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ON CERTAIN (f, g, u, v, λ)-STRUCTURES

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§ 0. Introduction.

Yano and Okumura introduced what they call an (f, g, u, v, λ)-structure, where f is a tensor field of type (1, 1), g a Riemannian metric, u and v 1-forms and λ is function satisfying

$$f^2=-I+u\otimes U+v\otimes V,$$

$$u\circ f=\lambda v, \qquad v\circ f=-\lambda u,$$

$$fU=-\lambda V, \qquad fV=\lambda U,$$

$$u(U)=1-\lambda^2, \qquad v(V)=1-\lambda^2,$$

$$u(V)=0, \qquad v(U)=0$$

and

$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y)$$

for arbitrary vector fields X and Y, where U and V are vector fields associated with 1-forms u and v respectively.

Submanifolds of codimension 2 in an almost Hermitian manifold or hypersurfaces in an almost contact metric manifold admit an (f, g, u, v, λ)-structure ([3], [2]).

If an (f, g, u, v, λ)-structure satisfies $S=0$, where S is a tensor field of type (1,2) defined by

$$S(X, Y)=[f, f](X, Y)+(du)(X, Y)U+(dv)(X, Y)V$$

for arbitrary vector fields X and Y, $[f, f]$ being the Nijenhuis tensor formed with f, the structure is said to be normal ([3]). We put

$$T(X, Y, Z)=g(S(X, Y), Z).$$

If

$$T(X, Y, Z)-\{(dw)(fX, Y, Z)-(dw)(fY, X, Z)=0,$$

w being a tensor field of type (0, 2) defined by $w(X, Y)=g(fX, Y)$ for arbitrary vector fields X, Y and Z, then we say that the (f, g, u, v, λ)-structure is quasi-normal ([2]).

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A typical example of a differentiable manifold with a normal (or quasi-normal) (f, g, u, v, λ)-structure is an even-dimensional sphere S^{2n}. Yano and one of the present authors proved the following two theorems from this point of view.

THEOREM 0.1. *In a manifold with* (f, g, u, v, λ)*-structure such that the function* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, the conditions*

$$\mathcal{L}_U g = -2\alpha\lambda g \quad \text{and} \quad dv = 2\alpha w$$

are equivalent, where $\mathcal{L}_U$ *denotes the operator of Lie differentiation with respect to the vector field* U *and* α *is a function. In particular, if* α *is non-zero constant, then the structure is normal.*

THEOREM 0.2. *Let* M *be a complete normal (or quasi-normal)* (f, g, u, v, λ)*-structure satisfying*

$$\mathcal{L}_U g = -2c\lambda g \quad \text{or} \quad dv = 2cw,$$

c being a non-zero constant. If $\lambda(1-\lambda^2)$ *is almost everywhere non-zero function and* $\dim M > 2$, *then* M *is isometric with an even-dimensional sphere* S^{2n}.

The main purpose of the present paper is to prove the following

THEOREM A. *Let* M *be a complete quasi-normal* (f, g, u, v, λ)*-structure satisfying one of the following conditions:*

$$\mathcal{L}_U g = 2\alpha\lambda g, \tag{0.1}$$

$$du = 2\beta w, \tag{0.2}$$

$$\mathcal{L}_V g = 2\gamma\lambda g, \tag{0.3}$$

$$dv = 2\delta w, \tag{0.4}$$

α, β, γ *and* δ *being non-zero functions. If* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero function and* $\dim M > 2$, *then* M *is isometric with an even-dimensional sphere* S^{2n}.

In the sequel, we assume that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and use the index notation.

In section 1, we prove that a quasi-normal (f, g, u, v, λ)-structure satisfying (0.1) and (0.3) implies $dw=0$.

In section 2, we prove theorem A and its corollary.

In the last section 3, as an application of theorem A, we study a totally umbilical submanifold of codimension 2 with a quasinormal (f, g, u, v, λ)-structure in almost Tachibana manifold.

§ 1. Quasi-normal (f, g, u, v, λ)-structure.

