POSITIVE HARMONIC FUNCTIONS ON AN END

By Mitsuru Ozawa

It is well known that the Martin theory on positive harmonic functions plays an important role in the theory of open Riemann surfaces. Its whole theory depends upon the potential theory and the so-called Martin compactification of the given surface. In the present paper we shall give a proof of it, especially the representation theorem on positive harmonic functions on an end, which seems extremely simple. In our proof we shall introduce a suggestive functional and make use of a variational method.

Let W be an open Riemann surface and $\{W_m\}$ be its exhaustion in the usual sense. Let $HP(W-\overline{W}_m)$ be a class of positive harmonic functions on $W-\overline{W}_m$ vanishing continuously on ∂W_m. Evidently $HP(W-\overline{W}_m)$ is a positively linear space.

Lemma 1. *The space $HP(W-\overline{W}_m)$ is a metric space with the metric*

$$\rho(u, v)=\int_{\partial W_m} \frac{\partial}{\partial n} | u-v | ds.$$

Lemma 2. *The uniform convergence in the wider sense in $W-\overline{W}_m$ in the class $HP(W-\overline{W}_m)$ is equivalent to the ρ-convergence in the space $HP(W-\overline{W}_m)$.*

Lemma 3. *The unit sphere U_P in the space $HP(W-\overline{W}_m)$ is a ρ-compact convex set.*

Proof. Let $v \in U_P$, then

$$\int_{\partial W_m} \frac{\partial}{\partial n} v(p)\, ds=1.$$

Let $\omega_q(p)$ be the harmonic measure $\omega(p, \partial W_q, W_q-\overline{W}_m)$ and $M=\max v(p)$, $d=\min v(p)$ on ∂W_q. Then we have $d\omega_q(p) \leqq v(p)$ on $W_q-\overline{W}_m$ and hence on ∂W_m

$$d \frac{\partial}{\partial n} \omega_q(p) \leqq \frac{\partial}{\partial n} v(p).$$

Let l denote the value of the integral

$$\int_{\partial W_m} \frac{\partial}{\partial n} \omega_q(p)\, ds,$$

Received June 15, 1960.

then we have an inequality $dl \leqq 1$. By this inequality and the Harnack inequality, we have $M \leqq kd \leqq kl^{-1}$, where k depends solely upon the configuration of ∂W_q. Therefore $v(p)$ in U_P is uniformly bounded on $W_q - \overline{W}_m$, which implies that we can select a subsequence converging uniformly in $W_q - \overline{W}_m$. Since q is arbitrary, this remains valid in $W - \overline{W}_m$ in the wider sense. The limit function V thus obtained satisfies the normalization $V \in U_P$. Let v_ν converge uniformly to V in the wider sense, then $\partial v_\nu / \partial n$ converges uniformly to $\partial V / \partial n$ on ∂W_m. Thus we have

$$\lim_{\nu \to \infty} \rho(v_\nu, V) = \lim_{\nu \to \infty} \int_{W_m} \frac{\partial}{\partial n} | v_\nu - V | ds = 0,$$

which implies that U_P is ρ-compact. Convexity is obvious.

Existence of Martin's minimal positive harmonic function and the possibility of approximation of any element in $HP(W - \overline{W}_m)$ by a positively linear combination of minimals can be obtained by the following two facts: (1) Any extreme point of U_P coincides with a suitable minimal positive harmonic function in Martin's sense up to a positive constant factor and vice versa, and (2) there is at least one extreme point on U_P and the ρ-closed convex hull spanned by all the extreme points coincides with the original sphere U_P. The last statement is nothing but the so-called Kreĭn-Mil'man theorem on the existence of extreme point on compact convex set. The former fact (1) is easy to prove. However, we shall avoid to use these two facts.

