mathbf{T}$,

$$\limsup_{\varphi(z_n)\to\omega} |u(z_n)| \ge m - \varepsilon.$$

This means that $\overline{\varphi(D_{m-\varepsilon}(u))} \supset \mathbf{T}$. Now we get $d \ge m - \varepsilon$. Since ε is arbitrary, we have that $d \ge m$.

To complete our proof, we will show that $d \le m$. Suppose that $d > 0$. For $0 < \delta < d$, we have that $\overline{\varphi(D_\delta(u))} \supset \mathbf{T}$. For all $\omega \in \mathbf{T}$, there exists a sequence $\{z_n\} \in D_\delta(u)$ such that $\varphi(z_n) \to \omega$. Moreover we have that

$$\limsup_{\varphi(z_n)\to\omega} |u(z_n)| \geq \delta.$$

This implies that $d \leq m$. This completes our proof. $\square$

Considering the special cases, we have the following.

COROLLARY 2.7. *Let $u \in H^\infty$ and $\varphi \in S(\mathbf{D})$.*
(i) $j(M_u) = \|u\|_{-\infty,\mathbf{T}}$.
(ii) *If $\overline{\varphi(\mathbf{D})} \supset \mathbf{T}$, then $j(C_\varphi) = 1$. Otherwise, $j(C_\varphi) = 0$.*

Next we state the characterization of the closedness of $\operatorname{Ran} uC_\varphi$. We denote by $\hat{f}$ the Gelfand transform of $f \in H^\infty$. Let $M(H^\infty)$ be the maximal ideal space of H^∞. Then the adjoint C_φ^* of C_φ induces a continuous map Φ from $M(H^\infty)$ into $M(H^\infty)$. More precisely we can see that $\widehat{C_\varphi f}(x) = \hat{f}(\Phi(x))$ for $x \in M(H^\infty)$. Let S be the Shilov boundary of $M(H^\infty)$ and $\Delta_\delta(u) = \{x \in S : |\hat{u}(x)| \geq \delta\}$. Hence, combining our result and the result of [3], we get the following corollary.

COROLLARY 2.8. *Let $u \in H^\infty$ and $\varphi \in S(\mathbf{D})$. The followings are equivalent;*
(i) $\operatorname{Ran} uC_\varphi$ *is closed in* H^∞.
(ii) *there exists $\delta > 0$ such that $\overline{\varphi(D_\delta(u))} \supset \mathbf{T}$.*
(iii) *there exists $\delta > 0$ such that $\Phi(\Delta_\delta(u)) \supset S$.*

Now we give a typical example which shows what affects the estimation of the injectivity modulus.

Example 2.9. *Let $u(z) = 1 - z$. Let $\varphi(z) = z$ and $\psi(z) = z^2$. Then $j(uC_\varphi) = 0$ and $j(uC_\psi) = \sqrt{2}$.*

Proof. Indeed, $j(uC_\varphi) = j(M_u) = \|1 - z\|_{-\infty,\mathbf{T}} = 0$.
On the other hand, we have that

$$\begin{aligned} j(uC_\psi) &= \inf_{\omega\in\mathbf{T}} \max\{|1-\zeta| : \zeta^2 = \omega\} \\ &= \inf_{\theta\in[0,\pi]} \max\{|1-e^\theta|, |1+e^\theta|\} = \sqrt{2} \end{aligned}$$ $\square$

In the last of this section, we give the comparison between some norms and minimum moduli of C_φ and M_u. The essential norm $\|T\|_e$ of T is the distance from T to the closed ideal of compact operators, that is, $\|T\|_e = \inf\{\|T + K\| : K \text{ is compact}\}$. It is trivial that T is compact if and only if $\|T\|_e = 0$. It is known that C_φ is compact on H^∞ if and only if $\overline{\varphi(\mathbf{D})} \cap \mathbf{T} \neq \emptyset$. Moreover if C_φ is not compact on H^∞, then $\|C_\varphi\|_e = 1$ (see [7]). On the other hand, in [6], it is estimated that $\|M_u\|_e = \|M_u\| = \|u\|_\infty$. Hence we have the following inequalities.

COROLLARY 2.10. *Let $u \in H^\infty$ and $\varphi \in S(\mathbf{D})$.*
(i) $0 \le k(C_\varphi) \le j(C_\varphi) \le \|C_\varphi\|_e \le \|C_\varphi\| = 1$ *and each of these quantities above is zero or one.*
(ii) $0 \le k(M_u) \le j(M_u) \le \|M_u\|_e = \|M_u\| = \|u\|_\infty$.

3. Minimum moduli of weighted composition operators on A

In this section, we consider weighted composition operators on the disc algebra A. We remark that the phenomena observed through the estimation of the minimum moduli of weighted composition operators on A and H^∞ are very similar. We can prove the following results in the similar method in the case of H^∞. More precisely, we can prove them only in term of the subset of $\mathbf{T}$, without Lemma 2.5. Here we omit the proof.

We start on the trivial cases.

PROPOSITION 3.1. *If $u \equiv 0$ or $\varphi \equiv p \in \overline{\mathbf{D}}$, then $j(uC_\varphi) = k(uC_\varphi) = 0$.*

We suppose that uC_φ is non-trivial, that is, $u \not\equiv 0$ and $\varphi \in S(\overline{\mathbf{D}})$ is not constant. If uC_φ is non-trivial, then uC_φ is injective on A. Hence we can prove the following results as the same way of the cases of H^∞.

THEOREM 3.2. *Let uC_φ be a non-trivial weighted composition operator on A. Then $k(uC_\varphi) > 0$ if and only if u has no zero on $\overline{\mathbf{D}}$ and φ is an automorphism of $\mathbf{D}$. Moreover, in such cases, $k(uC_\varphi) = j(uC_\varphi) = \|u\|_{-\infty,\mathbf{D}}$.*

COROLLARY 3.3. *Let $u \in A$ and $\varphi \in S(\overline{\mathbf{D}})$.*
(i) $k(M_u) = \|u\|_{-\infty,\mathbf{D}}$.
(ii) *If φ is an automorphism of $\mathbf{D}$, $k(C_\varphi) = 1$. Otherwise, $k(C_\varphi) = 0$.*

Next we estimate $j(uC_\varphi)$ on A.

DEFINITION 3.4. *Denote that $T_\delta(u) = \{z \in \mathbf{T} : |u(z)| \ge \delta\}$.*

Since uC_φ is injective, $j(uC_\varphi) > 0$ if and only if Ran uC_φ is closed in A. We can get the following theorem by the similar proof of Theorem 2.6 replacing D_δ by T_δ.

THEOREM 3.5. *Let uC_φ be a non-trivial weighted composition operator on A. Then we have*

$$j(uC_\varphi) = \sup\{\delta : \varphi(T_\delta(u)) \supset \mathbf{T}\} \tag{3}$$

$$= \inf_{\omega \in \mathbf{T}} \sup\{|u(\zeta)| : \varphi(\zeta) = \omega\} \tag{4}$$

where we define the supremum in (3) *is equal to* 0 *if such a constant δ does not exist, and we define also the infimum in* (4) *is equal to* 0 *if $\varphi(\mathbf{T}) \not\supset \mathbf{T}$.*

COROLLARY 3.6. *Let* $u \in A$ *and* $\varphi \in S(\overline{\mathbf{D}})$.
(i) $j(M_u) = \|u\|_{-\infty,\mathbf{T}}$.
(ii) *If* $\varphi(\mathbf{T}) \supset \mathbf{T}$, *then* $j(C_\varphi) = 1$. *Otherwise*, $j(C_\varphi) = 0$.

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