We consider a C^∞ differentiable manifold M with an (f, g, u, v, λ)-structure, that is, a Riemannian manifold with metric tensor g which admits a tensor field f of type (1, 1), two 1-forms u and v (or two vector fields associated with them), and a function λ satisfying

$$
\begin{aligned}
& f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h, \\
& f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\
(1.1) \qquad & u_t f_i{}^t = \lambda v_i \quad \text{or} \quad f_t{}^h u^t = -\lambda v^h, \\
& v_t f_i{}^t = -\lambda u_i \quad \text{or} \quad f_t{}^h v^t = \lambda u^h, \\
& u_t u^t = 1-\lambda^2, \qquad v_t v^t = 1-\lambda^2, \qquad u_t v^t = 0,
\end{aligned}
$$

$$(1.2) \qquad f_{ji} = g_{ti} f_j{}^t$$

being skew-symmetric. Such an M is even-, say, $2n$-dimensional.

We put

$$
\begin{aligned}
(1.3) \qquad S_{ji}{}^h = & f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h \\
& + u_{ji} u^h + v_{ji} v^h,
\end{aligned}
$$

where

$$u_{ji} = \nabla_j u_i - \nabla_i u_j, \qquad v_{ji} = \nabla_j v_i - \nabla_i v_j,$$

and ∇_j denotes the operator differentiation with respect to the Riemannian connection.

If the tensonsor $S_{ji}{}^h$ vanishes, the (f, g, u, v, λ)-structure is said to be normal. If the condition

$$(1.4) \qquad S_{jih} - (f_j{}^t f_{tih} - f_i f_{tjh}) = 0,$$

where

$$S_{jih} = g_{th} S_{ji}{}^t \quad \text{and} \quad f_{jih} = \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji},$$

is satisfied, then we say that the (f, g, u, v, λ)-structure is quasi-normal ([2]).

Yano and one of the present authors derived the following general formulas in a manifold with an (f, g, u, v, λ)-structure

$$
\begin{aligned}
(1.5) \qquad v_t S_{ji}{}^t = & v_{ji} - f_j{}^t f_i{}^s v_{ts} - \lambda (f_j{}^t u_{ti} - f_i{}^t u_{tj}) \\
& - (f_j{}^t u_i - f_i{}^t u_j) \nabla_t \lambda + \lambda \{(\nabla_j \lambda) v_i - (\nabla_i \lambda) v_j\},
\end{aligned}
$$

$$
\begin{aligned}
(1.6) \qquad & S_{jih} - (f_j{}^t f_{tjh} - f_i{}^t f_{tjk}) \\
& = -(f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj}) + u_j (\nabla_i u_h) - u_i (\nabla_j u_h) + v_j (\nabla_i v_h) - v_i (\nabla_j v_h),
\end{aligned}
$$

$$
\begin{aligned}
(1.7) \qquad & u^j [S_{jih} - (f_j{}^t f_{tih} - f_i{}^t f_{tjh})] \\
& = \mathcal{L}_u g_{ih} - u_i u^t \mathcal{L}_u g_{th} + \lambda f_i{}^t \mathcal{L}_v g_{th} - \lambda^2 u_{ih} - (\lambda f_i{}^t + v_i u^t) v_{th},
\end{aligned}
$$

$$
\begin{aligned}
&v^j[S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})]\\
&\quad =\mathcal{L}_v g_{ih}-v_i v^t \mathcal{L}_v g_{th}-\lambda f_i{}^t \mathcal{L}_u g_{th}-\lambda^2 v_{ih}+(\lambda f_i{}^t-u_i v^t)u_{th},
\end{aligned} \tag{1.8}
$$

where $\mathcal{L}_u$ and $\mathcal{L}_v$ denote Lie differentiation with respect to u^h and v^h respectively.

LEMMA 1.1. *In a manifold with quasi-normal (f, g, u, v, λ)-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, we have* [2]

$$\lambda(1-\lambda^2)u_{ji}=u_i f_j{}^s u^t \mathcal{L}_v g_{st}-\{\lambda u_i v^t+(1-\lambda^2)f_i{}^t\}\mathcal{L}_v g_{jt}, \tag{1.9}$$

$$\lambda(1-\lambda^2)v_{ji}=-v_i f_j{}^s v^t \mathcal{L}_u g_{st}-\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}\mathcal{L}_u g_{jt}, \tag{1.10}$$

and consequently

$$(\mathcal{L}_u g_{ji})u^j v^i=0, \qquad (\mathcal{L}_v g_{ji})u^j v^i=0. \tag{1.11}$$

We now prove

LEMMA 1.2. *A quasi-normal (f, g, u, v, λ)-structure such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero and satisfies*

$$\mathcal{L}_u g_{ji}=2\alpha\lambda g_{ji}, \tag{1.12}$$

$$\mathcal{L}_v g_{ji}=2\beta\lambda g_{ji}, \tag{1.13}$$

α and β being functions implies $f_{ji}=0$.