LEMMA 4. *Any structure of the positively linear space $HP(W - \overline{W}_1)$ with ρ-metric depends solely upon the ideal boundary of W, that is, there exist two positively linear mappings T_m and S_m between $HP(W - \overline{W}_1)$ and $HP(W - \overline{W}_m)$:*

$$HP(W - \overline{W}_1) \underset{T_m}{\overset{S_m}{\rightleftarrows}} HP(W - \overline{W}_m)$$

such that both T_m and S_m are one-to-one onto and $T_m S_m = S_m T_m =$ the identity mapping. These mappings are commutative to the ρ-convergency in two respective spaces. However, in general, ρ-metric is not invariant under T_m and S_m.

It is not difficult to prove the Lemma 4. See [1] or [2]. We should remark that, if ρ-metric is invariant under the mappings T_m and S_m, $W \in O_G$ and vice versa.

Let $K_n^{(m)}$ be the class of harmonic functions u_n on $W_n - \overline{W}_m$ such that $u_n(p) = 0$ on ∂W_n and $= f$ on ∂W_m, where f is a sufficiently smooth function on ∂W_m. Let $K^{(m)}$ be the class of limit functions $\lim u_n$. Evidently this limit exists uniformly in the wider sense in $W - \overline{W}_m$. For later use, we remark that

$$v(p) = T_m v(p) + b_m(p), \qquad v \in HP(W - \overline{W}_1),\ T_m v \in HP(W - \overline{W}_m)$$

and $b_m(p) \in K^{(m)}$ equals $v(p)$ on ∂W_m. Let $\omega(M)$ belong to $K^{(1)}$ such that

$\omega(M)=1$ on ∂W_1. Let $G_m(p, M)$ be the Green function on $W_m-\overline{W}_1$ and $G(p, M)$ its limit $\lim_{m\to\infty}G_m(p, M)$. Let $g(p, M)$ be a normalized harmonic function defined by

$$g(p, M)=\frac{G(p, M)}{2\pi\omega(M)},$$

satisfying the normalization condition

$$\int_{\partial W_1}\frac{\partial}{\partial n}g(p, M)\,ds=1.$$

If there exists a limit function $\lim_{n\to\infty}g(p, M_n)$ uniformly in $W-\overline{W}_1$ in the wider sense along a suitable sequence $\{M_n\}$ tending to the ideal boundary, then we say that $\{M_n\}$ determines an ideal boundary point M and is a fundamental sequence. The limit function is denoted by $g(p, M)$. We say that two fundamental sequences $\{M_n\}$ and $\{M_n'\}$ are mutually equivalent if

$$\lim_{n\to\infty}g(p, M_n)=\lim_{n\to\infty}g(p, M_n')$$

for any $p\in W-\overline{W}_1$. Let $\varDelta$ be the set of these equivalence classes and $\mathfrak{D}$ be $W-\overline{W}_1+\varDelta$. In $\mathfrak{D}$ we introduce a metric by the integral

$$\rho(M, M')=\int_{\partial W_1}\frac{\partial}{\partial n}\,|\,g(p, M)-g(p, M')\,|\,ds,$$

which is evidently bounded for any M, M' in $\mathfrak{D}$, that is, $\rho(M, M')\leqq 2$. We can easily prove that $\overline{\mathfrak{D}-\overline{W}_2}$ is compact, $W-\overline{W}_1$ is open, $\varDelta$ is bounded, ρ-closed and ρ-compact and $\partial\mathfrak{D}=\varDelta+\partial W_1$ with respect to the ρ-topology and that the relative topology in $W-\overline{W}_2$ induced by the ρ-metric is equivalent to the original topology there. Further we see that the function $g(p, M)$, for a fixed p, is ρ-continuous as a function of M on $\overline{\mathfrak{D}-\overline{W}_2}$, except at $M=p$, especially on $\varDelta$. The notion of the ρ-Borel set can thus be introduced in $\overline{\mathfrak{D}-\overline{W}_2}$, especially in $\varDelta$ and the theory of Radon-Stieltjes-Lebesgue type integrals over $\varDelta$ can now be developed.

