J. SUZUKI AND N. TODA
KODAI MATH. SEM. REP.
26 (1974), 69–74

SOME NOTES ON PICARD CONSTANT

BY JUNJI SUZUKI AND NOBUSHIGE TODA

1. Introduction. Let R be an open Riemann surface, $M(R)$ the set of non-constant meromorphic functions on R and $P(f)$ the number of lacunal values for f belonging to $M(R)$. Introducing the quantity

$$P(R)=\sup_{f\in M(R)} P(f)\,,$$

which is called the Picard constant of R, Ozawa [3] discussed the existence of analytic mappings from R into another Riemann surface. When R is the proper existence domain of an n-valued algebroid function in $|z|<\infty$, it is known that $2\leqq P(R)\leqq 2n$ ([3]). Further, when $n=2$ or 3, there are many interesting results on $P(R)$ ([2], [3], [4], etc.). Recently, Aogai [1] has treated a general case and given the following result:

"Let R be an n-sheeted regularly branched covering surface of $|z|<\infty$. If $P(R)>3n/2$, then $P(R)=2n$ and R can be represented by an algebroid function y such that

$$y^n=(e^H-\alpha)(e^H-\beta)^{n-1}\,,$$

where H is a non-constant entire function and α, β are constants satisfying $\alpha\beta(\alpha-\beta)\neq 0$."

An n-sheeted covering surface is called regularly branched when it has no branch point other than those of order $n-1$.

We should like to generalize some of these results in this paper. Let $w(z)$ be an n-valued transcendental algebroid function in $|z|<\infty$ defined by an irreducible equation

$$A_0(z)w^n+A_1(z)w^{n-1}+\cdots+A_n(z)=0 \tag{1}$$

where $A_0(\not\equiv 0),\cdots,A_n$ are entire functions without common zero and at least one of A_j/A_0 $(j=1,\cdots,n)$ is not rational, and let R_w be the proper existence domain of $w(z)$, which is n-sheeted covering surface of $|z|<\infty$.

Let f be a non-constant meromorphic function on R_w, then there are entire functions $S_0(z)(\not\equiv 0),\cdots,S_n(z)$ without common zero such that

$$S_0(z)f^n+S_1(z)f^{n-1}+\cdots+S_n(z)=0\,.$$

Received Apr. 13, 1973.

We say that f is transcendental if there exists at least one ratio S_j/S_0 $(1 \leqq j \leqq n)$ which is not rational. Let $T(R_w)$ be the set of transcendental meromorphic fuuctions on R_w and

$$F(R_w)=\sup_{f \in T(R_w)} F(f),$$

where $F(f)$ is the number of Picard exceptional values for f. A value w_0 is called Picard exceptional when there are at most a finite number of roots of $f=w_0$ on R_w. By a result of Selberg [6] and definitions, it is clear that

$$2 \leqq P(R_w) \leqq F(R_w) \leqq 2n.$$

As in the case of $P(R)$, it is an interesting problem to determine the quantity $F(R_w)$, which is conformally invariant.

In § 2, we give some cases of R_w such that $F(R_w) \leqq n$ and in § 3, following the method used in [1], we characterize the surface R_w when it is regularly branched and $F(R_w)=2n$.

2. Some cases of R_w such that $F(R_w) \leqq n$. Let $w(z)$ be an n-valued transcendental algebroid function in $|z|<\infty$ defined by (1) and R_w the proper existence domain of $w(z)$. We should like to determine $F(R_w)$. As a first step to this problem, we give the following.

LEMMA 1. *If $w(z)$ has at least $n+1$ Picard exceptional values, then $w(z)$ is of regular growth and the order of $w(z)$ is positive and integral or infinite* ([7], Th. 7).

Let $N(r, R_w)$ be the quantity $N(r, \mathfrak{X})$ defined by Selberg [6] for R_w. We call

$$\limsup_{r \to \infty} \frac{\log N(r, R_w)}{\log r}$$

the order of the branch points of R_w.

