NIINO, K.
KŌDAI MATH. SEM. REP.
25 (1973), 385—391

ON THE GROWTH OF ALGEBROID FUNCTIONS OF FINITE LOWER ORDER

BY KIYOSHI NIINO

Dedicated to Professor Yukinari Tôki on his 60th birthday

1. In 1932 Paley [5] conjectured that
an entire function $g(z)$ of order λ satisfies

$$\varliminf_{r\to\infty}\frac{\log M(r, g)}{T(r, g)}\leqq\begin{cases}\dfrac{\pi\lambda}{\sin\pi\lambda} & \left(\lambda\leqq\dfrac{1}{2}\right),\\ \pi\lambda & \left(\lambda>\dfrac{1}{2}\right).\end{cases}$$

This conjecture was proved by Valiron [7] for $\lambda<1/2$. The first complete proof was given by Govorov [2]. A little later Petrenko [6] proved this conjecture for meromorphic functions of finite lower order. And D. F. Shea (cf. [1]) gave an improvement of Petrenko's theorem.

The purpose of this paper is to extend Shea's theorem to n-valued algebroid functions of finite lower order. Let $f(z)$ be an n-valued algebroid function, $f_j(z)$ the j-th determination of $f(z)$ and $T(r, f)$ the characteristic function of $f(z)$. We set

$$M(r, a, f)=\max_{|z|=r}\max_{1\leqq j\leqq n}\frac{1}{|f_j(z)-a|},\qquad a\neq\infty,$$

$$M(r, f)=M(r, \infty, f)=\max_{|z|=r}\max_{1\leqq j\leqq n}|f_j(z)|$$

and

$$\beta(a, f)=\varliminf_{r\to\infty}\frac{\log M(r, a, f)}{T(r, f)}.$$

We shall prove the following extension of Shea's theorem:

THEOREM 1. *Let $f(z)$ be an n-valued transcendental algebroid function of finite lower order μ and $\varDelta(\infty)=\varDelta$ the Valiron deficiency of $f(z)$ at ∞. Then we have*

$$\beta(\infty, f)\leqq n\pi\mu\{\varDelta(2-\varDelta)\}^{1/2}$$

if $\mu\geqq1/2$ or $\mu<1/2$ and $\sin(\pi\mu/2)\geqq(\varDelta/2)^{1/2}$, *and*

Received October 31, 1972.

$$\beta(\infty, f) \leqq n\pi\mu\{\Delta \cot \pi\mu + \tan(\pi\mu/2)\}$$

if $\mu < 1/2$ *and* $\sin(\pi\mu/2) < (\Delta/2)^{1/2}$.

As an immediate consequence of Theorem 1, we have the following extension of Petrenko's theorem:

THEOREM 2. *If $f(z)$ is an n-valued transcendental algebroid function of finite lower order μ, then for arbitrary complex a we have*

$$\beta(a, f) \leqq \begin{cases} \dfrac{n\pi\mu}{\sin \pi\mu} & \left(\mu \leqq \dfrac{1}{2}\right), \\ n\pi\mu & \left(\mu > \dfrac{1}{2}\right). \end{cases}$$

Finally we shall obtain

THEOREM 3. *For every fixed complex number a, every fixed numbers μ and λ such that $1/2 < \mu \leqq \lambda \leqq \infty$ and every fixed integer n such that $2 \leqq n \leqq 5$, there is an n-valued algebroid function $f_{\mu,\lambda,a}(z)$ of lower order μ and order λ such that*

$$\beta(a, f_{\mu,\lambda,a}) = n\pi\mu.$$

2. Preliminaries. Let $f(z)$ be an n-valued transcendental algebroid function defined by an irreducible equation

$$A_0(z)f^n + A_1(z)f^{n-1} + \cdots + A_{n-1}(z)f + A_n(z) = 0,$$

where $A_0, \cdots, A_n$ are entire functions without common zeros. Let $f_j(z)$ be the j-th determination of $f(z)$. We put

$$A(z) = \max_{0 \leqq j \leqq n} |A_j(z)|, \qquad \mu(r, A) = \frac{1}{2n\pi}\int_0^{2\pi} \log A(re^{i\theta})d\theta$$

and

$$f^*(z) = \max_{1 \leqq j \leqq n} |f_j(z)|.$$

Then Valiron [8, p. 21, 22] showed that

$$T(r, f) = \mu(r, A) + O(1) \tag{2.1}$$

and

$$\sum_{j=1}^{n} \overset{+}{\log} |f_j(z)| \leqq \log\left|\frac{A(z)}{A_0(z)}\right| + O(1). \tag{2.2}$$

