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NOTES ON CONFORMAL CHANGES OF RIEMANNIAN METRICS

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§ 0. Introduction.

Let M be an n-dimensional connected differentiable manifold of class C^∞ and g a Riemannian metric on M. We denote by (M, g) the Riemannian manifold with metric tensor g. If the angles between two vectors with respect to g and g^* are always equal at each point of the manifold, the Riemannian metrics g and g^* on M are said to be conformally related, or to be conformal to each other. It is known that the necessary and sufficient condition for g and g^* of M to be conformal to each other is that there exists a function ρ on M such that $g^*=e^{2\rho}g$. We call such a change of metric $g\to g^*$ a conformal change of Riemannian metric.

Let (M, g) and (M', g') be two Riemannian manifolds and π: $M\to M'$ a diffeomorphism. Then $g^*=\pi^{-1}g'$ is a Riemannian metric on M. When g and g^* are conformally related, that is, when there exists a function ρ on M such that $g^*=e^{2\rho}g$, we call π: $(M, g)\to(M', g')$ a conformal transformation. In particular, ρ=constant, then π is called a homothetic transformation or a homothety and if $\rho=0$, π is called an isometric transformation or an isometry.

The group of all conformal transformations of (M, g) on itself is called a conformal transformation group, that of all homothetic transformations a homothetic transformation group and that of all isometric transformations an isometry group. Let M be covered by a system of coordinate neighborhoods $\{U; x^h\}$ and g_{ji} components of the metric tensor g of M with respect to this coordinate system, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$. Let ∇_i, $K_{kji}{}^h$, K_{ji} and K be the operator of covarient differentiation with respect to Christoffel symbols $\left\{{h \atop ji}\right\}$ formed with g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature respectively.

If a vector field v^h defines an infinitesimal conformal transformation, then v^h satisfies

$$\mathcal{L}_v g_{ji}=\nabla_j v_i+\nabla_i v_j=2\rho g_{ji},$$

where $\mathcal{L}_v$ denotes the operator of Lie differentiation with respect to v^h, $v_i=g_{ih}v^h$ and

$$\rho=\frac{1}{n}\nabla_i v^i.$$

If v^h defines an infinitesimal homothetic transformation, then ρ is a constant

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and if v^h defines an infinitesimal isometry, then ρ is zero.

One of the present authors (Yano [12]) proved

THEOREM A. *If a compact orientable Riemannian manifold M of dimension $n>2$ with $K=$constant admits an infinitesimal non-homothetic conformal transformation v^h: $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, $\rho \neq$ constant such that*

$$\int_M G_{ji}\rho^j\rho^i dV \geqq 0,$$

where

$$G_{ji}=K_{ji}-\frac{1}{n}Kg_{ji}$$

and $\rho_i=\nabla_i\rho$, $\rho^h=\rho_i g^{ih}$, dV being the volume element of M, then M is isometric to a sphere.

THEOREM B. *If a compact Riemannian manifold M of dimension $n>2$ with scalar curvature $K=$constant admits an infinitesimal non-homothetic conformal transformation v^h such that*

$$\mathcal{L}_v(G_{ji}G^{ji})=0,$$

or

$$\mathcal{L}_v(Z_{kjih}Z^{kjih})=0,$$

where

$$Z_{kji}{}^h=K_{kji}{}^h-\frac{1}{n(n-1)}K(\delta_k^h g_{ji}-\delta_j^h g_{ki}),$$

then M is isometric to a sphere.

These theorems cover those of Goldberg and Kobayashi [2], [3], [4], Hsiung [6], [7], [8], and Lichnerowicz [10]. For further generalizations of Theorem A and B, see Yano and Sawaki [16].

One of the present authors (Yano [14]) proved

THEOREM C. *If M is a compact orientable Riemannian manifold of dimension $n>2$ and admits an infinitesimal non-homothetic conformal trasformation v^h: $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, $\rho \neq$ constant such that*

$$\mathcal{L}_v K=0$$

and

$$\int_M\left[K_{ji}\rho^j\rho^i-\frac{1}{n(n-1)}K^2\rho^2\right]dV \geqq 0,$$

then M is conformal to a sphere.

THEOREM D. *If M is a compact orientable Riemannian manifold of dimension $n>2$ and admits an infinitesimal non-homothetic conformal transformation v^h: $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, $\rho \neq$ constant such that*

$$\begin{cases} \mathcal{L}_v K=0, \qquad \mathcal{L}_v(G_{ji}G^{ji})=0, \\ \dfrac{1}{n-1}\displaystyle\int_M K^2\rho^2 dV \leqq \int_M K\rho_i\rho^i dV, \end{cases}$$

or

$$\begin{cases} \mathcal{L}_v K=0, \qquad \mathcal{L}_v(Z_{kjih}Z^{kjih})=0, \\ \dfrac{1}{n-1}\displaystyle\int_M K^2\rho^2 dV \leqq \int_M K\rho_i\rho^i dV, \end{cases}$$

then M is isometric to a sphere.

The Riemannian manifolds with scalar curvature not necessarily constant admitting an infinitesimal non-homothetic conformal transformation have been studied by the present authors (Yano and Sawaki [17]).