THEOREM 1. *Let $w(z)$ and R_w be as above. If $N(r, R_w)$ is of finite order and of irregular growth or of non-integral order or of order zero, then $F(R_w) \leqq n$.*

Proof. Suppose that there exists a transcendental meromorphic function f on R_w such that $F(f) \geqq n+1$. As in § 1, there are entire functions $S_0(z)(\not\equiv 0)$, $\cdots, S_n(z)$ without common zero such that

$$S_0(z)f^n+S_1(z)f^{n-1}+\cdots+S_n(z)=0.$$

The equation is irreducible and f is an n-valued algebroid function in $|z|<\infty$.

By the branch point theorem of Ullrich [8] and the second fundamental theorem of Selberg [6], we nave the following inequalities:

$$N(r, R_w) \leqq 2(n-1)T(r, f)+O(1), \tag{2}$$

$$(3) \qquad (n-1)T(r, f)<\sum_{\iota=1}^{n+1} N(r, a_\iota)+N(r, R_w)+S(r)$$

where a_ι ($\iota=1, \cdots, n+1$) are Picard exceptional values for f and $S(r)$ is the so-called error term in the second fundamental theorem. From (2), the order of f ($=\rho_f$) is not less than that of $N(r, R_w)$ ($=\rho_w$): $\rho_f \geqq \rho_w$.

1) When ρ_w is not integral or $\rho_w=0$. Now $\rho_f \geqq \rho_w$ and ρ_f is positive and integral by Lemma 1, so that $\rho_f > \rho_w$. On the other hand, as there exists a sequence $r_n \nearrow \infty$ such that

$$\lim_{n\to\infty} \frac{S(r_n)}{T(r_n, f)}=0,$$

from (3) we have

$$\rho_f=\lim_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \leqq \liminf_{n\to\infty} \frac{\log N(r_n, R_w)}{\log r_n} \leqq \rho_w$$

because f is of regular growth by Lemma 1 and

$$N(r, a_\iota)=O(\log r)=o(T(r, f)) \qquad (r\to\infty,\ \iota=1, \cdots, n+1).$$

This shows $\rho_f \leqq \rho_w$, which is absurd. That is, it must be $F(f)\leqq n$, so that $F(R_w)\leqq n$.

2) When $N(r, R_w)$ is of irregular growth and $\rho_w<\infty$. From the inequalities (2) and (3) and by Lemma 1, $N(r, R_w)$ must be of regular growth, which is a contradiction. That is, $F(R_w)\leqq n$.

Example 1. Let R be an ultrahyperelliptic surface defined by an equation $y^2=g(z)$ with an entire function $g(z)$ of non-integral order or of finite order and of irregular growth or of order zero. Then, $F(R)=2$. (In the first case, Ozawa [3] proved that $P(R)=2$.)

Example 2. Let R be a regularly branched three sheeted covering surface defined by an equation $y^3=g(z)$ with an entire function $g(z)$ (see [2]). If $g(z)$ is of non-integral order or of finite order and of irregular growth or of order zero, then $F(R)\leqq 3$.

Example 3. Let R be an n-sheeted covering surface defined by an equation $y^n=g(z)$ with an entire function $g(z)$ of non-integral order or of finite order and of irregular growth or of order zero. If the multiplicities of zeros of $g(z)$ are all less than n, then $F(R)\leqq n$.

Example 4. In Theorem 1, even if $w(z)$ is of non-integral order or of finite order and of irregular growth, the conclusion is not always true.