Let $Hg(W-\overline{W}_1)$ be a positively linear space each member of which has a form

$$\int_{\varDelta}g(p, M)\,d\sigma(M)$$

with a suitable non-negative mass-distribution σ on $\varDelta$ and its unit sphere by the ρ-metric be denoted by Ug. It is obvious that any element of $Hg(W-\overline{W}_1)$ belongs to $HP(W-\overline{W}_1)$ and hence Ug is a subset of U_P.

LEMMA 5. *Ug is ρ-closed in U_P and hence ρ-compact.*

Proof. Let v_n be a sequence belonging to Ug and $\lim_{n\to\infty}\rho(v_n, v)=0$ for a suitable $v\in HP(W-\overline{W}_1)$, then we have

$$v_n(p)=\int_{\varDelta}g(p, M)\,d\sigma_n(M),\qquad \int_{\varDelta}d\sigma_n(M)=1,\ d\sigma_n(M)\geqq 0.$$

We can then select a subsequence of mass-distributions $\{\sigma_{n_\nu}(M)\}$ converging weakly to a suitable mass-distribution $\sigma(M)$ on $\varDelta$ satisfying

$$\int_{\varDelta} d\sigma(M) = 1 \qquad \text{and} \qquad d\sigma(M) \geqq 0,$$

that is,

$$\lim_{\nu\to\infty}\int_{\varDelta} f(M)\, d\sigma_{n_\nu}(M) = \int_{\varDelta} f(M)\, d\sigma(M)$$

for any continuous function f on $\varDelta$. Since $g(p, M)$ is ρ-continuous on $\varDelta$,

$$v'(p) \equiv \int_{\varDelta} g(p, M)\, d\sigma(M)$$

belongs to Ug and

$$\left| \int_{\varDelta} g(p, M)\, d\sigma_{n_\nu}(M) - \int_{\varDelta} g(p, M)\, d\sigma(M) \right| < \varepsilon$$

holds uniformly on $W_2 - \overline{W}_1$ for any sufficiently large ν. Therefore we have

$$\begin{aligned} \rho(v_{n_\nu}, v') &= \int_{\partial W_1} \frac{\partial}{\partial n} \,|\, v_{n_\nu} - v' \,|\, ds \\ &= \int_{\partial W_1} \frac{\partial}{\partial n} \left| \int_{\varDelta} g(p, M)\, d\sigma_{n_\nu}(M) - \int_{\varDelta} g(p, M)\, d\sigma(M) \right| ds \\ &= \varepsilon \int_{\partial W_1} \frac{\partial}{\partial n} \omega(p; \partial W_2, W_2 - \overline{W}_1)\, ds, \end{aligned}$$

where $\omega(p, \partial W_2, W_2 - \overline{W}_1)$ is the harmonic measure. By the triangular inequality we have

$$\rho(v, v') \leqq \rho(v_{n_\nu}, v') + \rho(v_{n_\nu}, v) \leqq M\varepsilon$$

and hence $\rho(v, v') = 0$. Therefore we have $v \in Ug$. From this fact it follows that Ug is ρ-closed in U_P and hence ρ-compact by that of U_P.

Let u belong to $K^{(1)}$ and $u[v]$ be the integral

$$u[v] = \int_{\partial W_1} u(p) \frac{\partial}{\partial n} v(p)\, ds.$$

Let $L_u(M)$ be a limiting value

$$\lim_{n\to\infty} \frac{u(M_n)}{\omega(M_n)}$$

along a fundamental sequence $\{M_n\}$ tending to an ideal boundary point M in $\varDelta$. Let $v \in Ug$, then we have

$$\begin{aligned} u[v] &= \int_{\partial W_1} u(p) \frac{\partial}{\partial n} v(p)\, ds = \int_{\partial W_1} u(p) \frac{\partial}{\partial n} \int_{\varDelta} g(p, M)\, d\sigma_v(M)\, ds \\ &= \int_{\varDelta}\int_{\partial W_1} u(p) \frac{\partial}{\partial n} g(p, M)\, ds\, d\sigma_v(M) = \int_{\varDelta} L_u(M)\, d\sigma_v(M). \end{aligned}$$