On the other hand, Goldberg and one of the present authors (Goldberg and Yano [5]) proved

THEOREM E. *Let (M, g) be a compact Riemannian manifold with scalar curvature K=constant and admitting a non-homothetic conformal change $g^*=e^{2\rho}g$ of metric such that $K^*=K$. Then if*

$$\int_M u^{-n+1}G_{ji}u^j u^i dV \geqq 0,$$

where $u=e^{-\rho}$, $u_i=\nabla_i u$, $u^h=u_i g^{ih}$, then M is isometric to a sphere.

Generalizing this theorem, Obata and one of the present authors (Yano and Obata [15]) proved

THEOREM F. *If a compact orientable Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$\int_M (\Delta u)K dV=0, \qquad G^*_{ji}G^{*ji}=u^4 G_{ji}G^{ji},$$

where $\Delta u=g^{ji}\nabla_j\nabla_i u$, then M is conformal to a sphere.

THEOREM G. *If a compact Riemannian manifold M of dimension $n>2$ and with K=constant admits a conformal change of metric such that*

$$G^*_{ji}G^{*ji}=u^4 G_{ji}G^{ji},$$

then M is isometric to a sphere.

THEOREM H. *If a compact orientable Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$\int_M (\Delta u) K dV = 0, \qquad Z^*_{kjih} Z^{*kjih} = u^4 Z_{kjih} Z^{kjih},$$

then M is conformal to a sphere.

THEOREM I. *If a compact Riemannian manifold M of dimension $n>2$ and with K=constant admits a conformal change of metric such that*

$$Z^*_{kjih} Z^{*kjih} = u^4 Z_{kjih} Z^{kjih},$$

then M is isometric to a sphere.

THEOREM J. *If a compact orientable Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$\int_M (\Delta u) K dV = 0, \qquad W^*_{kjih} W^{*kjih} = u^4 W_{kjih} W^{kjih},$$

$$a+(n-2)b \neq 0,$$

where

$$W_{kji}{}^h = a Z_{kji}{}^h + b(\delta_k^h G_{ji} - \delta_j^h G_{ki} + G_k{}^h g_{ji} - G_j{}^h g_{ki}),$$

a and b being constant, then M is conformal to a sphere.

THEOREM K. *If a compact orientable Riemannian manifold M of dimension $n>2$ and with K=constant admits a conformal change of metric such that*

$$W^*_{kjih} W^{*kjih} = u^4 W_{kjih} W^{kjih}, \qquad a+(n-2)b \neq 0,$$

then M is isometric to a sphere.

THEOREM L. *If a compact Riemannian manifold M of dimension $n \geqq 2$ admits a conformal change of metric such that*

$$K^* = K, \qquad \mathcal{L}_{Du} K = 0, \qquad \int_M u^{-n+1} G_{ji} u^j u^i dV \geqq 0,$$

where $\mathcal{L}_{Du}$ denotes the Lie derivative with respect to $u^h = g^{hi} \nabla_i u$, then M is isometric to a sphere.

THEOREM M. *If a compact Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$K^* = K, \qquad \mathcal{L}_{Du} K = 0, \qquad G^*_{ji} G^{*ji} = G_{ji} G^{ji},$$

then M is isometric to a sphere.

THEOREM N. *If a compact Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$K^*=K, \qquad \mathcal{L}_{Du}K=0, \qquad Z^*{}_{kjih}Z^{*kjih}=Z_{kjih}Z^{kjih},$$

then M is isometric to a sphere.

THEOREM O. *If a compact Riemannian manifold M of dimension $n>2$ admits a conformal change of metric such that*

$$K^*=K, \qquad \mathcal{L}_{Du}K=0, \qquad W^*{}_{kjih}W^{*kjih}=W_{kjih}W^{kjih},$$

$$a+(n-2)b\neq 0,$$

then M is isometric to a sphere.

To prove these theorems, they used

THEOREM P. (Obata [11]) *If a complete Riemannian manifold M of dimension $n\geqq 2$ admits a non-constant function u such that*

$$\nabla_j\nabla_i u=-c^2 u g_{ji},$$

where c is a positive constant, then M is isometric to a sphere of radius $1/c$ in $(n+1)$-dimensional Euclidean space.

THEOREM Q. (Yano and Obata [15]) *If a complete Riemannian manifold M of dimension $n\geqq 2$ admits a non-constant function u such that*

$$\mathcal{L}_{Du}K=0, \qquad \nabla_j\nabla_i u-\frac{1}{n}\Delta u g_{ji}=0,$$

then M is isometric to a sphere.

The main purpose of the present paper is to get generalizations of Theorems F$\sim$O.

We assume that the Riemannian manifold M we consider is compact and orientable. If M is not orientable, we have only to take an orientable double covering space of M.

§ 1. Preliminaries (See also Yano and Obata [15]).

We consider a conformal change

$$g^*{}_{ji}=e^{2\rho}g_{ji} \tag{1.1}$$

of the metric of a Riemannian manifold M and denote by Ω^* the quantity formed with g^* by the same rule as that Ω is formed with g.