In fact, let $f_1(z)$ be an entire function of non-integral order ρ_1 (resp. of finite order and of irregular growth) greater than 1 and $f_2(z)=f_1(z)^2-e^z-1$. Let $w(z)$ be two-valued algebroid function defined by

$$w^2-2f_1(z)w+f_2(z)=0.$$

Then, $w(z)$ is of non-integral order ρ_1 (resp. of finite order and of irregular growth) and $N(r, R_w)=N(r, -1, e^z)$, so that $N(r, R_w)$ is of regular growth of order 1. The function $f=\sqrt{e^z+1}$ is analytic and transcendental on R_w and 1, -1 and ∞ are Picard exceptional for f. This shows $F(R_w)\geqq 3$.

3. Characterization. In this section, we shall give a characterization of some n-sheeted covering surface R defined by a transcendental algebroid function with $F(R)=2n$. Our method used here is essentially due to Aogai [1]. We start from the following lemma.

LEMMA 2. *Let f be an n-valued algebroid function in $|z|<\infty$ satisfying $F(f)=2n$. Then there exist a non-constant entire function $H(z)$, a rational function $R(z)(\not\equiv 0)$ and constants $a_1,a_2,\cdots,a_n$; $b_1,b_2,\cdots,b_n$ such that the defining equation of f has the following form:*

$$F(f,z)\equiv G_1(f)+R(z)e^{H(z)}G_2(f)=0\,, \tag{4}$$

where $G_1(f)=f^n+b_1f^{n-1}+\cdots+b_n$ and $G_2(f)=a_1f^{n-1}+a_2f^{n-2}+\cdots+a_n$.

Further, the two algebraic equations $G_1(z)=0$ and $G_2(z)=0$ have no common root, the roots of $G_1(z)=0$ are non-formal Picard exceptional values and those of $G_2(z)=0$ are formal in the sense of Rémoundos [5].

Proof. Let f be defined by

$$S_0(z)f^n+S_1(z)f^{n-1}+\cdots+S_n(z)=0\,, \tag{5}$$

where $S_0(z)(\not\equiv 0),\cdots,S_n(z)$ are entire functions without common zero. As $F(f)=2n$, we may suppose that ∞ is formal and according to Rémoundos [5], there are $n-1$ finite formal values $\alpha_1,\cdots,\alpha_{n-1}$ and n non-formal values $\beta_1,\cdots,\beta_n$. That is, in (5) $S_0(z)$ is equal to a polynomial $P_0(z)$ and

$$\sum_{k=0}^{n} S_k(z)\alpha_i^{n-k}=P_i(z) \qquad (i=1,\cdots,n-1) \tag{6}$$

$$\sum_{k=0}^{n} S_k(z)\beta_j^{n-k}=Q_j(z)e^{H_j(z)} \qquad (j=1,\cdots,n) \tag{7}$$

where P_i, Q_j $(i=1,\cdots,n-1;\ j=1,\cdots,n)$ are polynomials and H_j $(j=1,\cdots,n)$ are non-constant entire functions satisfying $H_j(0)=0$.

Picking up any two from (7) and all from (6) and eliminating $S_1,\cdots,S_n$, we have, as $S_0=P_0$,

$$c_0P_0+c_1P_1+\cdots+c_{n-1}P_{n-1}+d_\mu Q_\mu e^{H_\mu}+d_\nu Q_\nu e^{H_\nu}=0$$

where $c_0,\cdots,c_{n-1}$, d_μ, d_ν are non-zero constants $(1\leqq\mu\neq\nu\leqq n)$. By the impossibility of Borel's identity (see [5]),

$$H_\mu(z)-H_\nu(z)=\text{constant }(=0)$$

and

$$d_\mu Q_\mu=-d_\nu Q_\nu\,.$$

Noting that μ, ν are any two from $j=1, \cdots, n$ and $H_j(0)=0$ $(j=1, \cdots, n)$, we have

$$H_1=H_2=\cdots=H_n=H, \qquad H(0)=0$$

and

$$Q_j=-\frac{d_1}{d_j}Q_1 \qquad (d_1 d_2 \cdots d_n \neq 0,\ j=1, \cdots, n).$$

Thus, the equation (7) with respect to $S_1, \cdots, S_n$ can be written as follows:

$$\sum_{k=1}^{n} S_k \beta_j^{n-k}=d_j' Q_1 e^H-\beta_j^n P_0 \qquad (j=1, \cdots, n)$$

where $d_j'=-d_1/d_j$. Solving this, we have

$$S_j=a_j Q_1 e^H+b_j P_0 \qquad (j=1, \cdots, n).$$

Substituting these into (5),

$$\begin{aligned} &P_0 f^n+(a_1 Q_1 e^H+b_1 P_0) f^{n-1}+\cdots+(a_n Q_1 e^H+b_n P_0) \\ &=P_0(f^n+b_1 f^{n-1}+\cdots+b_n)+Q_1 e^H(a_1 f^{n-1}+\cdots+a_n)=0. \end{aligned}$$

Thus, putting $G_1(f)=f^n+b_1 f^{n-1}+\cdots+b_n$, $G_2(f)=a_1 f^{n-1}+\cdots+a_n$ and $R(z)=Q_1(z)/P_0(z)$, the defining equation of f becomes

$$G_1(f)+R(z)e^{H(z)}G_2(f)=0.$$

Naturally, $R\not\equiv 0$ by the assumption and the algebraic equations $G_1(z)=0$ and $G_2(z)=0$ have no common zero because of the irreducibility of the defining equation. It is trivial that the roots of $G_1(z)=0$ are non-formal, those of $G_2(z)=0$ are formal and f has no other finite exceptional values.

Using this lemma, we can prove the following

THEOREM 2. *Let R be an n-sheeted regularly branched covering surface of $|z|<\infty$ defined by a transcendental algebroid function in $|z|<\infty$. If $F(R)=2n$, then R can be represented by an algebroid function*

$$y^n=(R(z)e^{H(z)}-\alpha)(R(z)e^{H(z)}-\beta)^{n-1},$$

where $H(z)$ is a non-constant entire function, $R(z)(\not\equiv 0)$ is rational and α, β are constants with $\alpha\beta(\alpha-\beta)\neq 0$.

Proof. By the assumption $F(R)=2n$, there exists a transcendental meromorphic function f on R such that $F(f)=2n$. f may be regarded as an n-valued algebroid function in $|z|<\infty$ having $2n$ Picard exceptional values. By Lemma 2, we may suppose that f is defined by (4). The equation (4) is irreducible and the proper existence domain of f is conformally equivalent to R. Therefore, using that for any $\gamma\in\{0<|z|<\infty\}$, the equation $R(z)e^{H(z)}=\gamma$ has an infinite number of simple roots, we can prove this theorem as in the proof of Theorem 2 in [1].

References

[1] Aogai, H., Picard constant of a finitely sheeted covering surface. Kōdai Math. Sem. Rep. 25 (1973), 219-224.

[2] Hiromi, G. and K. Niino, On a characterization of regularly branched three sheeted covering Rimann surfaces. Kōdai Math. Sem. Rep. 17 (1965), 250-260.

[3] Ozawa, M., On complex analytic mappings. Kōdai Math. Sem. Rep. 17 (1965), 93-102.

[4] Ozawa, M., On ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. 17 (1965), 103-108.

[5] Rémoundos, G., Extensions aux fonctions algébroïdes multiformes du théorème du M. Picard et de ses généralisations. Mém. Sci. Math. Paris (1927).

[6] Selberg, H. L., Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale. Avh. Norske Vid. Akad. Oslo 8 (1934), 1-72.

[7] Toda, N., Sur la croissance de fonctions algébroïdes à valeurs déficientes. Kōdai Math. Sem. Rep. 22 (1970), 324-337.

[8] Ullrich, E., Uber den Einfluss der Verzweigtheit einer Algebroide auf ihre Wertverteilung. J. rei. und angw. Math. 167 (1931), 198-220.

Mie University and Nagoya University.