Let $v \in U_P$, then we have

$$u[v]=\int_{\partial W_1} u(p)\frac{\partial}{\partial n}v(p)\,ds=\int_{\partial W_m} u(p)\frac{\partial}{\partial n}T_m v(p)\,ds$$

and

$$1=\omega[v]\equiv\int_{\partial W_1}\frac{\partial}{\partial n}v(p)\,ds=\int_{\partial W_m}\omega(p)\frac{\partial}{\partial n}T_m v(p)\,ds.$$

Now we can select an exhaustion $\{W_m\}$ in such a way that any connected component $(\partial W_m)_j$ of ∂W_m is a dividing cycle of the surface W. Let σ_{mj} be the integral

$$\sigma_{mj}=\int_{(\partial W_m)_j}\omega(p)\frac{\partial}{\partial n}T_m v(p)\,ds,$$

then we have

$$\sum_{j=1}^{\nu(m)}\sigma_{mj}=1, \qquad \sigma_{mj}\geqq 0.$$

Further we have

$$u[v]=\sum_{j=1}^{\nu(m)}\frac{u(P_{mj})}{\omega(P_{mj})}\sigma_{mj}$$

for a suitable set of points $P_{mj}\in(\partial W_m)_j$. If $(\partial W_m)_j$ is homologous to $\sum_{k=1}^N(\partial W_l)_k$ on $\overline{W_l-\overline{W}_m}$ and if σ_{lk} corresponds to $(\partial W_l)_k$, then $\sigma_{mj}=\sum_{k=1}^N\sigma_{lk}$; indeed we can easily prove that $T_l v=T_{ml}T_m v$ for any $v\in HP(W-\overline{W}_1)$, where T_{ml} is a T map defined on $HP(W-\overline{W}_m)$ such that $T_{ml}HP(W-\overline{W}_m)=HP(W-\overline{W}_l)$. If the above topological condition on $(\partial W_m)_j$ and $(\partial W_l)_k$ holds, then we say that any such $(\partial W_l)_k$ is a successor of $(\partial W_m)_j$. If a sequence of the dividing cycles $(\partial W_m)_j$, $(\partial W_{m+1})_k, \cdots, (\partial W_{m+p})_l, \cdots$ satisfies successively the condition of successor, then we can select an ideal boundary point $M_{mj}\in\Delta$ from a corresponding sequence $P_{mj}, P_{m+1,k}, \cdots, P_{m+p,l}, \cdots, P_{rs}\in(\partial W_r)_s$. Many such M_{mj} may exist. Now we select any M_{mj}, say $M_{mj}{}^\circ$, and attach a value $L_{mj}(M_{mj}{}^\circ)\equiv u(P_{mj})/\omega(P_{mj})$ and a mass σ_{mj} at the point $M_{mj}{}^\circ$. Then we can write

$$u[v]=\sum_{j=1}^{\nu(m)}L_{mj}(M_{mj}{}^\circ)\,\sigma_{mj}.$$

On the other hand we can consider σ_{lk} as a mass attached at the $M_{lk}{}^\circ\in\Delta$. Then there is a sequence $\{\sigma_{lk}\}$ of mass-distributions on Δ_{mj}, a part of Δ homologous to $(\partial W_m)_j$. Evidently $\sum_{k=1}^N\sigma_{lk}=\sigma_{mj}$ and $\sigma_{lk}\geqq 0$. Thus we can select a subsequence converging weakly to a non-negative mass-distribution $\sigma_v(M)$ on Δ_{mj} such that