Since

$$\overset{+}{\log} f^*(z) \leqq \sum_{j=1}^{n} \overset{+}{\log} |f_j(z)|,$$

we have from (2.2)

$$\overset{+}{\log} M(r, f) = \max_{|z|=r} \overset{+}{\log} f^*(z) \leqq \log M\left(r, \frac{A}{A_0}\right) + O(1). \tag{2.3}$$

Therefore is follows from (2.1) and (2.3) that

$$\beta(\infty, f) = \varliminf_{r\to\infty} \frac{\overset{+}{\log} M(r, f)}{T(r, f)} \leqq \varliminf_{r\to\infty} \frac{\log M(r, A/A_0)}{\mu(r, A)}. \tag{2.4}$$

3. Proof of Theorem 1. Now we shall give a proof of Theorem 1 along Fuch's idea [1, pp. 23–32], borrowing his several estimates. In the first place we start from the following lemma, which is derived from Petrenko's formula;

LEMMA. ([1, p. 26]) *Let $g(z)$ be a meromorphic function and $\{b_j\}$ its poles. Then we have, for $\gamma > 1$ and $2S < u < R/2$,*

$$\log |g(u)| < \frac{\gamma^2}{2\pi} \int_S^R \frac{u^\gamma r^{\gamma-1}}{(u^\gamma + r^\gamma)^2} dr \int_{-\pi/\gamma}^{\pi/\gamma} \log |g(re^{i\theta})| d\theta$$

$$+ \sum_{S \leqq |b_j| \leqq R} \log \frac{|b_j|^\gamma + u^\gamma}{||b_j|^\gamma - u^\gamma|} + \gamma K \left\{ \left(\frac{S}{u}\right)^\gamma T(2S, g) + \left(\frac{u}{R}\right)^\gamma T(R, g) \right\},$$

where K is an absolute constant.

In the sequel K denote an absolute constant, not always the same at each occurence.

Applying Lemma to meromorphic functions $A_j(z)/A_0(z)$ and using $T(r, A_j/A_0) \leqq n\mu(r, A) + O(1)$, we have for $1 \leqq j \leqq n$

$$\log \left| \frac{A_j(u)}{A_0(u)} \right| < \frac{\gamma^2}{2\pi} \int_S^R \frac{u^\gamma r^{\gamma-1}}{(u^\gamma + r^\gamma)^2} dr \int_{-\pi/\gamma}^{\pi/\gamma} \log \left| \frac{A_j(re^{i\theta})}{A_0(re^{i\theta})} \right| d\theta$$

$$+ \sum_{S \leqq |b_j| \leqq R} \log \frac{|b_j|^\gamma + u^\gamma}{||b_j|^\gamma - u^\gamma|} + \gamma K n \left\{ \left(\frac{S}{u}\right)^\gamma \mu(2S, A) + \left(\frac{u}{R}\right)^\gamma \mu(R, A) \right\},$$

where b_j are zeros of $A_0(z)$. We increase the right-hand side by replacing

$$\frac{1}{2\pi} \int_{-\pi/\gamma}^{\pi/\gamma} \log \left| \frac{A_j(re^{i\theta})}{A_0(re^{i\theta})} \right| d\theta \quad \text{by} \quad m\left(r, \frac{A}{A_0}\right)$$

and take the maximum over j in the left-hand side. Then we obtain

$$\log \left| \frac{A(u)}{A_0(u)} \right| < \gamma^2 \int_S^R \frac{u^\gamma r^{\gamma-1} m(r, A/A_0)}{(u^\gamma + r^\gamma)^2} dr$$

$$+ \sum_{S \leqq |b_j| \leqq R} \log \frac{|b_j|^\gamma + u^\gamma}{||b_j|^\gamma - u^\gamma|} + \gamma K n \left\{ \left(\frac{S}{u}\right)^\gamma \mu(2S, A) + \left(\frac{u}{R}\right)^\gamma \mu(R, A) \right\}.$$