Proof. Substituting (1.4), (1.12) and (1.13) into (1.7) and (1.8), we have respectively

$$v_{ji}=-2\alpha f_{ji}, \qquad u_{ji}=2\beta f_{ji},$$

from which, using (1.12) and (1.13),

$$\nabla_j v_i=\beta\lambda g_{ji}-\alpha f_{ji}, \tag{1.14}$$

$$\nabla_j u_i=\alpha\lambda g_{ji}+\beta f_{ji}. \tag{1.15}$$

Differentiating $u_t u^t=1-\lambda^2$ covariantly and taking account of (1.15), we find

$$\nabla_j \lambda=-\alpha u_j-\beta v_j. \tag{1.16}$$

Using (1.4), (1.14) and (1.15), we get from (1.6)

$$
\begin{aligned}
&f_j{}^t \nabla_h f_{ti}-f_i{}^t \nabla_h f_{tj}\\
&=u_j(\alpha\lambda g_{ih}+\beta f_{ih})-u_i(\alpha\lambda g_{jh}+\beta f_{jh})\\
&\quad +v_j(\beta\lambda g_{ih}-\alpha f_{ih})-v_i(\beta\lambda g_{jh}-\alpha f_{jh}),
\end{aligned}
$$

or equivalently,

$$\nabla_h(f_j{}^t f_{ti})+2f_i{}^t\nabla_h f_{jt}$$
$$=\lambda(\alpha u_j+\beta v_j)g_{ih}-\lambda(\alpha u_i+\beta v_i)g_{jh}+(\beta u_j-\alpha v_j)f_{ih}-(\beta u_i-\alpha v_i)f_{jh},$$

from which, using (1.1), (1.14) and (1.15),

$$f_j{}^t\nabla_h f_{it}=-\lambda(\alpha u_j+\beta v_j)g_{ih}+(\beta u_i-\alpha v_i)f_{jk}. \tag{1.17}$$

Transvecting (1.17) with $f_k{}^j$ and using (1.1), we find

$$-\nabla_h f_{ik}+u_k u^t\nabla_h f_{it}+v_k v^t\nabla_h f_{it}$$
$$=\lambda^2(\beta u_k-\alpha v_k)g_{ih}+(\beta u_i-\alpha v_i)(-g_{kh}+u_k u_h+v_k v_h),$$

or, using (1.1) again

$$-\nabla_h f_{ik}+u_k(v_i\nabla_h\lambda+\lambda\nabla_h u_i-f_i{}^t\nabla_h u_t)-v_k(u_i\nabla_h\lambda+\lambda\nabla_h u_i+f_i{}^t\nabla_h v_t)$$
$$=\lambda^2(\beta u_k-\alpha v_k)g_{ih}+(\beta u_i-\alpha v_i)(-g_{kh}+u_k u_h+v_k v_h).$$

Substituting (1.14), (1.15) and (1.16) into the last equation, we find

$$\nabla_h f_{ik}=-g_{hi}(\beta u_k-\alpha v_k)+g_{kh}(\beta u_i-\alpha v_i),$$

from which, $f_{hik}=0$. This completes the proof of the lemma.

§2. Quasi-normal (f, g, u, v, λ)-structure satisfying $\mathcal{L}_u g_{ji}=2\alpha\lambda g_{ji}$.

In this section, we assume that the (f, g, u, v, λ)-structure is quasi-normal and satisfies

$$\mathcal{L}_u g_{ji}=2\alpha\lambda g_{ji}, \tag{2.1}$$

where α is a non-zero function.

