F. BRICKELL AND K. YANO
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CONCURRENT VECTOR FIELDS AND MINKOWSKI STRUCTURES

BY FREDERICK BRICKELL AND KENTARO YANO

§1. **Concurrent vector fields.** We make the general assumption that all the differentiable manifolds and geometric objects which we use are of class C^∞. Let M be a differentiable manifold and ∇ a linear connection on M. A vector field Λ on M is *concurrent* with respect to ∇ if

$$\nabla_u \Lambda = u$$

for all vectors u tangent to M. ([4])

Example. Let V be a real vector space of dimension n and choose a basis $E_1, \cdots, E_n$ for V. A vector $v \in V$ can be expressed uniquely as

$$v = \sum_i x^i(v) E_i, \qquad i=1, \cdots, n$$

and the *standard* chart $(x^1, \cdots, x^n)$ defines a manifold structure on V which is independent of the particular basis chosen. The vector field $\sum_i x^i(\partial/\partial x^i)$ also is independent of the chosen basis and we call it the *radial vector field* on V. The conditions

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = 0, \qquad i, j=1, \cdots, n$$

determine a complete linear connection on V which we call the *standard* connection on V. The radial vector field is concurrent with respect to the standard connection.

A riemannian metric g on M determines a unique connection on M called a riemannian connection. We say that Λ is concurrent with respect to g if it is concurrent with respect to the corresponding riemannian connection.

Example. Let $x^1, \cdots, x^n$ be a standard chart on the real vector space V. If $[a_{ij}]$ is a constant positive definite matrix then the conditions

$$g(\partial/\partial x^i, \partial/\partial x^j) = a_{ij}, \qquad i, j=1, \cdots, n$$

determine a riemannian metric g on V. The corresponding riemannian connection is the standard connection. Consequently the radial vector field is

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concurrent with respect to g. The riemannian metrics obtained in this way are all euclidean metrics. We will see in Theorem 2 that they are all the metrics on V for which the radial vector field is concurrent.

Our first theorem states necessary and sufficient conditions for a vector field to be concurrent with respect to a riemannian metric.

THEOREM 1. *A vector field Λ on a manifold M is concurrent with respect to a riemannian metric g if, and only if,*

$$\Lambda = \operatorname{grad} F, \qquad L_\Lambda g = 2g$$

where $F=(1/2)g(\Lambda, \Lambda)$ and L_Λ denotes the Lie derivative with respect to Λ.

Proof. We show that the conditions are both necessary conditions. Let X, Y be any two vector fields in M and let ∇ denote the riemannian connection. Because Λ is concurrent we find that

$$XF = \frac{1}{2} X(g(\Lambda, \Lambda)) = g(\nabla_X \Lambda, \Lambda) = g(X, \Lambda)$$

and therefore $\Lambda = \operatorname{grad} F$. Secondly we have, using $L_\Lambda X = \nabla_\Lambda X - X$ and $L_\Lambda Y = \nabla_\Lambda Y - Y$, that

$$\begin{aligned}(L_\Lambda g)(X, Y) &= \Lambda(g(X, Y)) - g(L_\Lambda X, Y) - g(X, L_\Lambda Y)\\ &= \nabla_\Lambda(g(X, Y)) - g(\nabla_\Lambda X - X, Y) - g(X, \nabla_\Lambda Y - Y)\\ &= 2g(X, Y).\end{aligned}$$

We show that together the conditions are sufficient. The first condition $XF = g(X, \Lambda)$ and the identity $[X, Y]F = X(YF) - Y(XF)$ lead to

$$\Phi(X, Y) = \Phi(Y, X) \tag{1}$$

where $\Phi(X, Y) = g(X, \nabla_Y \Lambda - Y)$. The second condition $(L_\Lambda g)(X, Y) = 2g(X, Y)$, written as $g(\nabla_X \Lambda, Y) + g(X, \nabla_Y \Lambda) = 2g(X, Y)$, gives at once

$$\Phi(X, Y) + \Phi(Y, X) = 0. \tag{2}$$

The relations (1), (2) together imply that $\Phi(X, Y) = 0$ for all vector fields X, Y in M. Consequently

$$\nabla_Y \Lambda - Y = 0$$

for all vector fields Y in M and therefore Λ is concurrent with respect to the riemannian metric g.