First of all we have

(1. 2) $$\left\{ {h \atop j\,i} \right\}^* = \left\{ {h \atop j\,i} \right\} + \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji} \rho^h,$$

where

$$\rho_i = \nabla_i \rho, \qquad \rho^h = \rho_i g^{ih},$$

from which

(1. 3) $$K^*{}_{kji}{}^h = K_{kji}{}^h - \delta_k^h \rho_{ji} + \delta_j^h \rho_{ki} - \rho_k{}^h g_{ji} + \rho_j{}^h g_{ki},$$

where

$$\rho_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2} \rho_t \rho^t g_{ji}, \qquad \rho_j{}^h = \rho_{ji} g^{ih},$$

and consequently

(1. 4) $$K^*{}_{ji} = K_{ji} - (n-2) \rho_{ji} - \rho_t{}^t g_{ji},$$

(1. 5) $$e^{2\rho} K^* = K - 2(n-1) \rho_t{}^t,$$

where

$$\rho_t{}^t = \Delta \rho + \frac{n-2}{2} \rho_t \rho^t, \qquad \Delta \rho = g^{ji} \nabla_j \rho_i.$$

We also have

(1. 6) $$G^*{}_{ji} = G_{ji} - (n-2)(\nabla_j \rho_i - \rho_j \rho_i) + \frac{n-2}{n} (\Delta \rho - \rho_t \rho^t) g_{ji},$$

(1. 7) $$\begin{aligned} Z^*{}_{kji}{}^h = Z_{kji}{}^h &- \delta_k^h (\nabla_j \rho_i - \rho_j \rho_i) + \delta_j^h (\nabla_k \rho_i - \rho_k \rho_i) - (\nabla_k \rho^h - \rho_k \rho^h) g_{ji} \\ &+ (\nabla_j \rho^h - \rho_j \rho^h) g_{ki} + \frac{2}{n} (\Delta \rho - \rho_t \rho^t)(\delta_k^h g_{ji} - \delta_j^h g_{ki}), \end{aligned}$$

(1. 8) $$\begin{aligned} W^*{}_{kji}{}^h = W_{kji}{}^h &+ \{a + (n-2)b\} \{ -\delta_k^h (\nabla_j \rho_i - \rho_j \rho_i) + \delta_j^h (\nabla_k \rho_i - \rho_k \rho_i) \\ &- (\nabla_k \rho^h - \rho_k \rho^h) g_{ji} + (\nabla_j \rho^h - \rho_j \rho^h) g_{ki} + \frac{2}{n} (\Delta \rho - \rho_t \rho^t)(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \}. \end{aligned}$$

If we put

(1. 9) $$u = e^{-\rho}, \qquad u_i = \nabla_i u,$$

then we have

(1. 10) $$\nabla_j u_i = -u (\nabla_j \rho_i - \rho_j \rho_i),$$

(1. 11) $$\Delta u = -u (\Delta \rho - \rho_t \rho^t),$$

and consequently

(1.12) $$K^* = u^2 K + 2(n-1)u\Delta u - n(n-1)u_t u^t,$$

(1.13) $$G^*_{ji} = G_{ji} + (n-2)P_{ji},$$

(1.14) $$Z^*_{kji}{}^h = Z_{kji}{}^h + Q_{kji}{}^h,$$

(1.15) $$W^*_{kji}{}^h = W_{kji}{}^h + \{a + (n-2)b\} Q_{kji}{}^h,$$

where

(1.16) $$P_{ji} = u^{-1}\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right),$$

(1.17) $$Q_{kji}{}^h = \delta_k^h P_{ji} - \delta_j^h P_{ki} + P_k{}^h g_{ji} - P_j{}^h g_{ki},$$

and

$$P_j{}^h = P_{ji} g^{ih}.$$

From (1.16) and (1.17), we find

(1.18) $$P_{ji}P^{ji} = u^{-2}\left\{(\nabla_j u_i)(\nabla^j u^i) - \frac{1}{n}(\Delta u)^2\right\}$$

and

(1.19) $$Q_{kjih}Q^{kjih} = 4(n-2)P_{ji}P^{ji}$$

respectively.

We also have, from (1.13), (1.14) and (1.15).

(1.20) $$G^*_{ji}G^{*ji} = u^4\{G_{ji}G^{ji} + 2(n-2)G_{ji}P^{ji} + (n-2)^2 P_{ji}P^{ji}\},$$

(1.21) $$Z^*_{kjih}Z^{*kjih} = u^4\{Z_{kjih}Z^{kjih} + 8G_{ji}P^{ji} + 4(n-2)P_{ji}P^{ji}\},$$

and

(1.22) $$W^*_{kjih}W^{*kjih} = u^4\{W_{kjih}W^{kjih} + 8(a+(n-2)b)^2 G_{ji}P^{ji} + 4(n-2)(a+(n-2)b)^2 P_{ji}P^{ji}\}$$

respectively.

For the expression $G_{ji}P^{ji}$, we have, from (1.16),

(1.23) $$G_{ji}P^{ji} = u^{-1}G_{ji}\nabla^j u^i,$$

where $\nabla^j = g^{ji}\nabla_i$.

First of all, we prove

PROPOSITION 1.1. *If a compact orientable Riemannian manifold M of dimension n with K=constant>0 admits a conformal change of metric such that*

(1. 24) $$K^* \geqq K = constant > 0,$$

then

(i) *for* $n \geqq 2$, *we have*

$$\cdots \geqq \int_M u^p dV \geqq \int_M u^{p-1} dV \geqq \cdots \geqq \int_M u\, dV \geqq \int_M dV \geqq \int_M u^{-1} dV.$$

If the equality holds somewhere then the conformal change is homothetic.