By Theorem 0.1, (2.1) is equivalent to

$$v_{ji}=-2\alpha f_{ji}. \tag{2.2}$$

Substituting (1.4), (2.1) and (2.2) into (1.5), we have

$$v^t(f_j{}^s f_{sit}-f_i{}^s f_{sjt})$$
$$=2\alpha\lambda(u_j v_i-u_i v_j)-2\lambda(f_j{}^t\nabla_t u_i-f_i{}^t\nabla_t u_j-2\alpha\lambda f_{ji})$$
$$-(f_j{}^t u_i-f_i{}^t u_j)\nabla_t\lambda+\lambda[(\nabla_j\lambda)v_i-(\nabla_i\lambda)v_j].$$

Transvecting this equation with $u^j v^i$ and taking account of the skew-symmetry of f_{jih} and u_{ji}, we find

$$0=2\alpha\lambda(1-\lambda^2)^2+2\lambda(1-\lambda)^2 u^t\nabla_t\lambda,$$

that is,

$$u^t \nabla_t \lambda = -\alpha(1-\lambda^2). \tag{2.3}$$

Moreover, differentiating $v_t v^t = 1-\lambda^2$ covariantly and using (2.2), we find $v^t(\nabla_t v_j - 2\alpha f_{jt}) = -\lambda \nabla_j \lambda$, that is,

$$v^t \nabla_t v_j = -\lambda \nabla_j \lambda - 2\alpha\lambda u_j. \tag{2.4}$$

Similarly we can prove from $u_t u^t = 1-\lambda^2$ and (2.1) that

$$u^t \nabla_t u_j = \lambda \nabla_j \lambda + 2\alpha\lambda u_j. \tag{2.5}$$

Substituting (2.1) and (2.2) into (1.7) and taking account of (1.4), we have $f_i{}^t \mathcal{L}_v g_{th} = \lambda u_{ih}$, or,

$$f_i{}^t(2\nabla_t v_h + v_{ht}) = \lambda(\mathcal{L}_u g_{ih} - 2\nabla_h u_i),$$

or, using (2.1) and (2.2) again,

$$\lambda \nabla_h u_i + f_i{}^t \nabla_t v_h = \alpha(1+\lambda^2) g_{ih} - \alpha(u_i u_h + v_i v_h). \tag{2.6}$$

Transvecting (2.6) with v^h and using (2.1), we have

$$\lambda v^t(-\nabla_i u_t + 2\alpha\lambda g_{ti}) - \lambda f_i{}^t \nabla_t \lambda = 2\alpha\lambda^2 v_i,$$

that is,

$$-v^t \nabla_i u_t = u^t \nabla_i v_t = f_i{}^t \nabla_t \lambda. \tag{2.7}$$

On the other hand, taking the symmetric part of (2.6) with respect to h and i, we obtain

$$\lambda(\nabla_h u_i + \nabla_i u_h) + f_i{}^t \nabla_t v_h + f_h{}^t \nabla_t v_i$$

$$= 2\alpha(1+\lambda^2) g_{ih} - 2\alpha(u_i u_h + v_i v_h).$$

Transvecting the last equation with u^i and using (2.4), (2.5) and (2.7), we get

$$-\lambda^2 \nabla_h \lambda + \lambda^2(\nabla_h \lambda + 2\alpha u_h) + \lambda^2(\nabla_h \lambda + 2\alpha u_h) + f_h{}^t f_t{}^s \nabla_s \lambda = 4\alpha^2 u_h,$$

that is,

$$(1-\lambda^2)\nabla_h \lambda = (u^s \nabla_s \lambda) u_h + (v^s \nabla_s \lambda) v_h,$$

or, using (2.3),

$$\nabla_j \lambda = -\alpha u_j + \phi v_j, \tag{2.8}$$

where, the functing ϕ is defined by $(1-\lambda^2)\phi = v^t \nabla_t \lambda$.

