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ON THE FUNCTIONAL EQUATION $f^n=e^{P_1}+\cdots+e^{P_m}$ AND RIGIDITY THEOREMS FOR HOLOMORPHIC CURVES

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Introduction and statement of results

For each positive integer N we set

$$E_N=\{e^{P_1}+\cdots+e^{P_m}\ P_j\in C[z],\ \deg P_j\leqq N(j=1,\ \cdots,\ m),\ m\in N\}.$$

In 1929 J. F. Ritt [4] showed the following theorem.

THEOREM A. *Let $g_0, g_1, \cdots, g_n$ be elements of E_1 and f be a holomorphic function on $\{z;\ \omega_1<\arg z<\omega_2\}$ $(\omega_2-\omega_1>\pi)$ satisfying $g_nf^n+g_{n-1}f^{n-1}+\cdots+g_0=0$. Then $f\in E_1$.*

It seems to be natural to ask whether Theorem A is valid with E_1 replaced by E_N $(N\geqq 2)$. However, if $g_n\neq 1$, the function $f(z)=\sin(\pi z^2)/\sin \pi z$ gives a negative answer to the above question.

Let $g: C\to P_m$ be a holomorphic curve of finite order, $D_0, D_1, \cdots, D_{m-1}$ be hyperplanes and D_m be a hypersurfac of degree n $(\geqq 2)$ satisfying $D_0\cap\cdots\cap D_m=\emptyset$, $g(C)\cap(D_0\cup\cdots\cup D_m)=\emptyset$. We ask whether the image of g is contained in the intersection of hypersurfaces of P_m. This problem is related to the functional equation $f^n+g_{n-1}f^{n-1}+\cdots+g_0=0$ $(g_0, \cdots, g_{n-1}\in E_N)$ for an entire function f. M. Green [1] treated the first non-trivial case $f^2=e^{2\varphi_1}+e^{2\varphi_2}+e^{2\varphi_3}$ $(\varphi_1, \varphi_2, \varphi_3\in C[z])$ and showed that f is a linear combination of $e^{\varphi_1}, e^{\varphi_2}, e^{\varphi_3}$. He also showed that, if $g: C\to P_2$ is a holomorphic curve of finite order omitting the two lines $\{Z_0=0\}$ and $\{Z_1=0\}$ and the conic $\{Z_0^2+Z_1^2+Z_2^2=0\}$, then the image of g lies in a line or a conic ([1]).

In this paper we shall show the following results.

THEOREM 1. *Let $P_1, \cdots, P_m$ be polynomials, $N=\max_j \deg P_j$, $N\geqq 2$, $A_j=P_j^{(N)}(0)/N!(j=1,\cdots, m)$, n $(\geqq 2)$ be an integer and f be a holomorphic function on $\{z;\ \omega_1<\arg z<\omega_2\}$ $(\omega_2-\omega_1>\pi/N)$. Assume that $\#\{j\ A_j=v,\ j=1,\ \cdots,\ m\}=1$ for every vertex v of the convex hull of $\{A_j\}_{j=1}^m$, and that $f^n=e^{P_1}+\cdots+e^{P_m}$ on $\{z;\ \omega_1<\arg z<\omega_2\}$. Then f is an element of E_N.*

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THEOREM 2. *Let f be an entire function,* $n(\geqq 2)$ *be an integer and* $P_1, \cdots, P_4$ *be polynomials satisfying that* $\sum_{j\in J} e^{P_j}\neq 0$ *for every subset* $J\subset\{1, \cdots, 4\}$ *with* $J\neq\emptyset$ *and that* $P_j-P_k\neq$ *const. for some* $j\neq k$. *Assume that* $f^{n}=e^{P_1}+\cdots+e^{P_4}$. *Then there are the following two possibilities:*

(1) $n=2$, 3 *and* $f=e^{P}+e^{Q}$, *where* P, *Q are polynomials.*

(2) $n=2$ *and* $f=e^{P}R(e^{Q})$, *where P, Q are polynomials and* $R(w)=w^{2}+\sqrt{2}\sigma w-\sigma^{2}$ *with* $\sigma\neq 0$.

In Theorem 2 the vertices of the convex hull of $\{A_j\}_{j=1}^{4}$ do not necessarily satisfy the assumption of Theorem 1. For example, $A_1, \cdots, A_4$ can be on the line segment $\{\alpha+x(\beta-\alpha)\ 0\leqq x\leqq 1\}$ and satisfy $\#\{j\,;A_j=\alpha\}=1$, $\#\{j\,;A_j=\beta\}=2$. In this case, however, it is verified that, if $A_j=A_k$, then $P_j-P_k=$const.. In Section 2 we prove a more general result (Theorem 5). From Theorem 2 we obtain the following theorems.

THEOREM 3. *Let* $g: \boldsymbol{C}\rightarrow\boldsymbol{P}_2$ *be a holomorphic curve of finite order,* D_0, D_1 *be distinct lines and* D_2 *be a conic. Assume that* $D_0\cap D_1\cap D_2=\emptyset$, $g(\boldsymbol{C})\cap(D_0\cup D_1\cup D_2)=\emptyset$. *Then there is a homogeneous polynomial* $Q(w_0, w_1, w_2)$ *of degree at most three satisfying* $g(\boldsymbol{C})\subset\{Q(w_0, w_1, w_2)=0\}$.

THEOREM 4. *Let* $g: \boldsymbol{C}\rightarrow\boldsymbol{P}_3$ *be a nonconstant holomorphic curve of finite order satisfying* $g(\boldsymbol{C})\cap(\{w_0=0\}\cup\{w_1=0\}\cup\{w_2=0\}\cup\{w_0{}^{n}+\cdots+w_3{}^{n}=0\})=0$, *where n* $(\geqq 2)$ *is an integer. Then there are homogeneous polynomials* $Q_1(w_0, \cdots, w_3)$, $Q_2(w_0, \cdots, w_3)$ *which are relatively prime to each other and satisfy* $1\leqq\deg Q_1\leqq 2$, $1\leqq\deg Q_2\leqq 4$ *and* $g(\boldsymbol{C})\subset(\{Q_1(w_0, \cdots, w_3)=0\}\cap\{Q_2(w_0,\cdots, w_3)=0\})$. *Further if* $n\geqq 4$, *then g has the reduced representation* (g_0, g_1, g_2, g_3) *such that* $\{g_j\}_{j=0}^{3}=\{a_0$, fli, $a_2, e^{P}\}$ *or* $\{g_j\}_{j=0}^{3}=\{a_0, a_1, a_2e^{P}, a_3e^{P}\}$, *where* a_j*'s are constants and P is a polynomial.*

The order p of a holomorphic curve $g: \boldsymbol{C}\rightarrow\boldsymbol{P}_m$ is defined by $p=\limsup_{r\rightarrow\infty}(\log T(r, g)/\log r)$, where $T(r, g)$ is the characteristic function of g. (Let $(g_0, g_1, \cdots, g_m)$ be a reduced representation of g. Then we define $T(r, g)=(1/2\pi)\int_0^{2\pi}\log(\max_j|g_j(re^{i\theta})|)d\theta-\log(\max_j|g_j(0)|)$.)

Remark. In Theorem 3 we cannot conclude that the degree of $Q(w_0, w_1, w_2)$ is at most two, since the curve $(1, e^{z}, (1+e^{z})e^{z/2})$ satisfies the assumption of Theorem 3 with $D_0=\{w_0=0\}$, $D_1=\{w_1=0\}$, $D_2=\{w_2{}^{2}-w_0w_1-2w_1{}^{2}=0\}$. (In this case $Q(w_0, w_1, w_2)=w_2{}^{2}w_0-(w_0+w_1)^{2}w_1$and the image lies neither in a line nor in a conic.)

1. Proof of Theorem 1.

For each $\theta\in\boldsymbol{R}$ and $\alpha\in\boldsymbol{C}$, the polynomials $P_1(e^{i\theta}z)+\alpha z^{N}, \cdots, P_m(e^{i\theta}z)+\alpha z^{N}$ satisfy the hypotheses of Theorem 1 with / replaced by $f(e^{i\theta}z)e^{(\alpha/n)z^{N}}$. Therefore we may assume that $\omega_1<0<\omega_2$ and that the following condition (A) is satisfied.

(A) *$n(\geqq 2)$ is an integer, $P_1, \cdots, P_m$ are polynomials, $P_j-P_k \neq$ const. $(j\neq k)$, $N=\max_j \deg P_j$, $N\geqq 2$, $A_j=P_j^{(N)}(0)/N!$ $(j=1, \cdots, m)$, U is the convex hull of $\{A_1, \cdots, A_m\}$, $\{A_1, \cdots, A_t\}$ is the set of the vertices of U, $t\geqq 2$, $\arg(A_1-c)<\arg(A_2-c)<\cdots<\arg(A_t-c)<\arg(A_1-c)+2\pi$ for all $c\in(C-\{A_1, \cdots, A_t\})$, $\operatorname{Re} A_1=\operatorname{Re} A_2$, $\operatorname{Im} A_2>\operatorname{Im} A_1$ and $U\subset\{z; \operatorname{Re} z\leqq \operatorname{Re} A_1\}$.*

For each $\nu\in\{1, \cdots, t\}$, let $\{p_{\nu,j}\}$ be the set of polynomials of degree at most N definedd by

$$
(1.1)\qquad \begin{gathered}
\exp(P_\nu/n)\Big(1+\sum_{j=1}\gamma_j\Big(\sum_{\mu\in\{1,\cdots,m\}-\{\nu\}}\exp(P_\mu-P_\nu)\Big)^j\Big)\equiv\sum\exp(p_{\nu,j}),\\
p_{\nu,j}-p_{\nu,k}\neq\text{const.}\ (j\neq k),\qquad \operatorname{Im}(p_{\nu,j}(0))\in[0, 2\pi),
\end{gathered}
$$

where $1+\sum_{j=1}^{\infty}\gamma_j w^j$ is the Taylor expansion of $(1+w)^{1/n}$ $(|w|<1)$. Put

$$
(1.2)\qquad a_{\nu,j}=p_{\nu,j}{}^{(N)}(0)/N!,
$$

$$
(1.3)\qquad \begin{gathered}
S_\nu=\{z;\ \arg((A_{\nu+1}-A_\nu)/n)\leqq\arg(z-(A_\nu/n))\leqq\arg((A_{\nu-1}-A_\nu)/n)\}\cup\{A_\nu/n\}\\
(A_0=A_t,\ \nu=1, \cdots, t-1),\\
S_t=\{z;\ \arg((A_1-A_t)/n)\leqq\arg(z-(A_t/n))\leqq\arg((A_{t-1}-A_t)/n)\}\cup\{A_t/n\}
\end{gathered}
$$

(see Figure 1 and 2).

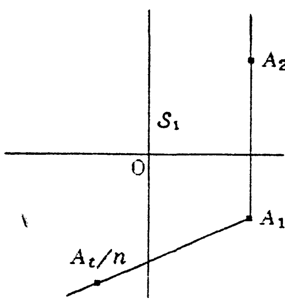

Fig. 1.

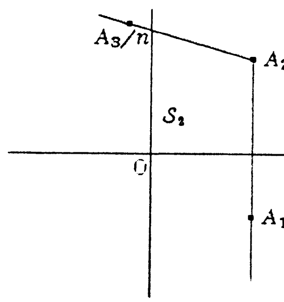

Fig. 2.

Put

$$
H_1=\{z;\ \operatorname{Re} z<0\}\cup i\boldsymbol{R}^+,\qquad H_2=\{z;\ \operatorname{Re} z<0\}\cup i\boldsymbol{R}^-,
$$

where $i\boldsymbol{R}^+=\{ix; x>0\}$, $i\boldsymbol{R}^-=\{ix; x<0\}$. For $\theta\in(0, \pi/2)$ and $d>0$, we set

$$
G_1(\theta, d)=\{z;\ 0<\arg z<\theta,\ \operatorname{Im} z>d\},
$$

$$
G_2(\theta, d)=\{z;\ 0>\arg z>-\theta,\ \operatorname{Im} z<-d\}.
$$

Further we denote by

$$C(p)$$

the leading coefficient of a polynomial p. Note that $C(p)=0$ if and only if $p=0$.