(ii) *for* $n \geqq 4$, *we have*

$$\int_M u dV \geqq \int_M u^{-3} dV.$$

If the equality holds, the conformal change is homothetic.

Proof. (i) We have, from (1. 12),

$$\Delta u = \frac{1}{2(n-1)}(K^* u^{-1} - Ku) + \frac{1}{2} n u^{-1} u_i u^i,$$

and conseqnently, using the assumption $K^* \geqq K$,

(1. 24) $$\Delta u \geqq \frac{1}{2(n-1)} K(u^{-1} - u) + \frac{1}{2} n u^{-1} u_i u^i.$$

Multiplying by $(1+u)^{-1}$ and integrating over M, we find

$$\int_M (1+u)^{-1} \Delta u dV \geqq \frac{1}{2(n-1)} \int_M K(u^{-1} - 1) dV + \frac{n}{2} \int_M u^{-1}(1+u)^{-1} u_t u^t dV.$$

Noting that

$$\int_M (1+u)^{-1} \Delta u dV = \int_M (1+u)^{-1} \nabla_i u^i dV$$
$$= \int_M (1+u)^{-2} u_t u^t dV,$$

we have

$$\int_M (1+u)^{-2} u_t u^t dV \geqq \frac{1}{2(n-1)} \int_M K(u^{-1} - 1) dV + \frac{n}{2} \int_M u^{-1}(1+u)^{-1} u_t u^t dV,$$

$$\int_M \left\{ u - \frac{n}{2}(1+u) \right\} u^{-1}(1+u)^{-2} u_t u^t dV \geqq \frac{K}{2(n-1)} \int_M (u^{-1} - 1) dV,$$

(1. 25) $$\int_M \left\{ -\frac{n-2}{2} u - \frac{n}{2} \right\} u^{-1}(1+u)^{-2} u_t u^t dV \geqq \frac{K}{2(n-1)} \int_M (u^{-1} - 1) dV,$$

from which

$$0 \geqq \int_M (u^{-1}-1)dV$$

that is,

$$\int_M (1-u^{-1})dV \geqq 0.$$

On the other hand, we have

$$\int_M \{(u^p-u^{p-1})-(u^{p-1}-u^{p-2})\}dV=\int_M u^{p-2}(u-1)^2 dV \geqq 0,$$

and consequently

$$\int_M (u^p-u^{p-1})\,dV \geqq \int_M (u^{p-1}-u^{p-2})dV.$$

But, we know that

$$\int_M (1-u^{-1})dV \geqq 0,$$

and consequently

$$\begin{aligned}\int_M (u^p-u^{p-1})\,dV &\geqq \int_M (u^{p-1}-u^{p-2})dV \geqq \cdots \geqq \int_M (u^2-u)dV \\ &\geqq \int_M (u-1)dV \geqq \int_M (1-u^{-1})dV \geqq 0,\end{aligned} \tag{1. 26}$$

from which

$$\int_M u^p dV \geqq \int_M u^{p-1}dV \geqq \cdots \geqq \int_M u dV \geqq \int_M dV \geqq \int_M u^{-1}dV.$$

We assume that the equality holds for a fixed p:

$$\int_M u^p dV = \int_M u^{p-1}dV.$$

Then, from (1. 26), we have

$$\int_M (1-u^{-1})dV=0$$

and consequently, from (1. 25),

$$u_t u^t=0, \qquad u_t=0$$

from which

$$u=\text{constant}$$

and hence the conformal change is homothetic.

(ii) Multiplying (1. 24) by $(1+u^{-2})$ and integrating over M, we find

$$\begin{aligned}\int_M u^{-2}\Delta u\,dV &\geqq \frac{K}{2(n-1)}\int_M (u^{-3}-u)dV+\frac{n}{2}\int_M (u^{-1}+u^{-3})u_tu^t\,dV,\\ &\quad -\frac{n-4}{2}\int_M u^{-3}u_tu^t dV-\frac{n}{2}\int_M u^{-1}u_tu^t\,dV \\ &\geqq \frac{K}{2(n-1)}\int_M (u^{-3}-u)dV,\end{aligned}\tag{1. 27}$$

from which, for $n\geqq 4$,

$$\int_M (u^{-3}-u)dV\leqq 0,$$

and consequently

$$\int_M u\,dV\geqq\int_M u^{-3}dV.$$

If the equality holds, we have, from (1. 27),

$$\int_M u^{-1}u_tu^t dV=0,$$

from which

$$u_tu^t=0, \qquad u_t=\text{constant}$$

and consequently

$$u=\text{constant}$$

and the conformal change is homothetic.

§ 2. Lemmas.