Defferentiating (2.8) covariantly, we find

$$\nabla_j \nabla_i \lambda = -\alpha_j u_i - \alpha \nabla_j u_i + \phi_j v_i + \phi \nabla_j v_i,$$

where $\alpha_j = \nabla_j \alpha$, $\phi_j = \nabla_j \phi$, from which,

$$0 = \alpha_j u_i - \alpha_i u_j + \alpha(\nabla_j u_i - \nabla_i u_j) - \phi_j v_i + \phi_i v_j + 2\alpha\phi f_{ji}. \tag{2.9}$$

Transvecting (2.9) with u^i, we have

$$0=(1-\lambda^2)\alpha_j-(\alpha_t u^t)u_j+\alpha(\nabla_j u_t-\nabla_t u_j)u^t+(\phi_t u^t)v_j+2\alpha\phi\lambda v_j,$$

or, using (2.5) and (2.8),

$$(1-\lambda^2)\alpha_j=(\alpha_t u^t)u_j-(\phi_t u^t)v_j. \tag{2.10}$$

Similarly, transvecting (2.9) with v^i and using (2.1), (2.7) and (2.8), we find

$$(1-\lambda^2)\phi_j=-(\alpha_t v^t)u_j+(\phi_t v^t)v_j. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we obtain

$$\begin{aligned}0=&(\alpha_t v^t+\phi_t u^t)(u_j v_i-u_i v_j)\\&+(1-\lambda^2)\alpha(\nabla_j u_i-\nabla_i u_j+2\phi f_{ji}),\end{aligned} \tag{2.12}$$

from which, transvecting u^j and taking account of (2.5) and (2.8),

$$\alpha_t v^t+\phi_t u^t=0. \tag{2.13}$$

Thus, (2.12) becomes

$$\nabla_j u_i-\nabla_i u_j=-2\phi f_{ji} \tag{2.14}$$

by virtue of $\alpha\neq 0$.

Adding (2.1) and (2.14), we find

$$\nabla_j u_i=\alpha\lambda g_{ji}-\phi f_{ji}.$$

Substituting this into (2.6), we obtain

$$f_i{}^t\nabla_t v_h=\lambda\phi f_{hi}+\alpha(g_{ih}-u_i u_h-v_i v_h).$$

from which, transvecting with $f_j{}^i$,

$$\begin{aligned}&-\nabla_j v_h+u_j u^t\nabla_t v_h+v_j v^t\nabla_t v_h\\&=\lambda\phi(g_{jh}-u_j u_h-v_j v_h)+\alpha(f_{jh}-\lambda v_j u_h+\lambda u_j v_h).\end{aligned}$$

or, using (2.4) and (2.5),

$$\nabla_j v_i=-\lambda\phi g_{ji}-\alpha f_{ji},$$

which implies that

$$\mathcal{L}_v g_{ji}=-2\lambda\phi g_{ji}. \tag{2.15}$$

Thus, we have $f_{jih}=0$ because of (2.1), (2.15) and Lemma 1.2. This means that the structure is normal. Taking account of Theorem 0.1, we have

THEOREM 2.1. *A quasi-normal (f, g, u, v, λ)-structure such that λ $(1-\lambda^2)$ is* *almost everywhere non-zero and satisfies one of the following:*

(1) $\mathcal{L}_u g_{ji}=2\alpha\lambda g_{ji}$, (2) $\mathcal{L}_v g_{ji}=2\gamma\lambda g_{ji}$, (3) $\nabla_j u_i-\nabla_i u_j=2\beta f_{ji}$,
(4) $\nabla_j v_i-\nabla_i v_j=2\delta f_{ji}$.

α, β, γ and δ being non-zero functions, is normal.

Now, differentiating (2.2) covariantly, we obtain

$$\nabla_k\nabla_j v_i-\nabla_k\nabla_i v_j=-2(\alpha_k f_{ji}+\alpha\nabla_k f_{ji}),$$

from which, using Ricci identity and $f_{kji}=0$,

$$\alpha_k f_{ji}+\alpha_j f_{ik}+\alpha_i f_{kj}=0. \tag{2.16}$$

Transvecting (2.16) with u^k, we find

$$(u^t\alpha_t)f_{ji}=\lambda\alpha_i v_j-\lambda\alpha_j v_i.$$

Thus, if $\dim M>2$, we have $u^t\alpha_t=0$ because the rank of f_{ji} is almost everywhere maximum.

Similarly, transvecting (2.16) with v^k again, we can verify that $v^t\alpha_t=0$. From the fact that $u^t\alpha_t=0$ and $v^t\alpha_t=0$, we see that $\alpha=$const. by virtue of (2.10) and (2.13).