$$\int_{\Delta_{mj}}d\sigma_v(M)=\sigma_{mj}.$$

Simultaneously we can select the limit distribution $\sigma_\nu(M)$ or a set of ideal boundary points $\{M\}$, $M\in\Delta_{mj}$ such that $L_{mj}(M_{mj}{}^\circ)\to L_u(M)$. This procedure can be done for all the Δ_{mj}. Resulting mass-distribution is denoted by $\sigma_v(M)$. Evidently the total mass of $\sigma_v(M)$ is equal to 1 and $d\sigma_v(M)\geqq 0$. Now we should remark here that M may depend upon the given $u(p) \in K^{(1)}$. Then we have

$$u[v] = \sum_{j=1}^{\nu(m)} L_{mj}(M_{mj}{}^{\circ}) \int_{\Delta_{mj}} d\sigma_v(M)$$
$$= \sum_{j=1}^{\nu(m)} \int_{\Delta_{mj}} L_{mj}(M_{mj}{}^{\circ})\, d\sigma_v(M) = \int_{\Delta} L_m(M_m{}^{\circ})\, d\sigma_v(M),$$

where $L_m(M_m{}^{\circ})$ is equal to $L_{mj}(M_{mj}{}^{\circ})$ on the jth Δ_{mj} and $M_m{}^{\circ}$ is a generic point of $\{M_{mj}{}^{\circ}\}$ on Δ. Since $L_m(M_m{}^{\circ})$ is uniformly bounded and the total mass is equal to 1, we have

$$\lim_{m\to\infty} \int_{\Delta} L_m(M_m{}^{\circ})\, d\sigma_v(M) = \int_{\Delta} L_u(M)\, d\sigma_v(M).$$

Thus we have the following

LEMMA 6. $\{u[v]\}_{v\in U_P} = \{u[v]\}_{v\in Ug}$.

Next step is our main part. Let $E(v)$ be a functional defined by the integral

$$E(v) = \int_{\partial W_1} \frac{\dfrac{\partial V}{\partial n}}{\dfrac{\partial v}{\partial n}} \frac{\partial V}{\partial n} ds$$

for a $V \in U_P$ and any $v \in Ug$. Suppose that $U_P \supset Ug$ but $U_P \neq Ug$, then there exists a member V in U_P which is not in Ug. Evidently $E(v)$ is ρ-continuous on Ug. By Lemma 5 Ug is ρ-compact, therefore there is at least one minimum value $E_0 = E(v_0)$, $v_0 \in Ug$. Then we have

$$\int_{\partial W_1} \left(1 - \frac{\dfrac{\partial V}{\partial n}}{\dfrac{\partial v}{\partial n}}\right) \frac{\partial v}{\partial n} ds > 0$$

for any $v \in Ug$. This implies the inequality $E(v) > 1$ for any $v \in Ug$. Let ε be any sufficiently small positive number, then $(1-\varepsilon)v_0 + \varepsilon v \in Ug$ is for any $v \in Ug$ a competing function and hence

$$E(v_0) \leqq E((1-\varepsilon)v_0 + \varepsilon v)$$
$$= E(v_0) - \varepsilon \int_{\partial W_1} \frac{\left(\dfrac{\partial V}{\partial n}\right)^2}{\left(\dfrac{\partial v_0}{\partial n}\right)^2} \left(\frac{\partial v}{\partial n} - \frac{\partial v_0}{\partial n}\right) ds$$
$$+ \varepsilon^2 \int_{\partial W_1} \frac{\left(\dfrac{\partial V}{\partial n}\right)^2}{\dfrac{\partial v_0}{\partial n}} \frac{\left(\dfrac{\partial v}{\partial n} - \dfrac{\partial v_0}{\partial n}\right)^2 \Big/ \left(\dfrac{\partial v_0}{\partial n}\right)^2}{1 + \varepsilon\left(\dfrac{\partial v}{\partial n} - \dfrac{\partial v_0}{\partial n}\right) \Big/ \dfrac{\partial v_0}{\partial n}} ds.$$