By applying this formula to $A(e^{ia}z)/A_0(e^{ia}z)$ (a: real) we see that $\log |A(u)/A_0(u)|$ may be replaced by $\log M(u, A/A_0)$:

$$\log M\left(u, \frac{A}{A_0}\right) < \gamma^2 \int_S^R \frac{u^\gamma r^{\gamma-1} m(r, A/A_0)}{(u^\gamma + r^\gamma)^2}\, dr$$

(3.1)

$$+ \sum_{S \leq |b_j| \leq R} \log \frac{|b_j|^\gamma + u^\gamma}{||b_j|^\gamma - u^\gamma|} + \gamma K n \left\{ \left(\frac{S}{u}\right)^\gamma \mu(2S, A) + \left(\frac{u}{R}\right)^\gamma \mu(R, A) \right\}.$$

We now choose $\gamma > \max(1, 2\mu)$. By the reasoning [1, pp. 27-29] we deduce from (3.1) that

$$\int_{2S}^{R/2} u^{-\mu-1} \log M\left(u, \frac{A}{A_0}\right) du$$

(3.2)

$$< \frac{\pi\mu}{\sin(\pi\mu/\gamma)} \int_{2S}^{R/2} u^{-\mu-1} m\left(u, \frac{A}{A_0}\right) du$$

$$+ \pi\mu \tan \frac{\pi\mu}{2\gamma} \int_{2S}^{R/2} u^{-\mu-1} N\left(u, \frac{1}{A_0}\right) du$$

$$+ \gamma K n \{S^{-\mu} \mu(2S, A) + R^{-\mu} \mu(2R, A)\}.$$

Note also that we have

$$m\left(u, \frac{A}{A_0}\right) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} \left| \frac{A(ue^{i\theta})}{A_0(ue^{i\theta})} \right| d\theta$$

(3.3)

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{A(ue^{i\theta})}{A_0(ue^{i\theta})} \right| d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log |A(ue^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |A_0(ue^{i\theta})| d\theta$$

$$= n\mu(n, A) - N(u, 1/A_0) + O(1)$$

and by the definition of Valiron deficiency

$$N\left(u, \frac{1}{A_0}\right) > (1 - \Delta(\infty) - \varepsilon) n\mu(u, A) \quad \text{for} \quad u > S_0(\varepsilon). \tag{3.4}$$

By our choice of γ, $\pi\mu/\gamma < \pi/2$, so that

$$\tan \frac{\pi\mu}{2\gamma} - 1 \Big/ \sin \frac{\pi\mu}{\gamma} = -\cot \frac{\pi\mu}{\gamma} < 0.$$

Therefore it follows from (3.2), (3.3) and (3.4) that

$$\int_{2S}^{R/2} u^{-\mu-1} \log M\left(u, \frac{A}{A_0}\right) du$$

(3.5)

$$< n\pi\mu \left\{ (\Delta(\infty) + \varepsilon) \cot \frac{\pi\mu}{\gamma} + \tan \frac{\pi\mu}{2\gamma} \right\} \int_{2S}^{R/2} u^{-\mu-1} \mu(u, A) du$$

$$+\gamma Kn\{S^{-\mu}\mu(2S, A)+R^{-\mu}\mu(2R, A)\}.$$

Hence applying to (3.5) the reasoning of [1, pp. 30–32] and taking (2.4) into account, we can see that the statement of Theorem 1 is true.

Thus the proof of Theorem 1 is complete.

4. Proof of Theorem 3. Let $h_{\mu,\lambda}(z)$ be an entire function of order λ and of lower order μ defined in the following manner:

$h_{\mu,\lambda}(z)$ is the Mittag-Leffler function

$$E_\lambda(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n/\lambda+1)} \qquad \left(\lambda>\frac{1}{2}\right)$$

if $\lambda=\mu>1/2$, is the entire function $E_{\mu,\lambda}(z)$ constructed by Petrenko [6, pp. 409–412] if $1/2<\mu<\lambda\leqq\infty$, and is $\exp\exp z$ if $\mu=\lambda=\infty$. Then it is known that

$$\beta(\infty, h_{\mu,\lambda})=\pi\mu \tag{4.1}$$

(cf. [6, pp. 408–413]).