LEMMA 2. 1. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric such that*

$$\mathcal{L}_{Du}K=0, \qquad \mathcal{L}_{Du}K^*=0,$$

then

$$\int_M u^{p-1}G_{ji}u^ju^i dV+\int_M u^{p+1}P_{ji}P^{ji}dV$$

$$\text{(i)}\qquad =-(n+p-2)\left[\int_M u^{p-2}(\nabla_j u_i)u^ju^i dV + \frac{1}{2n(n-1)}\int_M (u^{p-1}K-u^{p-2}K^*)u_tu^t dV+\frac{1}{2}\int_M u^{p-3}(u_tu^t)^2 dV\right].$$

$$\text{(ii)}\qquad \int_M u^{-n+1}G_{ji}u^ju^i dV+\int_M u^{-n+3}P_{ji}P^{ji}dV=0.$$

Proof. (i) From (1.18), we have

$$u^2P_{ji}P^{ji}=(\nabla_j u_i)(\nabla^j u^i)-\frac{1}{n}(\Delta u)^2,$$

from which, multiplying by u^{p-1} and integrating over M,

$$\begin{aligned}\int_M u^{p+1}P_{ji}P^{ji}dV&=\int_M u^{p-1}(\nabla_j u_i)(\nabla^j u^i)dV-\frac{1}{n}\int_M u^{p-1}(\nabla_t u^t)(\Delta u)dV\\
&=-(p-1)\int_M u^{p-2}(\nabla_j u_i)u^ju^i dV-\int_M u^{p-1}(\nabla^j\nabla_j u_i)u^i dV\\
&\quad+\frac{1}{n}(p-1)\int_M u^{p-2}u_iu^i\Delta u dV+\frac{1}{n}\int_M u^{p-1}u^i\nabla_i\Delta u dV\\
&=-(p-1)\int_M u^{p-2}(\nabla_j u_i)u^ju^i dV-\int_M u^{p-1}(K_{ji}u^ju^i+u^i\nabla_i\Delta u)dV\\
&\quad+\frac{1}{n}(p-1)\int_M u^{p-2}u_iu^i\Delta u dV+\frac{1}{n}\int_M u^{p-1}u^i\nabla_i\Delta u dV,\end{aligned}$$

where we have used

$$(\nabla^j\nabla_j u_i)u^i=K_{ji}u^ju^i+u^i\nabla_i\Delta u.$$

Therefore we have

$$\begin{aligned}\int_M u^{p+1}P_{ji}P^{ji}dV=&-(p-1)\int_M u^{p-2}(\nabla_j u_i)u^ju^i dV\\
&-\int_M u^{p-1}K_{ji}u^ju^i dV-\frac{n-1}{n}\int_M u^{p-1}u^i\nabla_i\Delta u dV\\
&+\frac{p-1}{n}\int_M u^{p-2}u_iu^i\Delta u dV.\end{aligned}\tag{2.1}$$

Substituting

$$\Delta u=\frac{1}{2(n-1)}(K^*u^{-1}-Ku)+\frac{1}{2}nu^{-1}u_iu^i$$

into (2. 1), we have

$$\int_M u^{p+1}P_{ji}P^{ji}dV$$

$$=-(p-1)\int_M u^{p-2}(\nabla_j u_i)u^ju^idV-\int_M u^{p-1}K_{ji}u^ju^idV$$

$$-\frac{n-1}{n}\int_M u^{p-1}u^i\nabla_i\left\{\frac{1}{2(n-1)}(K^*u^{-1}-Ku)+\frac{1}{2}nu^{-1}u_tu^t\right\}dV$$

$$+\frac{p-1}{n}\int_M u^{p-2}u_iu^i\left\{\frac{1}{2(n-1)}(K^*u^{-1}-Ku)+\frac{1}{2}nu^{-1}u_tu^t\right\}dV$$

$$=-(p-1)\int_M u^{p-2}(\nabla_j u_i)u^ju^idV-\int_M u^{p-1}K_{ji}u^ju^idV$$

$$-\frac{n-1}{n}\int_M u^{p-1}\left\{\frac{1}{2(n-1)}(-K^*u^{-2}-K)u_iu^i+nu^{-1}(\nabla_j u_i)u^ju^i-\frac{1}{2}nu^{-2}(u_iu^i)^2\right\}dV$$

$$-\frac{n-1}{n}\int_M u^{p-1}\left\{\frac{1}{2(n-1)}(u^{-1}\mathcal{L}_{Du}K^*-u\mathcal{L}_{Du}K)\right\}dV$$

$$+\frac{p-1}{n}\int_M u^{p-2}u_iu^i\left\{\frac{1}{2(n-1)}(K^*u^{-1}-Ku)+\frac{1}{2}nu^{-1}u_tu^t\right\}dV$$

$$=\{-(p-1)-(n-1)\}\int_M u^{p-2}(\nabla_j u_i)u^ju^idV-\int_M u^{p-1}\left(K_{ji}-\frac{1}{n}Kg_{ji}\right)u^ju^idV$$

$$+\left\{\frac{1}{2n}+\frac{p-1}{2n(n-1)}\right\}\int_M u^{p-1}(K^*u^{-2}-K)u_iu^idV$$

$$+\left\{\frac{n-1}{2}+\frac{p-1}{2}\right\}\int_M u^{p-3}(u_iu^i)^2dV,$$

and consequently

$$\int_M u^{p-1}G_{ji}u^ju^idV+\int_M u^{p+1}P_{ji}P^{ji}dV$$

$$=-(n+p-2)\left[\int_M u^{p-2}(\nabla_j u_i)u^ju^idV+\frac{1}{2n(n-1)}\int_M(Ku^{p-1}-K^*u^{p-3})u_iu^idV-\frac{1}{2}\int_M u^{p-3}(u_iu^i)^2dV\right].$$

(ii) We put $n+p-2=0$, that is

$$p-1=-n+1, \qquad p+1=-n+3$$

in the equation above, then we get the equation to be proved.