Therefore, taking account of Theorem 0.2, Theorem 2.1 and $\alpha=$const., we have Theorem A, that is,

THEOREM 2.2. *Let M be a complete manifold with quasi-normal (f, g, u, v, λ)-structure satisfying one of the following:*

(1) $\mathcal{L}_u g_{ji}=2\alpha\lambda g_{ji}$, (2) $\mathcal{L}_v g_{ji}=2\gamma\lambda g_{ji}$,
(3) $\nabla_j u_i-\nabla_i u_j=2\beta f_{ji}$, (4) $\nabla_j v_i-\nabla_i v_j=2\delta f_{ji}$,

α, β, γ and δ being non-zero functions. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero function and $\dim M>2$, *then M is isometric with an even-dimensional sphere S^{2n}.*

COROLLARY 2.3. *Let M be a complete manifold with normal (f, g, u, v, λ)-structure satisfying $\nabla_j f_{ih}+\nabla_i f_{hj}+\nabla_h f_{ji}=0$ and one of* (1)~(4) *in Theorem* 2.1. *If $\lambda(1-\lambda^2)$ is almost everywhere non-zero and* $\dim M>2$, *then M is isometric with an even-dimensional sphere S^{2n}.*

§ 3. An application of main theorem.

In this section, we consider totally umbilical submanifolds of codimension 2 with quasi-normal (f, g, u, v, λ)-structure in an almost Tachibana manifold.

Let $\tilde{M}$ be a $(2n+2)$-dimensional Tachibana manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$ $(\kappa, \lambda, \mu, \nu, \cdots=1, 2, \cdots, 2n+2)$, and let $(F_\lambda{}^\kappa, G_{\mu\lambda})$ be the almost Tachibana structure, that is, $F_\lambda{}^\kappa$ is the almost complex structure;

$$F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda^\kappa, \tag{3.1}$$

and $G_{\mu\nu}$ a Riemannian metric such that

$$G_{\alpha\beta} F_\mu{}^\alpha F_\nu{}^\beta = G_{\mu\nu}, \tag{3.2}$$

and

$$\nabla_\mu F_\lambda{}^\kappa + \nabla_\lambda F_\mu{}^\kappa = 0, \tag{3.3}$$

where we denote by $\{{}_\mu{}^\kappa{}_\lambda\}$ and ∇_μ the Christoffel symbols formed with $G_{\mu\lambda}$ and the operator of covariant differentiation with respect to $\{{}_\mu{}^\kappa{}_\lambda\}$ respectively.

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U; x^h\}$ $(h, i, j, \cdots = 1, 2, \cdots, 2n)$ and which is differentiably immersed in $\tilde{M}$ as a submanifold of codimension 2 by the equations

$$y^\kappa = y^\kappa(x^h).$$

We put

$$B_i{}^\kappa = \partial_i y^\kappa \qquad (\partial_i = \partial/\partial x^i),$$

then $B_i{}^\kappa$ is, for each i, a local vector field of $\tilde{M}$ tangent to M and the vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood.

If we assume that we can choose two mutually orthogonal unit vectors C^κ and D^κ of $\tilde{M}$ normal to M in such a way that $2n+2$ vectors $B_i{}^\kappa$, C^κ, D^κ give the positive orientation of $\tilde{M}$, then the transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$, $F_\lambda{}^\kappa C^\lambda$ of C^λ and $F_\lambda{}^\kappa D^\lambda$ of D^λ by $F_\lambda{}^\kappa$ can be respectively written in the forms

$$\begin{aligned} &F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa, \\ &F_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa + \lambda D^\kappa, \\ &F_\lambda{}^\kappa D^\lambda = -v^i B_i{}^\kappa - \lambda C^\kappa, \end{aligned} \tag{3.4}$$

where $f_i{}^h$ is a tensor field of type (1.1) and u_i, v_i are 1-forms on M, and λ is a function on M, which can easily verify that is globally defined on M. And we have put $u^i = u_t g^{ti}$, $v^i = v_t g^{ti}$, g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$.

Moreover, the aggregate (f, g, u, v, λ) is a so-called (f, g, u, v, λ)-structure, that is, satisfies (1.1).