LEMMA 1.1. *Let $p_1, \cdots, p_m$ be polynomials satisfying $C(p_j)\in H_1 (j=1, \cdots, m)$ or $C(p_j)\in H_2 (j=1, \cdots, m)$ and $\lambda_1, \cdots, \lambda_m$ be positive numbers. Then* $\deg(\lambda_1 p_1+\cdots+\lambda_m p_m)=\max_j \deg p_j$, *and $C(\lambda_1 p_1+\cdots+\lambda_m p_m)\in H_1$ or $C(\lambda_1 p_1+\cdots+\lambda_m p_m)\in H_2$ respectively.*

proof. Assume that $C(p_j)\in H_1 (j=1, \cdots, m)$. Put $D=\max_j \deg p_j$, $J=\{j; \deg p_j=D, j=1, \cdots, m\}$, $c=\sum_{j\in J}\lambda_j C(p_j)$. Then we have $c\in H_1$. Further $\lambda_1 p_1+\cdots+\lambda_m p_m=cz^D+q(z)$, where q is a polynomial of degree at most $D-1$. Thus $\deg(\lambda_1 p_1+\cdots+\lambda_m p_m)=D=\max_j \deg p_j$, $C(\lambda_1 p_1+\cdots+\lambda_m p_m)=c\in H_1$.

Let $\nu\in\{1, 2\}$ be fixed. When polynomials p, q satisfy $C(p-q)\in H_\nu$, we write $p<_\nu q$. Then, by Lemma 1.1, $(\boldsymbol{C}[z], <_\nu)$ is an ordered set. Further, if $p\neq q$, then $p<_\nu q$ *or* $q<_\nu p$. Therefore $(\mathbf{CM}, <_1)$, $(\mathbf{CM}, <_2)$ are totally ordered sets. Hence we have the following

LEMMA 1.2. *Let Π $(\neq\emptyset)$ be a finite subset of* CM. *Then there are $p_1, p_2 \in\Pi$ such that $C(p-p_1)\in H_1$ for every $p\in\Pi-\{p_1\}$ and that $C(p-p_2)\in H_2$ for every $p\in\Pi-\{p_2\}$.*

LEMMA 1.3. *Let p be a polynomial of degree $N(\geqq 1)$.*

(1) *If* Re $C(p)<0$, *then there are positive numbers K, θ, R such that*

$$|\exp p(z)|<\exp(-K'|z|^N) \quad on \ \{z; |\arg z|<\theta, \ |z|>R\}.$$

(2) *If $C(p)\in i\boldsymbol{R}^+$, then there are positive numbers K', θ', d' such that*

$$|\exp p(z)|<\exp(-K'|\mathrm{Im}\, z|\,|z|^{N-1}) \quad on \ G_1(\theta', d').$$

(3) *If $C(p)\in i\boldsymbol{R}^-$, then there are positive numbers K'', θ'', d'', such that*

$$|\exp p(z)|<\exp(-K''|\mathrm{Im}\, z|\,|z|^{N-1}) \quad on \ G_2(\theta'', d'').$$

Proof. We shall prove only (2). Put $C(p)=iA\,(A>0)$, $q(z)=p(z)-iAz^N$. Then for $\zeta\in(0, \pi/4)$ we have

$$|\exp(p(z))|=|\exp(iAz^N+q(z))|=|\exp(iA(x^N+iNyx^{N-1}+\cdots+(iy)^N)+q(z))|$$

$$<\exp(-ANyx^{N-1}(1+O(y/x))+B|z|^{N-1}) \quad \text{on} \ \{|\arg z|<\zeta\},$$

where B is a positive constant and $z=x+iy$. Thus we have the desired result.

LEMMA 1.4. *Let m be a positive integer and $\Delta(\neq 0)$ be a subset of $(\boldsymbol{N}\cup\{0\})^{m}$. Then there exist $\alpha_{1}, \cdots, \alpha_{\tau}\in\Delta$ $(\tau<\infty)$ such that $\Delta\subset\{\alpha_{j}+\beta\, j=1, \cdots, \tau, \beta\in(\boldsymbol{N}\cup\{0\})^{m}\}$.*

Proof. By induction on m. For $\alpha_{1}, \cdots, \alpha_{p}\in(\boldsymbol{N}\cup\{0\})^{q}$ $(p, q\in\boldsymbol{N})$ we set

$$\langle\alpha_{1}, \cdots, \alpha_{p}\rangle=\{\alpha_{j}+\beta ; j=1, \cdots, p, \beta\in(\boldsymbol{N}\cup\{0\})^{q}\}.$$

Further for $\alpha=(\lambda_{1}, \cdots, \lambda_{q})\in(\boldsymbol{N}\cup\{0\})^{q}$ and $\lambda\in\boldsymbol{N}\cup\{0\}$, we denote by (α, λ) the element $(\lambda_{1}, \cdots, \lambda_{q}, \lambda)$ of $(\boldsymbol{N}\cup\{0\})^{q+1}$. It is easily seen that Lemma 1.4 holds for $m=1$. Assume that Lemma 1.4 holds for $m=\nu$. Let J be a subset of $(\boldsymbol{N}\cup\{0\})^{\nu+1}$ satisfying the assumption with m replaced by $\nu+1$. Put

$$\tilde{\Delta}=\{(\lambda_{1}, \cdots, \lambda_{\nu}) ; (\lambda_{1}, \cdots, \lambda_{\nu+1})\in\Delta\}.$$

Then, by the induction assumption, there exist $\tilde{\alpha}_{1}, \cdots, \tilde{\alpha}_{\rho}\in\tilde{\Delta}$ $(\rho\in\boldsymbol{N})$such that

$$\tilde{\Delta}\subset\langle\tilde{\alpha}_{1}, \ldots, \tilde{\alpha}_{\rho}\rangle .$$

Let

$$\lambda^{(j)}=\min\{\lambda_{\nu+1} ; (\tilde{\alpha}_{j}, \lambda_{\nu+1})\in\Delta\} \qquad (j=1, \cdots, \rho),$$

$$M=\max_{j}\lambda^{(j)},$$

$$\Delta^{(\sigma)}=\{(\lambda_{1}, \cdots, \lambda_{\nu}) \ (\lambda_{1}, \cdots, \lambda_{\nu}, \sigma)\in\Delta\} \qquad (\sigma=0, 1, \cdots, M).$$

Then, for every $\sigma\in\{0, 1, \cdots, M\}$, there exist $\alpha_{1}^{(\sigma)}, \cdots, \alpha_{\rho_{\sigma}}^{(\sigma)}\in\Delta^{(\sigma)}$ $(\rho_{\sigma}\in\boldsymbol{N}\cup\{0\})$ such that

$$\Delta^{(\sigma)}\subset\langle\alpha_{1}^{(\sigma)}, \cdots, \alpha_{\rho_{\sigma}}^{(\sigma)}\rangle \quad (\sigma=0, 1, \cdots, M).$$

Let $\alpha=(\lambda_{1}, \cdots, \lambda_{\nu+1})\in\Delta$. Then $(\lambda_{1}, \cdots, \lambda_{\nu})\in\tilde{\Delta}$. Thus for some j we have tfi, $\cdots, \lambda_{\nu})\in\langle\tilde{\alpha}_{j}\rangle$. Therefore, if $\lambda_{\nu+1}\geqq M$, then $\alpha\in\langle(\tilde{\alpha}_{j}, \lambda^{(j)})\rangle$. If $\lambda_{\nu+1}\leqq M$, then $(\lambda_{1}, \cdots, \lambda_{\nu})\in\Delta^{(\lambda_{\nu+1})}$. Thus for some j we have $(\lambda_{1}, \cdots, \lambda_{\nu})\in\langle\alpha_{j}^{(\lambda_{\nu+1})}\rangle$. Therefore $\alpha\in\langle(\alpha_{j}^{(\lambda_{\nu+1})}, \lambda_{\nu+1})\rangle$. Put

$$\alpha_{j}=(\tilde{\alpha}_{j}, \lambda^{(j)}) \qquad (j=1, \cdots, \rho),$$

$$\alpha_{\sigma, j}=(\alpha_{j}^{(\sigma)}, \sigma) \qquad 0=1, \cdots, \rho_{\sigma}, \sigma=0, \cdots, M).$$

Then $\alpha_{j}\in\Delta$ $0=1. \cdots, \rho)$, $\alpha_{\sigma, j}\in\Delta$ $0=1, \cdots, \rho_{\sigma}, \sigma=0, \cdots$, M) and

$$\Delta\subset\langle\alpha_{1}, \cdots, \alpha_{\rho}, \alpha_{0,1}, \cdots, \alpha_{M, \rho_{M}}\rangle .$$

Lemma 1.4 is thus proved.

LEMMA 1.5. *Assume that* (A) *holds. Then $\{a_{\nu, j}\}_{j}\subset\mathcal{S}(\nu=1, \cdots, t)$ Further if $\#\{j ; A_{\nu}=A_{j}, j=1, \cdots, m\}=1$, then $\{a_{\nu, j}\}_{j}$ has no finite accumulotion point.*

Proof. We shall give the proof only for $\nu=1$. From (1.1)-(1.3) we have

$$\{a_{1,j}\}_j\subset\{(A_1/n)+\sum_{j\neq 1}\lambda_j(A_j-A_1)\,;\lambda_j\in\boldsymbol{N}\cup\{0\}\}\subset\mathcal{S}_1\,.$$

Further if $\#\{j\ A_1=A_j,j=1,\ \cdots,m\}=1$, then $A_j-A_1\neq 0$ for any $j\neq 1$. Thus $\{a_{1,j}\}_j$ has no finite accumulation point (see Figure 1 and 2).

For each polynomial q, we put

$$J_1=\{j\,;\ \mathrm{Re}\ C(p_{1,j}-q)>0,\ p_{1,j}-q\neq\text{const.}\},$$

$$J'_1=\{j\,;\ C(p_{1,j}-q)\in i\boldsymbol{R}^-,\ p_{1,j}-q\neq\text{const.}\},$$

$$J''_1=\{j\,;\ C(p_{1,j}-q)\in H_1,\ p_{1,j}-q\neq\text{const.}\},$$

$$J_2=\{j\quad \mathrm{Re}\ C(p_{2,j}-q)>0, p_{2,j}-q\neq\text{const.}\},$$

$$J'_2=\{j\,;\ C(p_{2,j}-q)\in i\boldsymbol{R}^+,\ p_{2,j}-q\neq\text{const.}\},$$

$$J''_2=\{j\,;\ C(p_{2,j}-q)\in H_2,\ p_{2,j}-q\neq\text{const.}\},$$

$$R_1[q]=\sum_{\in J_1}\exp(p_{1,j}),\quad S_1[q]=\sum_{\in J'_1}\exp(p_{1,j}),\quad T_1[q]=\sum_{\in J''_1}\exp(p_{1,j}),$$

$$R_2[q]=\sum_{j\in J_2}\exp(p_{2,j}),\quad S_2[q]=\sum_{j\in J'_2}\exp(p_{2,j}),\quad T_2[q]=\sum_{j\in J''_2}\exp(p_{2,j}),$$

$$b_1(q)=\begin{cases}\exp(p_{1\ j}(0)-q(0)) & \text{if } p_{1\ j}-q=\text{const. for some } j,\\ 0 & \text{if } p_{1,j}-q\neq\text{const. for all } 7,\end{cases}$$

$$b_2(q)=\begin{cases}\exp(p_{2\ j}(0)-q(0)) & \text{if } p_{2\ j}-q=\text{const. for some } j,\\ 0 & \text{if } p_{2,j}-q\neq\text{const. for all } j.\end{cases}$$

Then

$$\sum_j\exp(p_{1,j})=b_1(q)e^q+R_1[q]+S_1[q]+T_1[q]\,,$$

$$\sum_j\exp(p_{2,j})=b_2(q)e^q+R_2[q]+S_2[q]+T_2[q]\,.$$

We see that $b_1(q)=1$ if and only if $q\in\{p_{1,j}\}_j$ and that $b_2(q)=1$ if and only if $q\in\{p_{2,j}\}_j$. Thus, if $q\in(\{p_{1,j}\}_j-\{p_{2,j}\}_j)\cup(\{p_{2,j}\}_j-\{p_{1,j}\}_j)$, then $b_1(q)\neq b_2(q)$.