LEMMA 2. 2. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric such that*

(i) $\mathcal{L}_{Du}K=0$, *then*

$$\int_M (u^{-3}\lambda^*-u\lambda)dV=(n-2)^2\int_M uP_{ji}P^{ji}dV,$$

(ii) $\mathcal{L}_{Du}K=0$, $\mathcal{L}_{Du}K^*=0$, *then*

$$\int_M (u^{-n+3}\lambda-u^{-n-1}\lambda^*)dV=(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV,$$

where

$$\lambda=G_{ji}G^{ji}, \qquad \lambda^*=G^*_{ji}G^{*ji}.$$

Proof. (i) From (1. 20) and (1. 23), we find

$$u^{-3}\lambda^*-u\lambda=2(n-2)G_{ji}\nabla^j u^i+(n-2)^2uP_{ji}P^{ji}. \tag{2. 2}$$

Integrating (2. 2) over M, we obtain

$$\int_M (u^{-3}\lambda^*-u\lambda)dV=2(n-2)\int_M G_{ji}\nabla^j u^i dV+(n-2)^2\int_M uP_{ji}P^{ji}dV$$

$$=-\frac{(n-2)^2}{n}\int_M \mathcal{L}_{Du}KdV+(n-2)^2\int_M uP_{ji}P^{ji}dV$$

by virtue of

$$\nabla^j G_{ji}=\frac{n-2}{2n}\nabla_i K,$$

and consequently, under the assumption $\mathcal{L}_{Du}K=0$, we have

$$\int_M (u^{-3}\lambda^*-u\lambda)dV=(n-2)^2\int_M uP_{ji}P^{ji}dV.$$

(ii) Multiplying (2. 2) by u^{-n+2} and integrating over M, we find

$$\int_M (u^{-n-1}\lambda^*-u^{-n+3}\lambda)dV=2(n-2)\int_M u^{-n+2}G_{ji}\nabla^j u^i dV+(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV$$

$$=2(n-2)^2\int_M u^{-n+1}G_{ji}u^j u^i dV-\frac{(n-2)^2}{n}\int_M u^{-n+2}\mathcal{L}_{Du}KdV+(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV$$

from which, substituting

$$\int_M u^{-n+1} G_{ji} u^j u^i dV = -\int_M u^{-n+3} P_{ji} P^{ji} dV$$

obtained from (ii) of Lemma 2. 1,

$$\int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*) dV = (n-2)^2 \int_M u^{-n+3} P_{ji} P^{ji} dV.$$

LEMMA 2. 3. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric such that*

(i) $\mathcal{L}_{Du}K=0$, *then*

$$\int_M (u^{-3}\mu^* - u\mu) dV = 4(n-2) \int_M u P_{ji} P^{ji} dV,$$

(ii) $\mathcal{L}_{Du}K=0$, $\mathcal{L}_{Du}K^*=0$, *then*

$$\int_M (u^{-n+3}\mu - u^{-n-1}\mu^*) dV = 4(n-2) \int_M u^{-n+3} P_{ji} P^{ji} dV$$

where

$$\mu = Z_{kjih} Z^{kjih}, \qquad \mu^* = Z^*{}_{kjih} Z^{*kjih}.$$

Proof. (i) From (1. 21) and (1. 23), we find

$$u^{-3}\mu^* - u\mu = 8 G_{ji} \nabla^j u^i + 4(n-2) u P_{ji} P^{ji}. \tag{2. 3}$$

Integrating over M, we obtain

$$\begin{aligned}\int_M (u^{-3}\mu^* - u\mu) dV &= 8\int_M G_{ji} \nabla^j u^i dV + 4(n-2)\int_M u P_{ji} P^{ji} dV \\ &= -\frac{4(n-2)}{n}\int_M \mathcal{L}_{Du} K dV + 4(n-2)\int_M u P_{ji} P^{ji} dV,\end{aligned}$$

from which, under the assumption $\mathcal{L}_{Du}K=0$,

$$\int_M (u^{-3}\mu^* - u\mu) dV = 4(n-2)\int_M u P_{ji} P^{ji} dV.$$

(ii) Multiplying (2. 3) by u^{-n+2} and integrating over M, we find

$$\begin{aligned}&\int_M (u^{-n-1}\mu^* - u^{-n+3}\mu) dV \\ &= 8\int_M u^{-n+2} G_{ji} \nabla^j u^i dV + 4(n-2)\int_M u^{-n+3} P_{ji} P^{ji} dV \\ &= 8(n-2)\int_M u^{-n+1} G_{ji} u^j u^i dV - \frac{4(n-2)}{n}\int_M u^{-n+2} \mathcal{L}_{Du} K dV + 4(n-2)\int_M u^{-n+2} P_{ji} P^{ji} dV,\end{aligned}$$

from which, using (ii) of Lemma 2. 1,

$$\int_M (u^{-n+3}\mu - u^{-n-1}\mu^*)\, dV = 4(n-2)\int_M u^{-n+3} P_{ji}P^{ji} dV.$$