For a moment we assume that an equation

$$f^n+h_{\mu,\lambda}(z)f^{n-1}+1=0 \tag{4.2}$$

is irreducible. Let $f_{\mu,\lambda}(z)$ be an entire algebroid function defined by (4.2). Then we have

$$\begin{aligned} T(r, f_{\mu,\lambda})+O(1) &=\mu(r, A)=\frac{1}{2n\pi}\int_0^{2\pi}\log\max\{1, |h_{\mu,\lambda}(re^{i\theta})|\}d\theta \\ &=\frac{1}{n}T(r, h_{\mu,\lambda}). \end{aligned} \tag{4.3}$$

Hence the order of $f_{\mu,\lambda}(z)$ is λ and its lower order is μ. We denote by $f_j(z)$ the j-th determination of $f_{\mu,\lambda}(z)$ and put $f^*(z)=\max\{|f_j(z)|: 1\leqq j\leqq n\}$. Since $h_{\mu,\lambda}(z)=-\sum f_j(z)$, we have $|h_{\mu,\lambda}(z)|\leqq nf^*(z)$ and consequently

$$\overset{+}{\log} M(r, h_{\mu,\lambda})\leqq\overset{+}{\log} M(r, f_{\mu,\lambda})+\log n. \tag{4.4}$$

Therefore it follows from (4.1), (4.3) and (4.4) that

$$\pi\mu=\varliminf_{r\to\infty}\frac{\overset{+}{\log} M(r, h_{\mu,\lambda})}{T(r, h_{\mu,\lambda})}\leqq\varliminf_{r\to\infty}\frac{\overset{+}{\log} M(r, f_{\mu,\lambda})}{nT(r, f_{\mu,\lambda})}=\frac{1}{n}\beta(\infty, f_{\mu,\lambda})$$

and consequently

$$\beta(\infty, f_{\mu,\lambda})\geqq n\pi\mu$$

On the other hand Theorem 2 implies $\beta(\infty, f_{\mu,\lambda})\leqq n\pi\mu$. Thus we obtain

$$\beta(\infty, f_{\mu,\lambda})=n\pi\mu,$$

which is the desired.

For $a\neq\infty$, we consider the following algebroid function:

$$f_{\mu,\lambda,a}(z)=\frac{1}{f_{\mu,\lambda}(z)}+a.$$

Then it is clearly deduced that

$$\beta(a, f_{\mu,\lambda,a})=n\pi\mu.$$

Now, in order to complete our proof of Theorem 3 we have to show that for $n=2$ to 5 the equations (4.2) are irreducible. We first have

LEMMA. *A function $f_{\mu,\lambda}(z)$ satisfying the equation* (4.2) *is neither single-valued nor n-1-valued.*

Proof. Suppose, to the contrary, that a function $f_{\mu,\lambda}(z)$ satisfying (4.2) is single-valued or n-1-valued. Then the equation (4.2) is reducible and we have the following factorization:

$$f^n+h_{\mu,\lambda}f^{n-1}+1=(f+e^g)(f^{n-1}+a_{n-2}f^{n-2}+\cdots+a_1f+e^{-g}),$$

where g and a_j $(j=1,\cdots,n-2)$ are suitable entire functions. By factorization theorm we have

$$h_{\mu,\lambda}(z)=e^{-(n-1)g(z)}\{e^{ng(z)}+(-1)^n\}$$
$$(4.5)\qquad =e^{g(z)}+(-1)^ne^{-(n-1)g(z)}.$$

Let $G(z)$ be the function in the right-hand side. If $g(z)$ is transcendental, then $G(z)$ is of infinite order and of regular growth. Hence by the definition of $h_{\mu,\lambda}$ we have $h_{\mu,\lambda}(z)=\exp\exp\ z$, which has no zero. However $G(z)$ has zeros (cf. [4, p. 103]), which is a contradiction. If $g(z)$ is a polynomial, then $G(z)$ is of finite order and of regular growth. Hence $h_{\mu,\lambda}(z)$ is the Mittag-Leffler function $E_\lambda(z)$, which is bounded for $\pi/2\lambda<|\arg z|<\pi$ (cf. [3, p. 19]). However $G(z)$ is unbounded there. In fact we put $g(z)=a_pz^p+\cdots$ $(a_p\neq 0)$ and $a_p=|a_p|e^{i\phi}$, $z=re^{i\theta}$. Then we have for every fixed θ satisfying $\cos(p\theta+\phi)\neq 0$

$$\operatorname{Re}\ g(z)=|a_p|r^p\cos(p\theta+\phi)\{1+o(1)\}\qquad(r\to\infty).$$