LEMMA 1.6. *Let q be a polynomial of degree at most N. Assume that* (A) *holds. If $\#\{j\,;A_1=A_j,\ j=1,\ \cdots,\ m\}=l$ or $\#\{j\ A_2=A_j,j=1,\ \cdots,\ m\}=1$, then we have $S_1[q]\in E_N$ or $S_2[q]\in E_N$ respectively.*

Proof. Put

$$\alpha_0=q^{(N)}(0)/N!\,,$$

$$\mathcal{L}_1=\mathcal{S}_1\cap\{z\,;\ \mathrm{Re}\ z=\mathrm{Re}\ \alpha_0,\ \mathrm{Im}\ z\leqq\mathrm{Im}\ \alpha_0\},$$

$$\mathcal{L}_2=\mathcal{S}_2\cap\{z\quad \mathrm{Re}\ z=\mathrm{Re}\ \alpha_0,\ \mathrm{Im}\ z\geqq\mathrm{Im}\ \alpha_0\}.$$

Then by the definitions of J'_1, J'_2

$$\{a_{1,j}\ j\in J'_1\}\subset \mathcal{L}_1, \qquad \{a_{2,j};\ j\in J'_2\}\subset \mathcal{L}_2\,.$$

Further $\mathcal{L}_1$, $\mathcal{L}_2$ are compact sets (see Figure 3 and 4). Therefore, if $\#\{j\,;\,A=A_j,\ j=1,\cdots,m\}=1$, then by Lemma 1.5 we have $\#\{a_{1,j}\,;\ \in J'_1\}<\infty$. Thus $S_1[q]\in E_N$. Similarly, if $\#\{j\,;\,A_2=A_j,\ j=1,\ \cdots,\ m\}=1$, then we have $S_2[q]\in E_N$. Lemma 1.6 is thus proved.

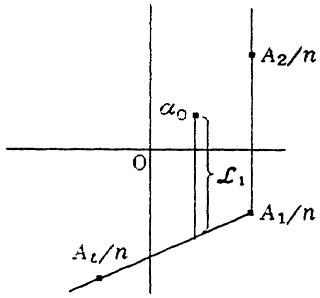

Fig. 3.

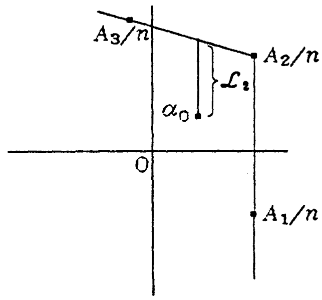

Fig. 4.

LEMMA 1.7. *Let q be a polynomial of degree at most N. Assume that* (A) *holds and that $S_1[q]\in E_N$, $S_2[q]\in E_N$. Then there exist positive constants $\theta'(q)$, $d'(q)$, h_1, h_2 such that*

$$|e^{-q(z)}S_2[q](z)|\leqq \exp(-h_1|\mathrm{Im}\,z|) \qquad on\ G_1(\theta'(q),\ d'(q)),$$

$$|e^{-q(z)}S_1[q](z)|\leqq \exp(-h_2|\mathrm{Im}\,z|) \qquad on\ G_2(\theta'(q),\ d'(q)).$$

Proof. By the definitions of $S_1[q]$, $S_2[q]$ and Lemma 1.3, we easily have the desired result.

LEMMA 1.8. *Let q be a polynomial of degree at most N. Assume that* (A) *holds, $C(P_\mu-P_1)\in H_1$ for every $\mu\neq 1$ and that $C(P_\mu-P_2)\in H_2$ for every $\mu\neq 2$. Then there exist positive constants $\theta(q)$, $d(q)$, k_1, k'_1, k_2, k'_2 such that*

(1) $|e^{-q(z)}T_1[q](z)|\leqq \exp(-k_1|\mathrm{Im}\,z|)+\exp(-k'_1|z|) \qquad on\ \ G_1(\theta(q),\ d(q)),$

(2) $|e^{-q(z)}T_2[q](z)|\leqq\exp(-k_2|\mathrm{Im}\,z|)+\exp(-k'_2|z|)$ *on* $G_2(\theta(q), d(q))$.

Proof. We shall prove only (1). We may assume $P_1=0$. Then

(1.4) $$\deg P_\mu\geqq 1, \qquad C(P_\mu)\in H_1 \quad \text{for every } \mu\neq 1.$$

For each $\lambda=(\lambda_2, \lambda_3, \cdots, \lambda_m)\in(N\cup\{0\})^{m-1}$ we put

$$\|\lambda\|=\lambda_2+\lambda_3+\cdots+\lambda_m,$$

$$\delta(\lambda)=\gamma_{\|\lambda\|}\frac{\|\lambda\|!}{\lambda_2!\lambda_3!\cdots\lambda_m!},$$

$$q^{(\lambda)}=\lambda_2P_2+\lambda_3P_3+\cdots+\lambda_mP_m,$$

where $(1+w)^{1/n}=\sum_{j=0}^{\infty}\gamma_jw^j$ $(|w|<1)$. Let k be a positive number such that

$$(m-1)k<1.$$

Then by (1.4) and Lemma 1.3, for suitable θ, d,

$$|\exp(P_\mu(z))|<k \quad \text{on } G_1(\theta,d) \quad (\mu=2, \text{ - }, m).$$

Hence

$$\sum_{j=0}^{\infty}\gamma_j(\sum_{\mu=2}^{m}\exp(P_\mu))^j=\sum_{\|\lambda\|\geqq 0}\delta(\lambda)\exp(q^{(\lambda)}),$$

$$\sum_j|\exp(p_{1,j}(z))|\leqq\sum_{\|\lambda\|\geqq 0}|\delta(\lambda)||\exp(q^{(\lambda)}(z))|\leqq\sum_{\|\lambda\|\geqq 0}|\delta(\lambda)|k^{\|\lambda\|}<\infty$$

on $G_1(\theta$, rf). Therefore $\sum_j\exp(p_{1,j}(z))$ is absolutely convergent and holomorphic on $G_1(\theta, d)$.

Put

$$\Delta=\{\lambda\in(N\cup\{0\})^{m-1};\ C(q^{(\lambda)}-q)\in H_1, \text{ tf\^{}- } q\neq\text{const.}\}.$$

Then by Lemma 1.4, there exist $\alpha_1, \cdots, \alpha_\tau\in\Delta$ satisfying

(1.5) $$\Delta\subset\langle\alpha_1, \cdots, \alpha_\tau\rangle.$$

Put

$$\Gamma(\lambda_0, h)=|\delta(\lambda_0)|+\sum_{\|\lambda\|\geqq 1}|\delta(\lambda_0+\lambda)|h^{\|\lambda\|}.$$

Then $\Gamma(\lambda_0, h)<\infty$ for all $\lambda_0\in(N\cup\{0\})^{m-1}$ and all $h\in(0, 1/(m-1))$. By (1.5)

$$\sum_{j\in J''_1}|\exp(p_{1,j}-q)|\leqq\sum_{\lambda\in\Delta}|\delta(\lambda)||\exp(q^{(\lambda)}-q)|$$

$$\leqq\sum_{j=1}^{\tau}|\exp(q^{(\alpha_j)}-q)|(|\delta(\alpha_j)|+\sum_{\|\lambda\|\geqq 1}|\delta(\alpha_j+\lambda)||\exp(q^{(\lambda)})|)$$

$$\leqq\sum_{j=1}^{\tau}|\exp(q^{(\alpha_j)}-q)|\Gamma(\alpha_j, k) \quad \text{on } G_1(\theta, \mathrm{d}).$$

Since $C(q^{(\alpha_j)}-q)\in H_1$ and $q^{(\alpha_j)}-q\neq$const., by Lemma 1.3 there exist positive constants $\theta(q)$ ($<\theta$), $d(q)$ ($>d$), k_1, k'_1 such that

$$\sum_{j=1}^{\tau}|\exp(q^{(\alpha_j)}(z)-q(z))|\Gamma(\alpha_j,\ k)<\exp(-k_1|\mathrm{Im}\, z|)+\exp(-k'_1|z|)$$

on $G_1(\theta(q),\ d(q))$. Thus we have the desired result.

LEMMA 1.9. *Let f be a holomorphic function on* $\{z;|\arg z|<\omega_0\}$ ($\omega_0>0$). *Assume that* (A) *holds,* $\#\{j;A_1=A_j, j=1,\cdots,m\}=l$, $\#\{j;A_2=A_j, j=1,\cdots,m\}=1$ *and that* $f^n=e^{p_1}+\cdots+e^{p_m}$ *on* $\{z;|\arg z|<\omega_0\}$. *Then*

$$\{p_{1,j}\}_j=\{p_{2,j}\}_j.$$

Proof. Put $W=(\{p_{1,j}\}_j-\{p_{2,j}\}_j)\cup(\{p_{2,j}\}_j-\{p_{1,j}\}_j)$ and assume $W\neq\emptyset$. Then, by Lemma 1.5, $\{\mathrm{Re}\, a_{\nu,j};p_{\nu,j}\in W\}$ is a discrete set which is bounded from above. Thus there exists $\alpha_0\in\{a_{\nu,j};p_{\nu,j}\in W\}$ which satisfies

$$\mathrm{Re}\,\alpha_0=\max\{\mathrm{Re}\, a_{\nu,j};\ p_{\nu,j}\in W\}. \tag{1.6}$$

Put

$$W'=\{p_{\nu,j}\in W;\ a_{\nu,j}=\alpha_0\}.$$

Then, by Lemma 1.5, $\#W'<\infty$. Thus, by Lemma 1.2, there exists a polynomial q_0 in W' such that

$$\mathrm{Re}\, C(p-q_0)\leqq 0 \quad \text{for every } p\in W'.$$

On the other hand, by (1.6), $\mathrm{Re}\, C(p_{\nu,j}-q_0)=\mathrm{Re}(a_{\nu,j}-\alpha_0)\leqq 0$ for every $p_{\nu,j}\in W-W'$. Thus we have

$$\mathrm{Re}\, C(p-q_0)\leqq 0 \quad \text{for every } p\in W. \tag{1.7}$$

We define J_1, J'_1, J''_1, J_2, J'_2, J''_2 for $q=q_0$. Then by (1.7) we have $\{p_{1,j};j\in J_1\}\cap W=\emptyset$, $\{p_{2,j};j\in J_2\}\cap W=\emptyset$. Therefore, by the definitions of J_1, J_2 and W, $\{p_{1,j}\ j\in J_1\}=\{p_{2,j}\ j\in J_2\}$ and $\{a_{1,j}\ \in J_1\}=\{a_{2,j}\ j\in J_2\}$. Further, if $j\in J_1$, then by the definition of J_1 we have $a_{1,j}=\alpha_0$ or $\mathrm{Re}\, a_{1,j}>\mathrm{Re}\,\alpha_0$ Thus by Lemma 1.5

$$\{a_{1,j};\ j\in J_1\}\subset(\mathcal{S}_1\cap\mathcal{S}_2\cap\{z;\ \mathrm{Re}\, z\geqq\mathrm{Re}\,\alpha_0\}).$$

Put ff^cSiΓVSsjΠte; $\mathrm{Re}\, z\geqq\mathrm{Re}\,\alpha_0\}$. Since $\mathcal{T}$ is a compact set, by Lemma 1.5 we have $\#\{a_{1,j};\ j\in J_1\}<\infty$ (see Figure 5). Thus $R_1=R_2\in E_N$.

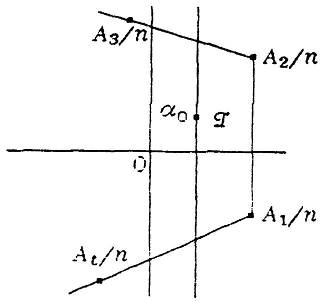

Fig. 5.