As an application of Theorem 1 we prove

THEOREM 2. *Let V be a real vector space of dimension n and origin O. The riemannian metrics on $V-\mathrm{O}$ for which the radial vector field is concurrent are given in terms of a standard chart by*

$$g=\sum_{i,j} g_{ij}dx^i dx^j$$

where the functions g_{ij} are positively homeogeneous of degree zero and satisfy the relations

$$\sum_{i,j}(\partial g_{ij}/\partial x^k)x^i x^j=0\,,\qquad i, j, k=1, \cdots, n\,.$$

The only riemannian metrics on V for which the radial vector field is concurrent are the euclidean metrics $\sum_{ij}a_{ij}dx^i dx^j$ where the a_{ij} are constants.

Proof. Let Λ denote the radial vector field. In terms of a standard chart $x^1, \cdots, x^n$ we have, for any riemannian metric $g=\sum_{i,j}g_{ij}dx^i dx^j$,

$$F=\frac{1}{2}g(\Lambda, \Lambda)=\frac{1}{2}\sum_{i,j}g_{ij}x^i x^j$$

and

$$(L_\Lambda g)_{ij}=\sum_k x^k(\partial g_{ij}/\partial x^k)+2g_{ij}\,.$$

Consequently the conditions in Theorem 1 translate to

$$\partial F/\partial x^k=\sum_j g_{kj}x^j\,,\qquad \sum_k x^k(\partial g_{ij}/\partial x^k)=0\,.$$

The first condition is equivalent to

$$\sum_{i,j}(\partial g_{ij}/\partial x^k)x^i x^j=0$$

and the second condition is equivalent to the condition that the functions g_{ij} be positively homegeneous of degree zero. It follows at once from the homogeneity condition that the only metrics which extend to V are those for which the functions g_{ij} are constants.

We describe some special metrics on $V-O$ for which the radial vector field is concurrent. Let L be a positive C^∞ function on $V-O$, positively homogeneous of degree one, and such that the matrix of elements

$$g_{ij}=\partial^2\left(\frac{1}{2}L^2\right)/\partial x^i\partial x^j$$

is positive definite. We extend the domain of L to V by defining $L(0)=0$ and call the pair (V, L) a *Minkowski structure.* If L is symmetric, that is $L(-v)=L(v)$, then L is a norm on V. The riemannian metric $\sum_{i,j}g_{ij}dx^i dx^j$ is defined on $V-O$ and is independent of the particular standard chart $x^1, \cdots, x^n$. We call it the riemannian metric associated with the Minkowski structure. When L is a euclidean norm we obtain the euclidean metrics for which the functions g_{ij} are constants. These metrics extend to complete metrics on V. We prove

THEOREM 3. *A riemannian metric g on $V-O$ is associated with a Minkowski structure on V if and only if:—*

(i) *The radial vector field Λ is concurrent with respect to g.*

(ii) *For any vector fields X, Y in $V-O$*

$$g(\tilde{\nabla}_X Y - \nabla_X Y, \Lambda)=0$$

where $\tilde{\nabla}$ is the riemannian connection determined by g and ∇ is the standard connection on V.

Proof. We work with a standard chart $x^1, \cdots, x^n$ on V. The necessity of the conditions follows easily from the Euler relations for a homogeneous function. To prove the sufficiency we begin with condition (i). We write

$$g=\sum_{i,j} g_{ij}dx^i dx^j, \qquad F=\frac{1}{2}\sum_{i,j} g_{ij}x^i x^j$$

and apply Theorem 2 to show that $\partial F/\partial x^i = \sum_k g_{ik}x^k$, and that each function g_{ij} is positively homogeneous of degree zero. Consequently, using the fact that $\partial^2 F/\partial x^i \partial x^j$ is symmetric in i, j, it follows that

$$\sum_k \frac{\partial g_{ik}}{\partial x^j}x^k = \sum_k \frac{\partial g_{jk}}{\partial x^i}x^k \text{ and that } \sum_k \frac{\partial g_{ij}}{\partial x^k}x^k = 0. \tag{3}$$