Therefore for every fixed θ satisfying $\cos(p\theta+\phi)>0$ we have

$$|G(z)|\geqq e^{\operatorname{Re}g(z)}-e^{-(n-1)\operatorname{Re}g(z)}\to\infty\qquad(r\to\infty)$$

and for every fixed θ satisfying $\cos(p\theta+\phi)<0$

$$|G(z)|\geqq e^{-(n-1)\operatorname{Re}g(z)}-e^{\operatorname{Re}g(z)}\to\infty\qquad(r\to\infty).$$

Thus we have a contradiction. Q. E. D.

We continue the proof of Theorem 3. It follows from this Lemma that for $n=2,3$ the equations (4.2) are irreducible.

Assume that for $n=4$ the equation (4.2) is reducible. Then by Lemma we have that

$$f^4+h_{\mu,\lambda}f^3+1=(f^2+af+e^g)(f^2+bf+e^{-g}),$$

where a, b and g are suitable entire functions. It follows from this identity that

$$b=-ae^{-2g},\ a^2=e^{3g}(1+e^{-2g}) \quad \text{and} \quad h_{\mu,\lambda}=a+b.$$

Since the function $1+e^{-2g(z)}$ has simple zeros if $g(z)\not\equiv$const. (cf. [4, p. 103]), we obtain $g(z)\equiv$const., and consequently $h_{\mu,\lambda}\equiv$const., which is a contradiction.

Next assume that for $n=5$ the equation (4.2) is reducible. Then by Lemma we have

$$f^5+h_{\mu,\lambda}f^4+1=(f^2+af+e^g)(f^3+b_2f^2+b_1f+e^{-g}),$$

where a, b_j and g are suitable entire functions. This identity yields that

$$b_1=-ae^{-2g},\ b_2=a^2e^{-3g}-e^{-2g},\ a^3-2e^ga+e^{4g}=0 \quad \text{and} \quad h_{\mu,\lambda}=a+b_2.$$

Hence the function $a(z)$ has no zero. Since $a(z)$ is single-valued, we have $a(z)=e^{H(z)}$ with a suitable entire function $H(z)$. Therefore it follows that

$$e^{3H(z)}-2e^{H(z)}e^{g(z)}+e^{4g(z)}=0,$$

that is,

$$e^{2H(z)-g(z)}+e^{3g(z)-H(z)}=2,$$

which is unable if $2H(z)-g(z)\not\equiv$const. or $3g(z)-H(z)\not\equiv$const.. Hence we have $H(z)\equiv$const., $g(z)\equiv$const. and consequently $h_{\mu,\lambda}(z)\equiv$const., which is a contradiction.

Thus the proof of Theorem 3 is complete.

REFERENCES

[1] FUCHS, W. H. J., Topics in Nevanlinna theory. Proceeding of the NRL Conference on Classical Function Theory (1970), 1–32.

[2] GOVOLOV, N. V., Paley's hypothesis. Functional Analysis and its Application **3** (1969), 115–118.

[3] HAYMAN, W. K., Meromorphic functions. Oxford Math. Monogr. (1964).

[4] OZAWA, M., On ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **17** (1965), 103–108.

[5] PALEY, R.E.A.C., A note on integral functions. Proc. Cambridge Philos. Soc. **28** (1932), 262–265.

[6] PETRENKO, V. P., Growth of meromorphic functions of finite lower order. Math. USSR-Izvestija **3** (1969), 391–432.

[7] VALIRON, G., Sur le minimum du module des fonctions entières d'ordre inférieur à un. Mathematica **11** (1935), 264–269.

[8] VALIRON, G., Sur la dérivée des fonctions algébroïdes. Bull. Soc. Math. France **59** (1931), 17–39.

FACULTY OF ENGINEERING,
YOKOHAMA NATIONAL UNIVERSITY.