Let $\theta(q_0)$, $\theta'(q_0)$, $d(q_0)$, $d'(q_0)$ be positive constants for which Lemma 1.7 and 1.8 hold with q replaced by q_0, and let θ_0, d_0 be positive constants satisfying $0<\theta_0<\min(\omega_0, \theta(q_0), \theta'(q_0))$, $d_0>\max(d(q_0), d'(q_0))$. Put

$$R=R_1[q_0]=R_2[q_0], \qquad F=(f-R-S_1[q_0]-S_2[q_0]-b_2(q_0)e^{q_0})e^{-q_0}.$$

Then, by Lemma 1.6, F is a holomorphic function on $\{|\arg z|<\omega_0\}$ satisfying

$$(1.8)\quad \begin{aligned} F(z)&=(b_1(q_0)-b_2(q_0))-S_2[q_0](z)e^{-q_0(z)}+T_1[q_0](z)e^{-q_0(z)} \quad \text{on } G_1(\theta_0, d_0),\\ F(z)&=-S_1[q_0](z)e^{-q_0(z)}+T_2[q_0](z)e^{-q_0(z)} \quad \text{on } G_2(\theta_0, d_0)\end{aligned}$$

Therefore, by Lemma 1.7 and 1.8, there are positive constants K_1, K_2 such that for every $y_0>d_0$ we have

$$(1.9)\quad \begin{aligned} |F(x+iy_0)-(b_1(q_0)-b_2(q_0))|&\leqq\exp(-K_1y_0)+o(1) \quad (x\to+\infty),\\ |F(x-iy_0)|&\leqq\exp(-K_2y_0)+o(1) \quad (x\to+\infty).\end{aligned}$$

Put $L_0=y_0\tan^{-1}\theta_0$. Then F is bounded on $\partial\{z;\operatorname{Re}z\geqq L_0, |\operatorname{Im}z|\leqq y_0\}$ and satisfies $|F(z)|<\exp(A|z|^N)$on $\{z$ $\operatorname{Re}z\geqq L_0$, $|\operatorname{Im}z|\leqq y_0\}$ with a positive constant A Therefore by the Phragmén-Lindelöf theorem (see [3; p. 43]) it is verified that F is a bounded function in $\{z;\operatorname{Re}z\geqq L_0, |\operatorname{Im}z|\leqq y_0\}$. Let L $(>L_0)$ be a positive number. Then

$$\frac{1}{L}\int_{L_0}^{L}F(x+iy_0)dx=\frac{1}{L}\int_{L_0}^{L}F(x-iy_0)dx -\frac{i}{L}\int_{-y_0}^{y_0}F(L_0+iy)dy+\frac{i}{L}\int_{-y_0}^{y_0}F(L+iy)dy$$

(see Figure 6).

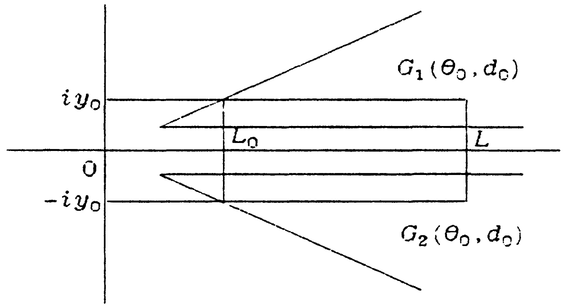

Fig. 6.

Since F is bounded on $\{z; \mathrm{Re}\, z \geqq L_0,\ |\mathrm{Im}\, z| \leqq y_0\}$, by using (1.9) we have

$$|b_1(q_0)-b_2(q_0)| \leqq \exp(-K_1 y_0)+\exp(-K_2 y_0)+o(1)+O(1/L) \qquad (L\to+\infty).$$

Since $q_0 \in W$, we have $b_1(q_0) \neq b_2(q_0)$. Thus for y_0 sufficiently large, we have a contradiction. Thus $W=\emptyset$, namely $\{p_{1,j}\}_j=\{p_{2,j}\}_j$. Lemma 1.9 is thus proved.

From Lemma 1.5 and 1.9 we have the following

COROLLARY. *Under the hypotheses of Lemma* 1.9, *assume that* $\mathcal{S}_1 \cap \mathcal{S}_2$ *is a bouuded set. Then* f *is an element of* E_N.

Now we can complete the proof of Theorem 1. For each polynomial p and $\theta \in \boldsymbol{R}$, we set $(p)_\theta(z)=p(ze^{i\theta})$. Then $(\cdot)_\theta : \boldsymbol{C}[z] \to \boldsymbol{C}[z]$ is a linear bijection which leaves every element of C $(\subset \boldsymbol{C}[z])$ fixed. Therefore for every $\nu \in \{1, \cdots, t\}$ and $\theta \in \boldsymbol{R}$, we have

$$\exp((P_\nu)_\theta/n)\Big(1+\sum_{j=1}\gamma_j\big(\sum_{\mu\in\{1,\ \ ,m\}-\{\nu\}}\exp((P_\mu)_\theta-(P_\nu)_\theta)\big)^j\Big)=\sum_j \exp(p_{\nu,j})_\theta.$$

Let $\nu \in \{1, \cdots, t\}$ be fixed. Then there exists $\theta_\nu \in (-\pi/N, \pi/N]$ such that

$$\mathrm{Re}\,(A_\nu e^{iN\theta_\nu})=\mathrm{Re}(A_{\nu+1}e^{iN\theta_\nu}), \qquad \mathrm{Im}\,(A_\nu e^{iN\theta_\nu})<\mathrm{Im}(A_{\nu+1}e^{iN\theta_\nu}),$$

$$\mathrm{Re}\,(A_\mu e^{iN\theta_\nu}) \leqq \mathrm{Re}\,(A_\nu e^{iN\theta_\nu}) \qquad (\mu=1, - , m).$$

Therefore (A) is fulfilled with $P_1, - , P_m, A_1, A_2$ replaced by $(P_1)_{\theta_\nu}, - , (P_m)_{\theta_\nu}, A_\nu e^{iN\theta_\nu}, A_{\nu+1}e^{iN\theta_\nu}$ respectively. Thus, if $\theta_\nu \in (\omega_1, \omega_2)$, then by Lemma 1.9 $\{(p_{\nu,j})_{\theta_\nu}\}_j=\{(p_{\nu+1,j})_{\theta_\nu}\}_j$. Therefore $\{p_{\nu,j}\}_j=\{p_{\nu+1,j}\}_j$, $\{a_{\nu,j}\}_j=\{a_{\nu+1,j}\}_j \subset \mathcal{S}_\nu \cap \mathcal{S}_{\nu+1})$. Let $\nu_0 \in \{1, \cdots, t\}$ be the integer such that $\{\nu; \theta_\nu \in (\omega_1, \omega_2)\}-\{\nu_0, \nu_0+1, \cdots, \nu_0+s\}$ (mod t). Then $\{a_{\nu_0,j}\}_j=\{a_{\nu_0+1,j}\}_j=\cdots=\{a_{\nu_0+s+1,j}\}_j \subset (\mathcal{S}_{\nu_0} \cap \mathcal{S}_{\nu_0+1} \cap \cdots \cap \mathcal{S}_{\nu_0+s+1})$. (We set $a_{\nu+t,j}=a_{\nu,j}$, $\mathcal{S}_{\nu+t}=\mathcal{S}_\nu$ $(\nu \in \{1, \cdots, t\})$) Since $\omega_2-\omega_1>\pi/N$, $\mathcal{S}_{\nu_0} \cap \mathcal{S}_{\nu_0+1} \cap \cdots \cap \mathcal{S}_{\nu_0+s+1}$ is a bounded set. Thus $\{a_{\nu_0,j}\}_j$ is a finite set Therefore by Lemma 1.5 $\{p_{\nu_0,j}\}_j$ is so. Thus / is an element of E_N. Theorem 1 is thus proved.

2. Proof of Theorem 2.

We begin with the proof of the following

THEOREM 5. *Let f be a holomorphic function on* $\{z; |\arg z|<\omega_0\}$ $(\omega_0>0)$ *and g be an element of E_{N-1}. Assume that* (A) *holds,* $\#\{j; A_1=A_j, j=1, \cdots, m\}=1$, $\#\{j; A_2=A_j, 7=1, \cdots, m\}=1$ *and that* $f^n=ge^{P_1}+e^{P_2}+\cdots+e^{P_m}$ *on* $\{z; |\arg z|<\omega_0\}$. *Then $g=h^n$ for some $h\in E_{N-1}$.*

LEMMA 2.1. *Let $n(\geqq 2)$ be an integer, $P_1, \cdots, P_s$ be polynomials, $P_\mu-P_\nu\neq$ const. $(\mu\neq\nu)$, $C(P_\mu-P_1)\in H_1(\mu=2, \cdots, s)$ and $\{r_j\}_j$ be the set of polynomials defined by*

$$\exp(P_1/n)\Big(1+\sum_{j=1}\gamma_j\big(\sum_{\mu=2}\exp(P_\mu-P_1)\big)^j\Big)\equiv\sum_j\exp(r_j),$$

$$r_j-r_k\neq const.\ (j\neq k),\qquad \mathrm{Im}(r_j(0))\in[0, 2\pi),$$

uhere $1+\sum_{j=l}^{\infty}\gamma_j w^j=(1+w)^{1/n}$ $(|w|<1)$. *Let Π $(\neq\emptyset)$ be a subset of $\{r_j\}_j$. Then there exists a polynomial $p_0\in\Pi$ such that*

$$C(p-p_0)\in H_1 \quad \text{for every } p\in\Pi-\{p_0\}.$$

Proof. We may assume $P_1=0$. Then

$$\deg P_\mu\geqq 1,\qquad C(P_\mu)\in H_1\ (\mu=2, \cdots, s).$$

For each polynomial p we set

$$(p)^*(z)=p(z)-p(0),$$

and for each $\lambda=(\lambda_2, \cdots, \lambda_s)\in(\boldsymbol{N}\cup\{0\})^{s-1}$

$$q^{(\lambda)}=\lambda_2P_2+\cdots+\lambda_sP_s.$$

Then $(\cdot)^*: \boldsymbol{C}[z]\to\boldsymbol{C}[z]$ is a linear mapping. By the definition of $\{r_j\}_j$ and Lemma 1.1, we have

$$\{(r_j)^*\}_j\subset\{(q^{(\lambda)})^*; \lambda\in(\boldsymbol{N}\cup\{0\})^{s-1}\}, \tag{2.1}$$

$$\deg q^{(\beta)}\geqq 1,\quad C((q^{(\beta)})^*)\in H_1 \quad \text{for every } \beta\neq(0, \cdots, 0). \tag{2.2}$$

Put

$$\Pi^*=\{(p)^*; p\in\Pi\},\qquad \Delta=\{\lambda; (q^{(\lambda)})^*\in\Pi^*\}. \tag{2.3}$$

Then by (2.2)

$$\Pi^*=\{(q^{(\lambda)})^*; \lambda\in\Delta\}.$$

Further by Lemma 1.4 there exist $\alpha_1, \cdots, \alpha_\tau\in\Delta$ such that

$$\Delta\subset\langle\alpha_1, \cdots, \alpha_\tau\rangle .$$

Put

$$\tilde{\Pi}=\{(q^{(\alpha_j+\beta)})^* ; j=1, \ldots, \tau, \beta\in(\boldsymbol{N}\cup\{0\})^{s-1}\}.$$

Then

$$\Pi^*\subset\tilde{\Pi}. \tag{2.4}$$

By Lemma 1.2 we may assume

$$C(q^{(\alpha_j)}-q^{(\alpha_1)})\in H_1$$

for every $q^{(\alpha_j)}$ satisfying $(q^{(\alpha_j)})^*\neq(q^{(\alpha_1)})^*$. Note that $C((q^{(\alpha_j+\beta)})^*-(q^{(\alpha_1)})^*)=$ $C((q^{(\alpha_j+\beta)})^*-(q^{(\alpha_j)})^*+(q^{(\alpha_j}$ Therefore by (2.2) and Lemma 1.1

$$C(p-(q^{(\alpha_1)})^*)\in H_1 \qquad \text{for every } p\in\tilde{\Pi}-\{(q^{(\alpha_1)})^*\}.$$

Thus by (2.4)

$$C(p-(q^{(\alpha_1)})^*)\in H_1 \quad \text{for every } p\in\Pi^*-\{(q^{(\alpha_1)})^*\}. \tag{2.5}$$

Since $\alpha_1\in\Delta$, by (2.3) there is an element p_0 of Π such that

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If $p\in\Pi-\{p_0\}$, then $\deg(p-p_0)\geqq 1$ and $(p)^*\in\Pi^*-\{(p_0)^*\}$. Therefore by (2.5)

$$C(p-p_0)=C((p)^*-(p_0)^*)=C((p)^*-(q^{(\alpha_1)})^*)\in H_1 \quad \text{for every } p\in\Pi-\{p_0\}.$$

Thus we have the desired result.