LEMMA 2. 4. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric such that*

(i) $\mathcal{L}_{Du}K=0$, *then*

$$\int_M (u^{-3}\nu^* - u\nu) dV = 4(n-2)\{a+(n-2)b\}^2 \int_M u P_{ji}P^{ji} dV,$$

(ii) $\mathcal{L}_{Du}K=0$, $\mathcal{L}_{Du}K^*=0$, *then*

$$\int_M (u^{-n+3}\nu - u^{-n-1}\nu^*) dV = 4(n-2)\{a+(n-2)b\}^2 \int_M u^{-n+3} P_{ji}P^{ji} dV,$$

where

$$\nu = W_{kjih}W^{kjih}, \qquad \nu^* = W^*{}_{kjih}W^{*kjih}.$$

Proof. (i) From (1. 22) and (1. 23), we find

$$u^{-3}\nu^* - u\nu = 8\{a+(n-2)b\}^2 G_{ji}\nabla^j u^i + 4(n-2)\{a+(n-2)b\}^2 u P_{ji}P^{ji}. \tag{2. 4}$$

Integrating over M, we obtain

$$\int_M (u^{-3}\nu^* - u\nu) dV = 8\{a+(n-2)b\}^2 \int_M G_{ji}\nabla^j u^i dV + 4(n-2)\{a+(n-2)b\}^2 \int_M u P_{ji}P^{ji} dV$$

$$= -\frac{4(n-2)\{a+(n-2)b\}^2}{n}\int_M \mathcal{L}_{Du}K dV + 4(n-2)\{a+(n-2)b\}^2 \int_M u P_{ji}P^{ji} dV$$

from which, under the assumption $\mathcal{L}_{Du}K=0$,

$$\int_M (u^{-3}\nu^* - u\nu) dV = 4(n-2)\{a+(n-2)b\}^2 \int_M u P_{ji}P^{ji} dV.$$

(ii) Multiplying (2. 4) by u^{-n+2} and integrating

$$\int_M (u^{-n-1}\nu^* - u^{-n+3}\nu) dV = 8\{a+(n-2)b\}^2 \int_M u^{-n+2} G_{ji}\nabla^j u^i dV$$

$$+ 4(n-2)\{a+(n-2)b\}^2 \int_M u^{-n+3} P_{ji}P^{ji} dV$$

$$= 8(n-2)\{a+(n-2)b\}^2 \int_M u^{-n+1} G_{ji}u^j u^i dV - \frac{4(n-2)\{a+(n-2)b\}^2}{n}\int_M u^{-n+2}\mathcal{L}_{Du}K dV$$

$$+ 4(n-2)\{a+(n-2)b\}^2 \int_M u^{-n+3} P_{ji}P^{ji} dV,$$

from which, using (ii) of Lemma 2. 1,

$$\int_M (u^{-n+3}\nu - u^{-n-1}\nu^*)\,dV = 4(n-2)\{a+(n-2)b\}^2 \int_M u^{-n+3} P_{ji}P^{ji}\,dV.$$

§ 3. Theorems.

THEOREM 3. 1. *If a compact Riemannian manifold M of dimension $n \geqq 3$ admits a conformal change of metric $g^* = e^{2\rho}g$ such that*

$$\mathcal{L}_{Du}K = 0, \qquad \mathcal{L}_{Du}K^* = 0,$$

$$u^p\lambda = \{(u-1)\varphi + 1\}\lambda^*$$

where p is a real number such that $p \leqq 4$ and φ a differentiable non-negative function of M, then M is isometric to a sphere.

In particular, for the case $p=4$ and $\varphi=0$, we have the same conclusion without the condition $\mathcal{L}_{Du}K^*=0$.

Proof. We first compute

$$\begin{aligned} &\int_M (u\lambda - u^{-3}\lambda^*)\,dV - \int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV \\ &= \int_M [u^{-n+3}(u^{n-2}-1)\lambda - u^{-n-1}(u^{n-2}-1)\lambda^*]\,dV \qquad (3.1) \\ &= \int_M (u^{n-2}-1)(u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_M (u^{n-2}-1)(u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV - \int_M u^{-n-1}(u^{n-2}-1)(u^{4-p}-1)\lambda^*\,dV \\ &= \int_M (u^{n-2}-1)(u^{-n+3}\lambda - u^{-n+3-p}\lambda^*)\,dV \\ &= \int_M u^{-n+3-p}(u^{n-2}-1)(u^p\lambda - \lambda^*)\,dV. \end{aligned}$$

But, we have by assumption

$$u^p\lambda - \lambda^* = (u-1)\varphi\lambda^*$$

and consequently

$$\int_M (u^{n-2}-1)(u^{-n+3}\lambda-u^{-n-1}\lambda^*)dV-\int_M u^{-n-1}(u^{n-2}-1)(u^{4-p}-1)\lambda^* dv$$

$$=\int_M u^{-n+3-p}(u^{n-2}-1)(u-1)\varphi\lambda^* dV\geqq 0,$$

because

$$u>0, \qquad (u^{n-2}-1)(u-1)\geqq 0, \qquad \varphi\geqq 0, \qquad \lambda^*\geqq 0,$$

Thus we have

$$\int_M (u^{n-2}-1)(u^{-n+3}\lambda-u^{-n-1}\lambda^*)dV\geqq\int_M u^{-n-1}(u^{n-2}-1)(u^{4-p}-1)\lambda^* dV. \tag{3.2}$$