The condition (ii) can be expressed as

$$\frac{1}{2}\sum_k\left(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right)x^k=0, \qquad i, j, k=1, \cdots, n.$$

The relations (3) enable us to deduce that

$$\sum_k (\partial g_{ik}/\partial x^j)x^k = 0.$$

But this fact implies that

$$\frac{\partial^2 F}{\partial x^i \partial x^j}=g_{ij}$$

and therefore g is associated with the Minkowski structure (V, L) where $L^2=2F$.

§ 2. Vector fields concurrent with respect to complete connections.

§ 2. Vector fields concurrent with respect to complete connections. The condition that a vector field is conncurrent with respect to a complete linear connection appears to be a very strong condition. We will prove two theorems concerning linear connections which are either riemannian or closely related to riemannian connections. We begin with two lemmas.

LEMMA 1. *Let M be a differentiable manifold and suppose that Λ is a vector field on M which is concurrent with respect to a complete linear connection. Then Λ is complete.*

Proof. Let m be any point in M. We have to show tuat the maximal integral curve of Λ starting from m is defined for all values of its parameter. Because ∇ is complete there exists a geodesic γ, defined for all values of its parameter s, and such that

$$\gamma(1)=m\,, \qquad \frac{d\gamma}{ds}(1)=\Lambda(m)\,.$$

Consider the vector field along γ defined by

$$U=s\frac{d\gamma}{ds}-\Lambda(\gamma(s))\,.$$

As γ is a geodesic and Λ is concurrent with respect to ∇ we find that

$$\nabla_{\frac{d\gamma}{ds}}U=\frac{d\gamma}{ds}-\nabla_{\frac{d\gamma}{ds}}\Lambda=\frac{d\gamma}{ds}-\frac{d\gamma}{ds}=0\,.$$

Therefore, because $U(1)=0$, it follows that U is the zero vector field among γ.

Consider the curve c defined by $c(t)=\gamma(e^t)$. We find that

$$c(0)=m\,,$$

$$\frac{dc}{dt}=e^t\frac{d\gamma}{ds}(e^t)=U(e^t)+\Lambda(c(t))=\Lambda(c(t))\,.$$

Consequently c is an integral curve of Λ starting from m. As c is defined for all values of its parameter t it follows that Λ is complete.

LEMMA 2. *Let E denote euclidean space and ∇ the corresponding riemannian connection. Then any vector field Λ on E which is concurrent with respect to ∇ has just one zero.*

Let O *denote this zero and regard E as a euclidean vector space V of origin* O. *Then Λ is the radial vector field on V.*

Proof. We use rectangular cartesian coordinates $x^1, \cdots, x^n$. Let $\Lambda=\sum_i \lambda^i\,\partial/\partial x^i$ be a vector field on E. Because

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j)=0\,, \qquad i, j,=1, \cdots, n$$

it follows that

$$\nabla_{\partial/\partial x^i}\Lambda=\sum_j(\partial\lambda^j/\partial x^i)\,\partial/\partial x^j\,.$$

Consequently Λ is concurrent if and only if $\partial\lambda^j/\partial x^i=\delta_i^j$ or $\lambda^j=x^j+a^j$ where $a^1, \cdots, a^n$ are constants. Therefore Λ has just one zero at the point of coordinates $(-a^1, \cdots, -a^n)$. The rest of the lemma follows easily.

THEOREM 4. *Suppose that M is a connected and complete riemannian manifold and that Λ is a vector field on M, concurrent with respect to the riemannian connection. Then Λ has just one zero and M is isometric with euclidean space.* (See [1] and [3])

Proof. Let g denote the riemannian metric on M. Lemma 1 shows that Λ is complete and therefore generates a one parameter group of transformation of M. According to Theorem 2, $L_\Lambda g=2g$ so that these transformations are all homothetic transformations of M. Apart from the identity transformation they are not isometries. Consequently Lemma 2 on p. 242 of [2], Vol. I, shows that M is locally euclidean.