LEMMA 2.2. *Assume that* (A) *holds,* $\{j\ A_1=A_j, j=1, -, m\}=\{1, t+1, \cdots, s\}$ $(t+1\leqq s\leqq m)$, $\#\{j ; A_2=A_j, j=1, \cdots, m\}=1$ *and that*

$$\exp(P_1)+\sum_{j=t+1}^{s}\exp(P_j)\neq h^n \quad \textit{for any } h\in E_N.$$

Let $\{p_{1,j}\}_j, \{p_{2,j}\}_j$ *be defined by* (1.1). *Then there exists* $q_0\in(\{p_{1,j}\}_j-\{p_{2,j}\}_j)\cup(\{p_{2,j}\}_j-\{p_{1,j}\}_j)$ *such that*

$$R_1[q_0]=R_2[q_0]\in E_N, \qquad S_1[q_0]\in E_N, \qquad S_2[q_0]\in E_N,$$

where $R_1[q_0]$, $R_2[q_0]$, $S_1[q_0]$, $S_2[q_0]$ *are defined in Section* 1.

Proof. We may assume

$$P_1=0, \qquad C(P_\mu)\in H_1 \qquad (\mu=2, \cdots, m).$$

Then

$$N-1\geqq\deg P_\mu\geqq 1 \qquad (\mu=t+1, \cdots, s),$$

$$\deg P_\mu=N \quad (\mu\in\{2, \cdots, m\}-\{t+1, \cdots, s\}).$$

Let $\{r_j\}_j$ be the set of polynomials defined by

$$1+\sum_{j=1}^{\infty}\gamma_i(\sum_{\mu=t+1}^{s}\exp(P_\mu))^j\equiv\sum_{\mathcal{J}}\exp(r_j),$$

$$r_j-r_k\neq\text{const.}\ (j\neq k),\qquad \operatorname{Im}(r_j(0))\in[0,\ 2\pi),$$

where $1+\sum_{j=1}^{\infty}\gamma_j w^j\equiv(l+w)^{1/n}$ $(|w|<1)$. Then by Lemma 1.1

$$\{p_{1,j}\,;\ a_{1,j}=0\}=\{r_j\}_j,$$

where $a_{\nu,j}=p_{\nu,j}^{(N)}(0)/N!$. Put

$$\Pi_1=\{p_{1,j}\}_j,\qquad \Pi_2=\{p_{2,j}\}_j,$$

$$\pi_1=\{p_{1,j}\,;\ a_{1,j}=0\},\qquad \pi_2=\{p_{2,j}\,;\ a_{2,j}=0\}.$$

By assumption we have $\#\pi_1=\infty$. Since $\#\{j\,;\ A_2=A_j,\ j=1,\ \cdots,\ m\}=1$, by Lemma 1.5 $\#\pi_2<\infty$. Therefore $(\pi_1-\pi_2)\neq\emptyset$. Thus by Lemma 2.1 there exists $q_1\in(\pi_1-\pi_2)$ such that

(2.6) $$C(q-q_1)\in H_1\quad\text{for every } q\in(\pi_1-\pi_2)-\{q_1\}.$$

Since $\#(\pi_2-\pi_1)<\infty$, by Lemma 1.2 there exists $q_2\in(\pi_2-\pi_1)$ such that

(2.7) $$C(q-q_2)\in H_1\quad\text{for every } q\in(\pi_2-\pi_1)-\{q_2\}$$

whenever $(\pi_2-\pi_1)\neq\emptyset$. Note that $C(q_1-q_2)\neq 0$. Put

$$q_0=\begin{cases} q_1 & \text{if } (\pi_2-\pi_1)\neq\emptyset \text{ or } C(q_2-q_1)\in H_1,\\ q2 & \text{if } (\pi_2-\pi_1)\neq\emptyset \text{ and } C(q_1-q_2)\in H_1.\end{cases}$$

Then $q_0\in((\pi_1-\pi_2)\cup(\pi_2-\pi_1))\subset((\Pi_1-\Pi_2)\cup(\Pi_2-\Pi_1))$. When $q_0=q_1$ and $(\pi_2-\pi_1)\neq\emptyset$, we have $C(q-q_0)=C((q-q_2)+(q_2-q_1))$. Thus, by (2.7) and Lemma 1.1,

$$C(q-q_0)\in H_1\quad\text{for every } q\in(\pi_2-\pi_1).$$

When $q_0=q_2$, we have $C(q-q_0)=C((q-q_1)+(q_1-q_2))$. Thus, by (2.6) and Lemma 1.1,

$$C(q-q_0)\in H_1\quad\text{for every } q\in(\pi_1-\pi_2).$$

Therefore, from (2.6), (2.7),

(2.8) $$C(q-q_0)\in H_1\quad\text{for every } q\in(\pi_1-\pi_2)-\{q_0\},$$

(2.9) $$\operatorname{Re} C(q-q_0)\leqq 0\quad\text{for every } q\in(\pi_1-\pi_2)\cup(\pi_2-\pi_1).$$

Since $\deg q_0\leqq N-1$, we have $C(p_{1,j}-q_0)=a_{1,j}(\neq 0)$, $C(p_{2,j}-q_0)=a_{2,j}$ for $p_{1,j}\in(\Pi_1-\pi_1)$, $p_{2,j}\in(\Pi_2-\pi_2)$. Note that $A_1=0$, $A_2\in i\boldsymbol{R}^+$. By Lemma 1.5 and (1.3), $a_{1,j}\in(\mathcal{S}_1-\{0\})\subset H_1$, $a_{2,j}\in\mathcal{S}_2\subset\{z\,;\ \operatorname{Re} z\leqq 0\}$ for $p_{1,j}\in(\Pi_1-\pi_1)$, $p_{2,j}\in(\Pi_2-\pi_2)$.

Therefore

(2.10) $$C(q-q_0)\in H_1 \quad \text{for every } q\in(\Pi_1-\pi_1),$$

(2.11) $$\mathrm{Re}\, C(q-q_0)\leqq 0 \quad \text{for every } q\in(\Pi_2-\pi_2).$$

Thus, from (2.8)-(2.11), we have

(2.12) $$C(q-q_0)\in H_1 \quad \text{for every } q\in(\Pi_1-(\pi_1\cap\pi_2))-\{q_0\},$$

(2.13) $$\mathrm{Re}\, C(q-q_0)\leqq 0 \quad \text{for every } q\in(\Pi_1\cup\Pi_2)-(\pi_1\cap\pi_2).$$

We define $J_1, J'_1, J''_1, J_2, J'_2, J''_2$ for $q=q_0$ as in Section 1. Then, from (2.13) and the definitions of J_1, J_2,

$$\{p_{1,j},\ j\in J_1\}=\{p_{2,j}\ \ j\in J_2\}\subset(\pi_1\cap\pi_2).$$

Since $\#(\pi_1\cap\pi_2)<\infty$, we have

$$R_1[q_0]=R_2[q_0]\in E_{N-1}.$$

From Lemma 1.6

$$S_2[q_0]\in E_N.$$

Further by (2.12) and the definition of J'_1

$$\{p_{1,j};\ j\in J'_1\}\subset(\pi_1\cap\pi_2).$$

Thus

$$S_1[q_0]\in E_{N-1}.$$

Lemma 2.2 is thus proved.

Proof of Theorem 5. We use the notations of Lemma 2.2. Assume that $g\neq h^n$ for any $h\in E_{N-1}$. Then by Lemma 2.2 there exists $q_0\in(\{p_{1,j}\}_j-\{p_{2,j}\}_j)\cup(\{p_{2,j}\}_j-\{p_{1,j}\}_j)$ such that

$$R_1[q_0]=R_2[q_0]\in E_N, \qquad S_1[q_0]\in E_N, \qquad S_2[q_0]\in E_N.$$

Therefore Lemma 1.7 and 1.8 hold for those $S_1[q_0]$, $S_2[q_0]$, $T_1[q_0]$, $T_2[q_0]$. Put

$$R=R_1[q_0]=R_2[q_0], \qquad F=(f-R-S[q_0]-S_2[q_0]-b_2(q_0)e^{q_0})e^{-q_0}.$$

Then F is a holomorphic function on $\{|\arg z|<\omega_0\}$ satisfying (1.8), (1.9), and $b_1(q_0)\neq b_2(q_0)$. Thus we have a contradiction as in Section 1. Theorem 5 is thus proved.

Now we prove Theorem 2. Put $N=\max_j \deg P_j, A_j=P_j^{(N)}(0)/N(j=1, \cdots, 4)$. We may assume that $\#\{A_j\}_j\geqq 2$. Then we have the following three cases.

Case 1): $\#\{A_j\}_j=2$. In this case, from the following Lemma 2.3, we have a contradiction.

LEMMA 2.3. *Let* n $(\geqq 2)$, N $(\geqq 1)$ *be integers,* A_1, A_2 *be distinct constants* *and* g_1, g_2 *be nonzero elements of* E_{N-1}. *Then*

$$f(z)^n \neq g_1(z)\exp\,(A_1z^N)+g_2(z)\exp\,(A_2z^N)$$

for any entire function f.

Proof. Assume that there exists an entire function f satisfying $f^n(z)=g_1(z)\exp\,(A_1z^N)+g_2(z)\exp\,(A_2z^N)$. Put

$$F(z)=g_1(z)\exp\,((A_1-A_2)z^N).$$

Then $T(r, g_2)=o(T(r, F))$ and

$$\Theta(0, F)=\Theta(\infty, F)=1, \qquad \Theta(-g_2, F)\geqq 1-(1/n).$$

Thus by the second fundamental theorem (see [2; p. 47]) we have a contradiction.

Case 2): $\#\{A_j\}_j=3$. Suppose that $A_1, \cdots, A_4$ do not lie on any streight line. Then we may assume that

$$A_1=0, \qquad A_2\in i\boldsymbol{R}^+, \qquad \mathrm{Re}\,A_3<0, \qquad \mathrm{Re}\,A_4<0.$$

Define $p_{\nu,j}$, $a_{\nu,j}$ and $\mathcal{S}_\nu$ ($\nu=1, 2$) as in Section 1. Then, by Lemma 1.9, $\{a_{1,j}\}_j=\{a_{2,j}\}_j$. Further from (1.1) we have $\{(P_1/n)+\nu(P_2-P_1)+\log\gamma_\nu\ \nu\in\boldsymbol{N}\}\subset\{p_{1,j}\}_j$. Therefore, by Lemma 1.5, $(\mathcal{S}_1\cap\mathcal{S}_2)\supset\{a_{1,j}\}_j\supset\{\nu A_2\,;\nu\in\boldsymbol{N}\}$. Thus $(\mathcal{S}_1\cap\mathcal{S}_2\cap i\boldsymbol{R})\supset\{\nu A_2\ \nu\in\boldsymbol{N}\}$. Since $\mathcal{S}_1\cap\mathcal{S}_2\cap i\boldsymbol{R}=\{ix\ 0\leqq x\leqq(\mathrm{Im}\,A_2)/n\}$, this is a contradiction. Thus $A_1, \cdots, A_4$ lie on a straight line. We assume that $A_2=A_3$ and $A_1\neq A_2\neq A_4$ (see Figure 7 and 8).