Since $p\leqq 4$, we have

$$(u^{n-2}-1)(u^{4-p}-1)\geqq 0$$

and consequently

$$\int_M u^{-n-1}(u^{n-2}-1)(u^{4-p}-1)\lambda^* dV\geqq 0.$$

Thus, from (3. 1) and (3. 2), we find

$$\int_M (u\lambda-u^{-3}\lambda^*)dV\geqq\int_M (u^{-n+3}\lambda-u^{-n-1}\lambda^*)dV.$$

But, from (i) of Lemma 2. 2, we have

$$0\geqq\int_M (u\lambda-u^{-3}\lambda^*)dV$$

and consequently

$$0\geqq\int_M (u^{-n+3}\lambda-u^{-n-1}\lambda^*)dV.$$

Thus from (ii) of Lemma 2. 2 and the above equation, we conclude

$$0\geqq(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV,$$

from which

$$P_{ji}=u^{-1}\left(\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}\right)=0,$$

or

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0.$$

Thus, by Theorem Q, M is isometric to a sphere.

If $p=4$ and $\varphi=0$, from (i) of Lemma 2. 2, we have immediately $P_{ji}=0$ and condition $\mathcal{L}_{Du}K^*=0$ is not necessary.

COROLLARY. *If a compact Riemannian manifold M of dimension $n\geqq 3$ admits a conformal change of metric $g^*=e^{2p}g$ such that*

$$\mathcal{L}_{Du}K=0, \qquad \mathcal{L}_{Du}K^*=0,$$

$$u^p\lambda=\lambda^* \qquad (p\leqq 4),$$

then M is isometric to a sphere.

Proof. In the condition

$$\{(u-1)\varphi+1\}\lambda^*=u^p\lambda$$

of the theorem, we put $\varphi=0$ and get

$$\lambda^*=u^p\lambda.$$

THEOREM 3. 2. *If a compact Riemannian manifold M of dimension $n\geqq 3$ admits a conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\mathcal{L}_{Du}K=0, \qquad \mathcal{L}_{Du}K^*=0$$

$$u^p\mu=\{(u-1)\varphi+1\}\mu^*,$$

where p is a real number such that $p\leqq 4$ and φ is a differentiable non-negative function of M, then M is isometric to a sphere.

In particular, for the case $p=4$ and $\varphi=0$, we have the same conclusion without the condition $\mathcal{L}_{Du}K^*=0$.

Proof. In the proof of Theorem 3. 1, we replace λ by μ and λ^* by μ^* and use Lemma 2. 3 instead of Lemma 2. 2.

COROLLARY. *If a compact Riemannian manifold M of dimension $n\geqq 3$ admits a conformal change of metric $g^*=e^{2p}g$ such that*

$$\mathcal{L}_{Du}K=0, \qquad \mathcal{L}_{Du}K^*=0,$$

$$u^p\mu=\mu^* \qquad (p\leqq 4),$$

then M is isometric to a sphere.

THEOREM 3.3. *If a compact Riemannian manifold M of dimension $n \geqq 3$ admits a conformal change of metric $g^* = e^{2p}g$ such that*

$$\mathcal{L}_{Du}K = 0, \qquad \mathcal{L}_{Du}K^* = 0,$$

$$u^p \nu = \{(u-1)\varphi + 1\}\nu^*, \qquad a + (n-2)b \neq 0$$

where p is a real number such that $p \leqq 4$ and φ is a differentiable non-negative function of M, then M is isometric to a sphere.

In particular, for the case $p = 4$ and $\varphi = 0$, we have the same conclusion without the condition $\mathcal{L}_{Du}K^* = 0$.

Proof. In the proof of Theorem 3.1, we replace λ by ν and λ^* by ν^* and use Lemma 2.4 instead of Lemma 2.2.

COROLLARY. *If a compact Riemannian manifold M of dimension $n \geqq 3$ admits a conformal change of metric $g^* = e^{2p}g$ such that*

$$\mathcal{L}_{Du}K = 0, \qquad \mathcal{L}_{Du}K^* = 0,$$

$$u^p \nu = \nu^* \qquad (p \leqq 4),$$

then M is isometric to a sphere.

THEOREM 3.4. *If a compact Riemannian manifold M of dimension $n \geqq 4$ with K = constant and λ (or μ, or ν) = constant > 0 admits a conformal change of metric such that*

$$\lambda \geqq \lambda^* \qquad (or\ \mu \geqq \mu^*\ or\ \nu \geqq \nu^*)$$

then the conformal change is homothetic.

Proof. From Lemma 2.1, (i), we have

$$\int_M (u^{-3} - u)\lambda dV \geqq 0, \qquad \int_M (u^{-3} - u)\lambda^* dV \geqq 0,$$

and consequently, if λ = constant > 0, then we have

$$\int_M (u^{-3} - u) dV \geqq 0.$$

On the other hand, we have, from Proposition 1. 1, (ii),

$$\int_M (u-u^{-3})dV \geqq 0,$$

and consequently

$$\int_M (u-u^{-3})dV=0.$$

Thus, again by Proposition 1. 1, (ii), the conformal change is homothetic.
We can prove the same conclusion for μ=constant>0 and ν=constant>0.

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