As M is complete it is covered isometrically by euclidean space E (see, for example, [2], Vol. II, pp. 102-105). The vector field Λ lifts to a vector field $\tilde{\Lambda}$ on E which is concurrent with respect to the riemannian metric on E. Lemma 2 shows that $\tilde{\Lambda}$ has just one zero. Therefore Λ has just one zero and E covers M just once. Consequently M is isometric with E.

Theorem 4 and Lemma 2 show that, to within isometries, the only examples of a vector field concurrent with respect to a complete riemannian metric are the radial vector fields on euclidean vector spaces. Together with Theorem 3 they provide a characterisation of Minkowski structures in terms of such a vector field.

Our final theorem is a slight generalisation of Theorem 4. Again it shows that essentially the only examples of a vector field concurrent with respect to a complete linear connection of a special type, are the radial vector fields on real vector spaces.

THEOREM 5. *Let M be a connected riemannian manifold with metric g. Let ∇ be a complete linear connection on M which preserves g and has the same geodesics as g. Suppose that Λ is a vector field on M which is concurrent with respect to ∇. Then Λ has just one zero and M is isometric with euclidean space.*

Let O denote this zero and regard M as a euclidean vector space V of origin O. Then Λ is the radial vector field on V.

Proof. Because ∇ has the same geodesics as g it follows that g is complete. Therefore Theorem 5 is an immediate consequence of Theorem 4 and Lemma 2, once we have shown that Λ is also concurrent with respect to the riemannian connection.

Let X, Y, Z be any vector fields in M and let T denote the torsion of ∇. The symmetric connection $\tilde{\nabla}$ associated with ∇ is given by

$$\tilde{\nabla}_X Y=\nabla_X Y-\frac{1}{2}T(X, Y)\,.$$

$\tilde{\nabla}$ has the same geodesics as ∇ and therefore coincides with the riemannian connection. Because both ∇ and $\tilde{\nabla}$ preserve the riemannian metric it follows that

$$g(T(X, Y), Z)+g(T(X, Z), Y)=0\,. \tag{4}$$

We put $F=(1/2)g(\Lambda, \Lambda)$ and calculate XF. We find by using (4) and the fact that Λ is concurrent with respect to ∇

$$\begin{aligned} XF &= g(\tilde{\nabla}_X A, A) \\ &= g(\nabla_X A, A) + g(T(X, A), A) \\ &= g(X, A) \end{aligned}$$

and therefore

$$A = \operatorname{grad} F. \tag{5}$$

A similar calculation gives

$$\begin{aligned} (L_A g)(X, Y) &= g(\tilde{\nabla}_X A, Y) + g(X, \tilde{\nabla}_Y A) \\ &= g(\nabla_X A, Y) + g(X, \nabla_Y A) + g(T(X, A), Y) + g(T(Y, A), X) \\ &= 2g(X, Y) \end{aligned}$$

and therefore

$$L_A g = 2g. \tag{6}$$

According to Theorem 1 the relations (5) and (6) together imply that A is concurrent with respect to $\tilde{\nabla}$.

We conclude with the remark that it is very easy to construct an example of a non-symmetric connection which satisfies the conditions in Theorem 5. For instance let V be a real vector space of dimension 4 and choose a standard chart $x^1, \cdots, x^4$. Define a connection ∇ by

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \sum_k C_{ijk}\partial/\partial x^k, \qquad i, j, k = 1, \cdots, 4$$

where the functions C_{ikj} are skew-symmetric in each pair of indices and

$$C_{123} = x^4, \qquad C_{124} = -x^3, \qquad C_{134} = x^2, \qquad C_{234} = -x^1.$$

This connection preserves the euclidean metric $g = (dx^1)^2 + \cdots + (dx^4)^2$ and has the same geodesics as g. The radial vector field is concurrent with respect to ∇.

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University of Waterloo,
Ontario, Canada