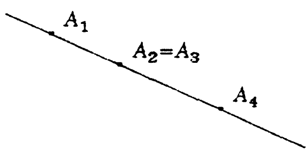

Fig. 7.

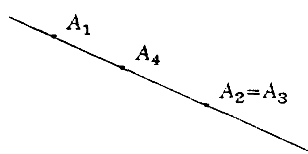

Fig. 8.

Subcase 2.1): $A_2\in\overline{A_1A_4}$. (We denote by $\overline{\alpha\beta}$ the line segment $\{\alpha+x(\beta-\alpha)\,;\ 0\leqq x\leqq 1\}$.) First we shall show the following

LEMMA 2.4.

(1) *Let* $Q_1, \cdots, Q_m$ *be polynomials satisfying* $Q_j-Q_k\neq$const. ($j\neq k$). *Then* $e^{Q_1}+\cdots+e^{Q_m}\not\equiv 0$.

(2) *Let* $P_1, \cdots, P_m$ *be polynomials. Assume that* $e^{P_1}+\cdots+e^{P_m}=0$ *and that* $\sum_{j\in J}e^{P_j}\neq 0$ *for any* $J\subsetneqq\{1, \cdots, m\}$ ($J\neq\emptyset$). *Then* $(P_1)^*=\cdots=(P_m)^*$. *If* $e^{P_1}+e^{P_2}=0$ *or* $e^{P_1}+e^{P_2}+e^{P_3}=0$, *then we always have* $(P_1)^*=(P_2)^*$ *or* $(P_1)^*=(P_2)^*=(P_3)^*$ *respectively.* (For each polynomial p we set $(p)^*(z)=p(z)-p(0)$.)

Proof. These are well-known results and immediate consequences of Lemma 1.3. We assume that $Q_1<_2Q_2<_2 \cdots <_2Q_m$. By Lemma 1.1, $C(Q_j-Q_m)\in H_2$ $(j=1, \cdots, m-1)$. Thus, by Lemma 1.3, there exist positive constants θ, d such that $|e^{(Q_1-Q_m)}+ \dots +e^{(Q_{m-1}-Q_m)}|<1/2$ on $G_2(\theta, d)$. Therefore $|(e^{Q_1}+\cdots+e^{Q_m})e^{-Q_m}|>1/2$ on $G_2(\theta, d)$. Thus (1) is proved. (2) follows from (1).

By Theorem 1 we have $f\in E_N$. We may assume that $A_1,\cdots, A_4\in \boldsymbol{R}$, $A_1<A_2=A_3<A_4$ and that $P_2\leqq_2P_3$. Then we have $f=e^{Q_1}+\cdots+e^{Q_m}$, where Q_j's are polynomials of degree at most N satisfying $Q_\mu-Q_\nu\neq$const. $(\mu\neq\nu)$, $Q_1<_2Q_2<_2 \cdots <_2Q_m$ and $Q_j^{(N)}(0)/N!\in[A_1/n, A_4/n]$ $(j=1, \cdots, m)$.

Put $(e^{Q_1}+\cdots+e^{Q_m})^n=\sum_{\mu_1+\cdots+\mu_m=n} n!(\mu_1!\cdots\mu_m!)^{-1}e^{\mu_1Q_1}\cdots e^{\mu_mQ_m}=\exp(\tilde{Q}_1)+\cdots+\exp(\tilde{Q}_k)$, where $\tilde{Q}_j$'s are polynomials satisfying $\tilde{Q}_\mu-\tilde{Q}_\nu\neq$const. $(\mu\neq\nu)$, $\tilde{Q}_1<_2\tilde{Q}_2<_2\cdots<_2\tilde{Q}_k$ It is easily seen that $m\geqq2$ and

$$\tilde{Q}_1=nQ_1,\quad \tilde{Q}_2=(n-1)Q_1+Q_2,\quad \tilde{Q}_{k-1}=Q_{m-1}+(n-1)Q_m,\quad \tilde{Q}_k=nQ_m.$$

We shall consider the following two cases.

1) $P_2-P_3=$const.. In this case we have $f^n=e^{P_1}+e^{P_2+c}+e^{P_4}=\exp(\tilde{Q}_1)+\cdots+\exp(\tilde{Q}_k)$ for some constant c. Therefore, by Lemma 2.4, we have $k=3$, $(n-1)Q_1+Q_2=Q_{m-1}+(n-1)Q_m=P_2+c$. Thus $R\equiv(n-1)(Q_1-Q_m)+(Q_2-Q_{m-1})=0$. If $m>2$, then $Q_1<_2Q_m$, $Q_2\leqq_2Q_{m-1}$ Therefore, by Lemma 1.1, we have $R<_20$. This is a contradiction. Thus $m=2$, $R=(n-2)(Q_1-Q_2)$. If $n>2$, then $R<_20$. This is again a contradiction. Thus $m=n=2, f=e^{Q_1}+e^{Q_2}$.

2) $P_2-P_3\neq$const.. By Lemma 2.4 we have

$$P_1=nQ_1,\quad P_2=(n-1)Q_1+Q_2,\quad P_3=Q_{m-1}+(n-1)Q_m,\quad P_4=nQ_m. \tag{2.14}$$

Put $B_j=Q_j^{(N)}(0)/N!(j=1, \cdots, m)$. Then by (2.14)

$$A_1=nB_1,\quad A_2=(n-1)B_1+B_2,\quad A_3=B_{m-1}+(n-1)B_m,\quad A_4=nB_m.$$

Since $A_1<A_2=A_3<A_4$, we have $B_1<B_2$, $B_{m-1}<B_m$, $B_j\leqq B_{j+1}$ $(j=1, \cdots, m-1)$, $(n-1)(B_1-B_m)+(B_2-B_{m-1})=0$. Therefore we have $n=m=2$ as in 1). Thus $f=e^{Q_1}+e^{Q_2}$. This implies $e^{P_2}+e^{P_3}=2e^{(Q_1+Q_2)}$, $P_2-P_3=$const., which contradicts the assumption.

Subcase 2.2): $A_2\notin\overline{A_1A_4}$. We may assume that $\mathrm{Re}\,A_1=\cdots=\mathrm{Re}\,A_4$, $\mathrm{Im}\,A_1>\mathrm{Im}\,A_4>\mathrm{Im}\,A_2=\mathrm{Im}\,A_3$. If $P_2-P_3\neq$const., then by Theorem 5 and Lemma 2.3 we have a contradiction. Thus $P_2-P_3=$const.. Therefore this case is reduced to Subcase 2.1).

Case 3): $\#\{A_j\}_j=4$. By Theorem 1 it is verified that $f\in E_N$. As in Case 2), we see that $A_1,\cdots, A_4$ lie on a straight line. We may assume that $A_1,\cdots, A_4\in\boldsymbol{R}$, $0=A_1<A_2<A_3<A_4$ and $P_1=0$. Then $f=c_1e^{Q_1}+\cdots+c_me^{Q_m}$, where c_j's are nonzero constants and Q_j's are polynomials such that $Q_j(0)=0$ $(j=1, \cdots, m)$, $\deg Q_j\leqq N$ $(j=2, \cdots, m)$, $Q_j^{(N)}(0)/N!\in[0, A_4/n]$ $(j=1, \cdots, m)$ and $Q_1<_2Q_2<_2\cdots<_2Q_m$.

Put $(c_1 e^{Q_1} + \cdots + c_m e^{Q_m})^n = \exp(\tilde{Q}_1) + \cdots + \exp(\tilde{Q}_k)$, where $\tilde{Q}_j$'s are polynomials satisfying $Q_\mu - \tilde{Q}_\nu \neq$ const. $(\mu \neq \nu)$, $Q_1 <_2 Q_2 <_2 <_2 Q_k$. By Lemma 2.4 we have fe=4 and $P_j = \tilde{Q}_j (j=1, \cdots, 4)$. Since $P_1 = 0$, we have $c_1{}^n e^{nQ_1} = 1$. Therefore we may assume $c_1 = 1$, $Q_1 = 0$. It is easily seen that ra^2 and

$$(\tilde{Q}_1)^* = nQ_1, \quad (\tilde{Q}_2)^* = (n-1)Q_1 + Q_2, \quad (\tilde{Q}_3)^* = Q_{m-1} + (n-1)Q_m, \quad (\tilde{Q}_4)^* = nQ_m.$$

Thus

$$0 = (P_1)^* = Q_1, \qquad (P_2)^* = (n-1)Q_1 + Q_2 = Q_2, \tag{2.15}$$
$$(P_3)^* = Q_{m-1} + (n-1)Q_m, \qquad (P_4)^* = nQ_m.$$

Put $B_j = Q_j{}^{(N)}(0)/N! (j = 1, -, m)$. Then

$$0 \leqq B_j \leqq B_{j+1} \leqq A_4/n \qquad (j = 1, \cdots, m-1). \tag{2.16}$$

Further by (2.15)

$$0 = A_1 = B_1, \quad A_2 = B_2, \quad A_3 = B_{m-1} + (n-1)B_m, \quad A_4 = nB_m. \tag{2.17}$$

Since $A_1 < A_2 < A_3 < A_4$, we have $B_1 < B_2$, $B_{m-1} < B_m$.

Assume $m \geqq 3$. Let p be the integer such that $B_2 = B_3 = \cdots = B_\rho < B_{\rho+1}$. Then

$$\{(\mu_1, \cdots, \mu_m) ; \sum_{j=1}^m \mu_j B_j \leqq B_2, \ \sum_{j=1}^m \mu_j = n, \ \mu_j \in N \cup \{0\}\}$$
$$= \{(n, 0, \cdots, 0)\} \cup \{(n-1, \underbrace{0, \cdots, 0,}_{\nu} 1, 0, \cdots, 0) ; \nu = 0, \cdots, \rho - 2\}.$$

Therefore $(\tilde{Q}_j)^* = Q_j (j = 1, \cdots, \rho)$ If $\rho \geqq 3$, then

$$f^n = 1 + e^{P_2} + e^{P_3} + e^{P_4} = 1 + n(c_2 e^{Q_2} + c_3 e^{Q_3} + \cdots + c_\rho e^{Q_\rho}) + \cdots + c_m{}^n e^{nQ_m}.$$

Therefore, by Lemma 2.4, $(P_2)^* = (Q_2)^* = Q_2$, $(P_3)^* = (\tilde{Q}_3)^* = Q_3$. Thus $A_2 = B_2$, $A_3 = B_3$. *Hence* $A_2 = A_3$. This is a contradiction. Thus $B_2 < B_3$. Similarly $B_{m-2} < B_{m-1}$. Therefore, if $m \geqq 3$, then

$$0 = B_1 < B_2 < B_3, \qquad B_{m-2} < B_{m-1} < B_m. \tag{2.18}$$

LEMMA 2.5. *There exist positive integers λ_j $(j = 2, \cdots, m)$ such that*

$$Q_j = \lambda_j Q_2 \qquad (j = 2, \cdots, \mathrm{m}). \tag{2.19}$$

Proof. By induction on j. For each polynomial Q we set

$$\mu(Q) = \{(\mu_1, \cdots, \mu_m) ; Q = \sum_{j=1}^m \mu_j Q_j, \ n = \sum_{j=1}^m \mu_j, \ \mu_j \in N \cup \{0\}\}.$$

Put

$$U=\{Q\,;\sum_{(\mu_1,\cdots,\mu_m)\in\mu(Q)}(n!/(\mu_1!\cdots\mu_m!))c_1^{\mu_1}\cdots c_m^{\mu_m}\neq 0\},$$

$$V_\nu=\{Q;\,Q=\sum_{j=1}^{\nu}\mu_jQ_j,\, n=\sum_{j=1}^{\nu}\mu_j,\,\mu_j\in N\cup\{0\}\}\qquad (\nu=2,\,\cdots,\,m).$$