It is also well known [3] that, from (3.1), (3.2), (3.3) and the equations of Gauss and Weingarten;

$$\begin{aligned} \nabla_j B_i{}^\kappa &= \partial_j B_i{}^\kappa + \left\{ {\kappa \atop \mu \ \lambda} \right\} B_j{}^\mu B_i{}^\lambda - B_h{}^\kappa \left\{ {h \atop j \ i} \right\} \\ &= h_{ji} C^\kappa + k_{ji} D^\kappa, \end{aligned}$$

$$\nabla_j C^\kappa = \partial_j C^\kappa + \left\{ {\kappa \atop \mu \ \lambda} \right\} B_j{}^\mu C^\lambda = -h_j{}^i B_i{}^k - l_j D^\kappa,$$

$$\nabla_j D^\kappa = \partial_j D^\kappa + \left\{ {\kappa \atop \mu\ \ \lambda} \right\} B_j{}^\mu D^\lambda = -k_j{}^i B_i{}^\kappa - l_j C^\kappa,$$

we have

$$(3.5)\qquad \nabla_j f_i{}^h + \nabla_i f_j{}^h = -2h_{ji}u^h + h_j{}^h u_i + h_i{}^h u_j - 2k_{ji}v^h + k_j{}^h v_i + k_i{}^h v_j,$$

$$(3.6)\qquad \nabla_j u_i + \nabla_i u_j = -h_{jt} f_i{}^t - f_{it} f_j{}^t - 2\lambda k_{ji} + l_j v_i + l_i v_j,$$

$$(3.7)\qquad \nabla_j v_i + \nabla_i v_j = -k_{jt} f_i{}^t - k_{it} f_j{}^t + 2\lambda h_{ji} - l_j u_i - l_j u_j,$$

where h_{ji} and k_{ji} are the second fundamental tensors of M with respect to the normals C^κ and D^κ respectively and $h_j{}^i = h_{jt}g^{ti}$, $k_j{}^i = k_{jt}g^{ti}$ and l_j is the third fundamental tensor.

Suppose that M is a non-minimal totally umbilical submanifold, that is,

$$(3.8)\qquad h_{ji} = \frac{1}{2n} h_t{}^t g_{ji}, \qquad k_{ji} = \frac{1}{2n} k_t{}^t g_{ji},$$

$$(3.9)\qquad (h_t{}^t)^2 + (k_t{}^t)^2 \neq 0.$$

Then, we have respectively from (3.6) and (3.7)

$$(3.10)\qquad \nabla_j u_i + \nabla_i u_j = -\frac{1}{n} k_t{}^t \lambda g_{ji} + l_j v_i + l_i v_j,$$

$$(3.11)\qquad \nabla_j v_i + \nabla_i v_j = \frac{1}{n} h_t{}^t \lambda g_{ji} - l_j u_i - l_i u_j$$

by virtue of (3.8).

We consider the set $M_1 = \{x \in M \mid \lambda^2(x) = 1\}$. Then $h_t{}^t = 0$ and $k_t{}^t = 0$ on M_1 because of (3.10) and (3.11). Since M is non-minimal, M_1 is a bordered set and hence $\lambda^2 \neq 1$ almost everywhere in M.

From (1.11), (3.10) and (3.11), we find

$$(3.12)\qquad l_t u^t = 0, \qquad l_t v^t = 0.$$

Substituting (3.10) and (3.12) into (1.10), we obtain

$$\lambda(1-\lambda^2)(\nabla_j v_i - \nabla_i v_j)$$
$$= \frac{1}{n} k_t{}^t \lambda(1-\lambda^2) f_{ij} + (1-\lambda^2)(v_j f_i{}^t l_t - v_i f_j{}^t l_t) - \lambda(1-\lambda^2) l_j u_i,$$

from which, taking the symmetric part, $l_j u_i + l_i u_j = 0$, which implies that $l_j = 0$. Thus (Thus (3.10) and (3.11) become

$$\mathcal{L}_u g_{ji} = -\frac{1}{n} k_t{}^t \lambda g_{ji}, \qquad \mathcal{L}_v g_{ji} = \frac{1}{h} h_t{}^t \lambda g_{ji}.$$

Using Theorem 2.1, we have

THEOREM 3.1. *Let M be a non-minimal totally umbilical submanifold of codimension* 2 *in an almost Tachibana manifold. If the induced (f, g, u, v, λ)-structure on M* ($\dim M>2$) *is quasi-normal and the function λ is almost everywhere non-zero, then M is isometric with an even-dimensional sphere.*

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