Since the last term is dominated by

$$\frac{\varepsilon^2}{1-\varepsilon}\int_{\partial W_1}\frac{\frac{\partial V}{\partial n}}{\frac{\partial v_0}{\partial n}}\left(1-\frac{\frac{\partial v}{\partial n}}{\frac{\partial v_0}{\partial n}}\right)^2\frac{\partial V}{\partial n}ds=O(\varepsilon^2),$$

we have

$$\int_{\partial W_1}\frac{\left(\frac{\partial V}{\partial n}\right)^2}{\left(\frac{\partial v_0}{\partial n}\right)^2}\left(\frac{\partial v}{\partial n}-\frac{\partial v_0}{\partial n}\right)ds\leqq 0,$$

that is,

$$E(v_0)\geqq\int_{\partial W_1}\frac{\left(\frac{\partial V}{\partial n}\right)^2}{\left(\frac{\partial v_0}{\partial n}\right)^2}\frac{\partial v}{\partial n}ds\equiv N(v)$$

for any $v\in Ug$. By Lemma 6 there exists at least one member $v_1\in Ug$ such that

$$E(v_0)=\int_{\partial W_1}\frac{\frac{\partial V}{\partial n}}{\frac{\partial v_0}{\partial n}}\frac{\partial v_1}{\partial n}ds,$$

considering

$$\frac{\partial V}{\partial n}\Big/\frac{\partial v_0}{\partial n}$$

as function of $K^{(1)}$ or more precisely as the boundary value of a function of $K^{(1)}$. Therefore we see that

$$0<N(v_1)\leqq E(v_0)\leqq N(v_1)^{1/2}\int_{\partial W_1}\frac{\partial v_1}{\partial n}ds=N(v_1)^{1/2}$$

by Schwarz' inequality. Hence we have $E(v_0)\leqq 1$. The discrepancy between $E(v_0)\leqq 1$ and $E(v_0)>1$ shows that the assumption $U_P\not\equiv Ug$ is untenable. Hence we get the following

THEOREM 1. *$U_P\equiv Ug$ and hence $HP(W-\overline{W}_1)\equiv Hg(W-\overline{W}_1)$.*

By this theorem any $v(p)\in HP(W-\overline{W}_1)$ can be written as follows:

$$v(p)=\int_{\Delta}g(p,M)\,d\sigma_v(M).$$

However, this mass-distribution $\sigma_v(M)$ is, in general, not unique.

Martin's other theorems remain valid with some modifications and the method of proofs for them is quite similar to the original one. So we shall only state the Martin's main theorem:

For any $v \in HP(W-\overline{W}_1)$, *there exists a unique mass-distribution* σ_v *on* $\varDelta_m \subset \varDelta$ *such that*

$$v=\int_{\varDelta_m} g(p, M)\, d\sigma_v(M),$$

where $\varDelta_m$ *is a subset of* $\varDelta$ *consisting of the minimal points* M *for which* $g(p, M)$ *is minimal in Martin's sense in* $HP(W-\overline{W}_1)$.

Heins' theorem 11.2 in [1] can be extended in our case, that is, if there is only one value $L_u(M)$ for any $u \in K^{(1)}$, then $\dim HP(W-\overline{W}_1)=1$ and hence W has only one ideal boundary component and vice versa. If further there is one $u \in K^{(1)}$ such that $u(M)>0$, $M \in \varDelta$, then $W \in O_G$ and hence $W-\overline{W}_1$ is a Heins' end and its Heins' harmonic dimension equals one.

References

[1] Heins, M., Riemann surfaces of infinite genus. Ann. of Math. 55 (1952), 296–317.

[2] Heins, M., On the Lindelöf principle. Ann. of Math. 61 (1955), 440–473.

[3] Martin, R. S., Minimal positive harmonic functions. Trans. Amer. Math. Soc. 49 (1941), 137–171.

Department of Mathematics,
Tokyo Institute of Technology