Then by (2.15) $U=\{(P_1)^*,\,\cdots,\,(P_4)^*\}=\{0,\,Q_2,\,Q_{m-1}+(n-1)Q_m, nQ_m\}$. (2.19) holds trivially for $j=2$. Assume that (2.19) holds for $j=2,\,\cdots,\,\nu\,(\nu<m)$. Further assume $Q_{\nu+1}\notin V_\nu$. If $(\mu_1,\cdots,\mu_m)\in\mu(Q_{\nu+1})$, then there exists an integer $\rho\geqq\nu+1$ such that $\mu_\rho\neq 0$. Since $Q_{\nu+1}=\sum_{j=1}^{m}\mu_jQ_j=\sum_{j=2}^{m}\mu_jQ_j$, we have $Q_{\nu+1}-\mu_\rho Q_\rho=\sum_{j=2}^{\rho-1}+\sum_{j=\rho+1}^{m}\mu_jQ_j$. By Lemma 1.1 $0\leqq_2(\sum_{j=2}^{\rho-1}+\sum_{j=\rho+1}^{m}\mu_jQ_j)$, $(Q_{\nu+1}-\mu_\rho Q_\rho)\leqq_2 0$. Therefore $\mu_2=\cdots=\mu_{\rho-1}=0$, $\mu_{\rho+1}=\cdots=\mu_m=0$, $Q_{\nu+1}=\mu_\rho Q_\rho$. Hence $\mu_\rho=1$, $\rho=\nu+1$. Thus

$$\mu(Q_{\nu+1})=\{(n-1,\underbrace{0\quad\quad}_{\nu-1},1,\,0,\,\cdots,\,0)\}.$$

Therefore $\#\mu(Q_{\nu+1})=1$. Thus $Q_{\nu+1}\in U$. On the other hand, by (2.16) and (2.18), we have $B_2<B_3\leqq B_{\nu+1}$, $B_{\nu+1}\leqq B_m$, $0<B_{m-1}$. (We assume $m\geqq 3$.) Therefore $0<B_2<B_{\nu+1}<(B_{m-1}+(n-1)B_m)<nB_m$. Hence $Q_{\nu+1}\notin U$. This is a contradiction. Thus $Q_{\nu+1}\in V_\nu$. By the induction assumption we have $V_\nu\subset\{\lambda Q_2\,;\lambda\in N\cup\{0\}\}$. Hence there is a positive integer $\lambda_{\nu+1}$ such that

$$Q_{\nu+1}-\lambda_{\nu+1}Q_2$$

Lemma 2.5 is thus proved.

By Lemma 2.5 there are polynomials $\mathcal{P}$, $\mathcal{R}$ satisfying that

$$f=\mathcal{P}(\exp(Q_2)),\qquad f^n=\mathcal{R}(\exp(Q_2)).$$

By Lemma 2.5 $B_m-B_{m-1}=\lambda(B_2-B_1)$ with $\lambda=\lambda_m-\lambda_{m-1}$. Similarly $B_2-B_1=\lambda'(B_m-B_{m-1})$ with a positive integer λ'. Thus $B_2-B_1=B_m-B_{m-1}$. Therefore from (2.17) we have $A_4-A_3=B_m-B_{m-1}=B_2-B_1=A_2-A_1$. This implies that, if $t>3$, then $\mathcal{R}^{(\nu)}(0)=0$ $(\nu=2,\,\cdots,\,t-2)$. Therefore $\mathcal{R}(w)=d_4w^t+d_3w^{t-1}+d_2w+1$, where $t\geqq 3$ and d_ν's are nonzero constants.

LEMMA 2.6. *Let $\mathcal{P}$, $\mathcal{R}$ be polynomials and $n(\geqq 2)$ be an integer such that*

$$\mathcal{P}^n=\mathcal{R},\qquad \mathcal{R}(w)=d_4w^t+d_3w^{t-1}+d_2w+1,\tag{2.20}$$

where $t\geqq 3$ and $d_\nu\neq 0$ for every ν. Then there are the following two possibilities:

(1) *$n=3$ and $\mathcal{P}(w)=\rho(w-\sigma)$ with ρ, $\sigma\neq 0$.*

(2) *$n=2$ and $\mathcal{P}(w)=\rho'(w^2+\sqrt{2}\,\sigma'w-\sigma'^2)$ with ρ', $\sigma'\neq 0$.*

Proof. Let α be a zero of $\mathcal{P}$. Then $\mathcal{R}(\alpha)=\mathcal{R}'(\alpha)=0$, This yields

$$(t-1)d_4d_2\alpha^2+((t-2)d_3d_2+td_4)\alpha+(t-1)d_3=0.$$

Therefore $\mathcal{P}$ has at most two distinct zeros α_1, α_2.

Case 1): $\alpha_1=\alpha_2$. Put $\sigma=$tfj$-\alpha_2$. In this case $\mathcal{P}(w)^n=\tau(w-\sigma)^{sn}-\mathcal{R}(w)$, where τ is a constant and $s=\deg \mathcal{P}$. From (2.20) we have $t-sn=3$. Thus $s=1$, $n=3$. Therefore we have the desired result.

Case 2): $\alpha_1\neq\alpha_2$. In this case

$$\mathcal{P}(w)^n=\tau'(w-\alpha_1)^{nu}(w-\alpha_2)^{nv}=\mathcal{R}(w) \tag{2.21}$$

where τ' is a constant and u, v are positive integers. On the other hand, from (2.20), we have

$$\mathcal{R}''(w)=\zeta w^{t-3}(w-\eta) \tag{2.22}$$

with ζ, $\eta\neq0$. Assume $n\geqq4$, then raw-2^2 and $nv-2\geqq2$. From (2.21), (2.22) we have a contradiction. Thus $n\leqq3$. Similarly we have $u=1$ or $v=l$. Assume that $u=1$, $v\geqq2$. Then, from (2.21) and (2.22), we obtain $n+nv=t$, $nv-2=t-3$. Thus $n=1$. This is a contradiction. Similarly $u-1$ whenever $v=1$. Thus $u=v=1$. If $n=3$ and $u=v=1$, then from (2.21) we have $t=6$. On the other hand (2.21) and (2.22) imply $t=4$. This is a contradiction. Thus $n=2$, $u=v=1$ and $t=4$. Therefore

$$\mathcal{P}(w)=\rho'(w^2-(\alpha_1+\alpha_2)w+\alpha_1\alpha_2),$$

where p' is a nonzero constant. Since $t=4$, from (2.20) and (2.21), the coefficient of w^2 of $\mathcal{P}(w^2)$ is equal to 0. Thus $(\alpha_1+\alpha_2)^2+2\alpha_1\alpha_2=0$. Hence

$$\mathcal{P}(w)=\rho'(w^2+\sqrt{2}\ \sigma' w-\sigma'^2),$$

where σ' is a nonzero constant. Lemma 2.6 is thus proved.

Lemma 2.5 and 2.6 complete the proof of Theorem 2.

3. Proof of Theorem 3.

Let $g=(g_0, g_1, g_2)$, where g_j's are entire functions without common zeros. We may assume that $D_0=\{w_0=0\}$, $D_1=\{w_1=0\}$. Let $P(w_0, w_1, w_2)$ be a homogeneous polynomial of degree two such that $D_2=\{P(w_0, w_1, w_2)=0\}$. Then, by the assumption, for suitable polynomials q_0, q_1, q

$$g_0=e^{q_0}, \qquad g_1=e^{q_1}, \qquad P(g_0, g_1, g_2)=e^q.$$

Since $D_0\cap D_1\cap D_2=\emptyset$, there exist constants a_0, fli, a_2 $(a_2\neq0)$, b_0, b_1, b_2 such that $P(w_0, w_1, w_2)=(a_0w_0+a_1w_1+a_2w_2)^2-(b_0w_0^2+b_1w_1^2+b_2w_0w_1)$.Since D_2 is not a line, we have $(b_0, b_1, b_2)\neq(0, 0, 0)$. Put

$$G=a_0g_0+a_1g_1+a_2g_2.$$

Then

$$G^2=b_0e^{2q_0}+b_1e^{2q_1}+b_2e^{(q_0+q_1)}+e^q\,. \tag{3.1}$$

If $q_0-q_1=$const., then $g_0=cg_1$ with a nonzero constant c. Thus, in what follows, we assume that $q_0-q_1\neq$const.. Further we may assume, without loss of generality, that

deg tfo^deg q_1, $\qquad C(q_0)\neq C(q_1)$.

If ft$_\beta$=ft$_{1=}$o, $b_2\neq 0$, then from (3.1) $G^2=b_2e^{q_0+q_1}+e^q$. Thus by Lemma 2.3 we have $G^2=ce^{q_0+q_1}$ with a constant c $(\neq b_2)$. Thus

$$(a_0g_0+a_1g_1+a_2g_2)^2=cg_0g_1\,, \qquad c\neq b_2\,.$$

Similarly, if $b_0=b_2=0$, $b_1\neq 0$, then $G^2=b_1e^{2q_1}+e^q$. Thus $G^2=c'e^{2q_1}$, $c'\neq b_1$. Therefore

$$a_0g_0+a_1g_1+a_2g2=\sqrt{c'}g_1\,, \qquad c'\neq b_1\,.$$

If ft$_{1==}$ft$_{2=}$o, $b_0\neq 0$, then $G^2=b_0e^{2q_0}+e^q$. Thus $G^2=c''e^{2q_0}$, $c''\neq b_0$. Therefore

$$a_0g_0+a_1g_1+a_2g_2=\sqrt{c''}g_0\,, \qquad c''\neq b_0\,.$$

Thus, in what follows, we assume that $\#\{j;$ ft$_y$=0, —0, 1, 2$\}\leqq 1$.

LEMMA 3.1. *Let*

$$\varphi_0=b_0e^{2q_0}, \qquad \varphi_1=b_1e^{2q_1}, \qquad \varphi_2=b_2e^{q_0+q_1}, \qquad \varphi_3=e^q\,.$$

Assume that there exists a subset J of $\{0, 1, 2, 3\}$ *satisfying*$\#J\geqq 2$ *and* $\sum_{j\in J}\varphi_j=0$. *Then there are the following three possibilities*

1) $$(a_0g_0+a_1g_1+a_2g_2)^2=b_2g_0g_1\,, \qquad b_2\neq 0\,,$$

2) $$a_0g_0+a_1g_1+a_2g_2=\sqrt{b_0}g_0\,, \qquad b_0\neq 0\,,$$

3) $$a_0g_0+a_1g_1+a_2g_2=\sqrt{b_1}g_1\,, \qquad b_1\neq 0\,.$$

Proof. We may assume that $\varphi_j\neq 0$ for all $\in J$. We shall consider the following three cases.

1) $\#J=2$. Put $J=\{j_2, j_3\}$. Then by Lemma 2.4 $\varphi_{j_2}/\varphi_{j_3}=$const.. If j_2, j_3 $\in\{0, 1, 2\}$, then $q_0-q_1=$const.. This is a contradiction. Thus we have $J\ni 3$. Let j_0, $\dot{}_1$ be integers such that $\{j_0, j_1\}=\{0, 1, 2\}-/$. Then from (3.1) $\varphi_{j_2}+\varphi_{j_3}=0$, $G^2=\varphi_{j_0}+\varphi_{j_1}$. If $\varphi_{j_0}\neq 0$ and $\varphi_{j_1}\neq 0$, then by Lemma 2.3 $\varphi_{j_0}/\varphi_{j_1}=$const.. Thus $q_0-q_1=$const.. This is a contradiction. Thus $\varphi_{j_0}=0$ or $\varphi_{j_1}=0$. Therefore one of the following three cases holds: $G^2=b_0g_0^2$ $(b_0\neq 0)$, $G^2=b_1g_1^2$ $(b_1\neq 0)$, $G^2=b_2g_0g_1$ $(b_2\neq 0)$. Thus we have the desired result.

2) $\#J=3$. Since $\#(\{0, 1, 2\}\cap J)\geqq 2$, by Lemma 2.4 we have $\varphi_j/\varphi_k=$const. for some j, $k\in\{0, 1, 2\}$. Thus $q_0-q_1=$const.. This is a contradiction.

3) $\#J=4$. We may assume, without loss of generality, that $\sum_{j\in J'}\varphi_j\neq 0$ for

any $J'\subsetneqq J$ $(J'\neq\emptyset)$. Since $\#(\{0,1,2\}\cap J)\geqq 2$, by Lemma 2.4 we have a contradiction as above.

In what follows, we assume that

$$\sum_{j\in J}\varphi_j\neq 0 \qquad \text{for any } J\subset\{0,1,2,3\} \text{ satisfying } \#J\geqq 2. \tag{3.2}$$

By Lemma 2.3 and (3.2), we have $\deg q_0=\deg q_1\geqq\deg q$. Put

$$N=\deg q_0=\deg q_1,$$

$$A_1=2q_0^{(N)}(0)/N!, \qquad A_2=2q_1^{(N)}(0)/N!,$$

$$A_3=(q_0+q_1)^{(N)}(0)/N!, \qquad A_4=q^{(N)}(0)/N!.$$

Then $A_1\neq A_2$, $A_3=(A_1+A_2)/2$. Therefore $\#\{A_j\}_{j=1}^4=3$ or 4.

Case 1): $b_0b_1b_2\neq 0$. In this case, we shall consider the following two subcases.

Subcase 1.1): $\#\{A_j\}_{j=1}^4=3$. There are the following three possibilities.

1) $A_4=A_1$. In this case, by Theorem 2, $G^2=ce^{2q_0}+b_1e^{2q_1}+b_2e^{q_0+q_1}$ with a constant c $(\neq b_0)$. By (3.2) we have $c\neq 0$. Therefore, by Theorem 2, $G-\sqrt{c}\,e^{q_0}+\sqrt{b_1}\,e^{q_1}$. Thus

$$(a_0-\sqrt{c})g_0+(a_1-\sqrt{b_1})g_1+a_2g_2=0, \qquad c\notin\{0, b_0\}.$$

2) $A_4=A_2$. In this case we have $G^2=b_0e^{2q_0}+c'e^{2q_1}+b_2e^{q_0+q_1}$ with a constant $c'\notin\{0, b_1\}$. Therefore, by Theorem 2, $G=\sqrt{b_0}\,e^{q_0}+\sqrt{c'}\,e^{q_1}$. Thus

$$(a_0-\sqrt{b_0})g_0+(a_1-\sqrt{c'})g_1+a_2g_2=0, \qquad c'\notin\{0, b_1\}.$$

3) $A_4=A_3$. In this case we have $G^2=b_0e^{2q_0}+b_1e^{2q_1}+c''e^{q_0+q_1}$ with a constant $c''\notin\{O, b_2\}$. Therefore, by Theorem 2, $G=\sqrt{b_0}\,e^{q_0}+\sqrt{b_1}\,e^{q_1}$. Thus

$$(a_0-\sqrt{b_0})g_0+(a_1-\sqrt{b_1})g_1+a_2g_2=0.$$

Subcase 1.2): $\#\{A_j\}_{j=1}^4=4$. In this case, by Theorem 2, $G=e^P(e^{2Q}+\sqrt{2}\,\sigma e^Q-\sigma^2)$ with polynomials P, Q and a nonzero constant σ. Thus, by Lemma 2.4 and (3.1), we have $\{A_1,\cdots,A_4\}=\{2\alpha, 2\alpha+\beta, 2\alpha+3\beta, 2\alpha+4\beta\}$, where $\alpha=P^{(N)}(0)/N!$, $\beta=Q^{(N)}(0)/N!$. This contradicts $A_1\neq A_2$, $A_3=(A_1+A_2)/2$ (see Figure 9).

Fig. 9.

Case 2): $b_0=0$, $b_1b_2\neq 0$. In this case

$$G^2=b_1e^{2q_1}+b_2e^{(q_0+q_1)}+e^q . \tag{3.3}$$

By *Lemma* 2.3 and (3.2), we have $A_2\neq A_3\neq A_4$. Therefore, by Theorem 2, we see that there are the following three possibilities.

1) $G=\sqrt{b_1}e^{q_1}+de^{q/2}$ $(d\in\{\pm 1\})$. We have $(G-\sqrt{b_1}e^{q_1})^2=e^q$. Therefore by (3.3) $2\sqrt{b_1}G-2b_1e^{q_1}-b_2e^{q_0}=0$. Thus

$$(2\sqrt{b_1}\,a_0-b_2)g_0+2(\sqrt{b_1}\,a_1-b_1)g_1+2\sqrt{b_1}\,a_2g_2=0 .$$

2) $G=\sqrt{b_2}\,e^{(q_0+q_1)/2}+d'e^{q/2}$ $(d'\in\{\pm 1\})$. Since $b_1e^{2q_1}=2d'\sqrt{b_2}\,e^{(q_0+q_1+q)/2}$, we have $e^q=(b_1{}^2/(4b_2))e^{3_1-q_0}$. Therefore, from (3.3), $G^2=b_1e^{2q_1}+b_2e^{(q_0+q_1)}+(b_1{}^2/(4b_2))\cdot e^{3q_1-q_0}$. Thus $G^2e^{q_0}=((b_1{}^2/(4b_2))e^{2q_1}+b_1e^{(q_0+q_1)}+b_2e^{2q_0})e^{q_1}=((b_1/(2\sqrt{b_2}))e^{q_1}+\sqrt{b_2}\,e^{q_0})^2$ e^{q_1}. Therefore

$$4b_2g_0(a_0g_0+a_1g_1+a_2g_2)^2=(2b_2g_0+b_1g_1)^2g_1 .$$

3) $G=\sqrt{b_1}\,e^{q_1}+\sqrt{b_2}\,e^{(q_0+q_1)/2}$. We have $(G-\sqrt{b_1}\,e^{q_1})^2=b_2e^{q_0+q_1}$. Thus

$$(a_0g_0+(a_1-\sqrt{b_1})g_1+a_2g_2)^2=b_2g_0g_1 .$$

Case 3): $b_2=0$, $b_0b_1\neq 0$. In this case

$$G^2=b_0e^{2q_0}+b_1e^{2q_1}+e^q .$$

There are the following three possibilities as above.

1) $G=\sqrt{b_0}e^{q_0}+\sqrt{b_1}\,e^{q_1}$. We have

$$(a_0-\sqrt{b_0})g_0+(a_1-\sqrt{b_1})g_1+a_2g_2=0 .$$

2) $G=\sqrt{b_0}\,e^{q_0}+de^{q/2}$ $(d\in\{\pm 1\})$. We have $(G-\sqrt{b_0}e^{q_0})^2=e^q=G^2-b_0e^{2q_0}-b_1e^{2q_1}$. Thus $2\sqrt{b_0}\,Ge^{q_0}=2b_0e^{2q_0}+b_1e^{2q_1}$. Therefore

$$2\sqrt{b_0}\,g_0(a_0g_0+a_1g_1+a_2g_2)=2b_0g_0{}^2+b_1g_1{}^2 .$$

3) $G=\sqrt{b_1}\,e^{q_1}+d'e^{q/2}$ $(d'\in\{\pm 1\})$. We have

$$2\sqrt{b_1}\,g_1(a_0g_0+a_1g_1+a_2g_2)=b_0g_0{}^2+2b_1g_1{}^2 .$$

Case 4): $b_1=0$, $b_0b_2\neq 0$. In this case there are the following three possibilities as in Case 2).

1) $G=\sqrt{b_0}\,e^{q_0}+de^{q/2}(d\in\{\pm 1\})$. We have

$$2(\sqrt{b_0}\,a_0-b_0)g_0+(2\sqrt{b_0}\,a_1-b_2)g_1+2\sqrt{b_0}\,a_2g_2=0 .$$

2) $G=\sqrt{b_2}\,e^{(q_0+q_1)/2}+d'e^{q/2}$ $(d'\in\{\pm 1\})$. We have

$$4b_2g_1(a_0g_0+a_1g_1+a_2g_2)^2=(b_0g_0+2b_2g_1)^2g_0 .$$

3) $G=\sqrt{b_0}\,e^{q_0}+\sqrt{b_2}\,e^{(q_0+q_1)/2}$. We have

$$((a_0-\sqrt{b_0})g_0+a_1g_1+a_2g_2)^2=b_2g_0g_1 .$$

Theorem 3 is thus proved.

4. Proof of Theorem 4.

Let $g=(g_0, g_1, g_2, g_3)$, where g_j's are entire functions without common zeros. Then, for suitable polynomials q_0, q_1, q_2, q,

$$g_0=e^{q_0},\qquad g_1=e^{q_1},\qquad g_2=e^{q_2},\qquad g_0{}^n+g_1{}^n+g_2{}^n+g_3{}^n=e^q .$$

Thus

(4.1) $$g_3{}^n=e^q-e^{nq_0}-e^{nq_1}-e^{nq_2} .$$

Put

$$q_{-1}=(q+i\pi)/n .$$

Then

(4.2) $$g_3{}^n=-\sum_{j=-1}^{2}\exp(nq_j) .$$

LEMMA 4.1. *Assume that there exists a subset* J $(\neq\emptyset)$ *of* $\{-1, 0, 1, 2\}$ *satisfying*

$$\sum_{j\in J}\exp(nq_j)=0 .$$

Then g has the reduced representation (h_0, h_1, h_2, h_3) *such that* $\{h_j\}_{j=0}^3=\{a_0,$ fli, $a_2, e^P\}$ *or* $\{h_j\}_{j=0}^3=\{a_0, a_1, a_2e^P,$ fls^p$\}$, *where* a_j*'s are constants and* P *is a polynomial.*

Proof. Since $\#J\geqq2$, we shall consider the following three cases.

1) $\#J=2$. Put $J=\{j_{-1}, j_0\}$. Let j_1, j_2 be integers such that $\{ \dot{}_1, j_2\}=\{-1, 0, 1, 2\}-J$. Then from (4.2)

$$\exp(nq_{j_{-1}})+\exp(nq_{j_0})=0 ,$$

$$g_3{}^n=-\exp(nq_{j_1})-\exp(nq_{j_2}) .$$

Then by Lemma 2.3 and 2.4

$$(q_{j_{-1}})^*=(q_{j_0})^*,\qquad \text{te.}_1)^*=(\hat{}_2)^*,\qquad g_3=c\exp(q_{j_1}),$$

where $(p)^*(z)=p(z)-p(0)$ for each polynomial p, and c is a constant. Since g is not a constant, $(q_{j_{-1}})^*\neq(q_{j_1})^*$. Thus we have the desired result.

2) $\#/=3$. Put $J=\{j_{-1}, j_0, j_1\}$ Let j_2 be an integer such that $\{j_2\}=\{-1, 0, 1, 2\}-J$. Then from (4.2)

$$\exp(nq_{j_{-1}})+\exp(nq_{j_0})+\exp(nq_{j_1})=0, \qquad g_3{}^n=-\exp(nq_{j_2}).$$

Thus by Lemma 2.4

toO*=(fco)*=foι>*> $g_3=c\exp$ fo $_2$)

with a constant c. Since g is not a constant, $(q_{j_{-1}})^*\neq(q_{j_2})^*$. Thus we have the desired result.

3) $\#J=4$. In this case we may assume, without loss of generality, that $\sum_{j\in J'}\exp(nq_j)\neq 0$ for any $J'\subsetneqq\{-1, 0, 1, 2\}$ $(J'\neq\emptyset)$. Then from (4.2)

$$\exp(nq_{j_{-1}})+\cdots+\exp(nq_{j_2})=0, \qquad g_3{}^n=0.$$

Thus by Lemma 2.4

$$(q_{j_{-1}})^*=\cdots=(q_{j_2})^*, \qquad g_3=0.$$

Thus g is a constant. This is a contradiction. Lemma 4.1 is thus proved.

If $n\geqq 4$, then by Theorem 2 $\sum_{j\in J}\exp(nq_j)=0$ for some $J\subset\{-1, 0, 1, 2\}$ $(J\neq 0)$. Therefore, in this case, Theorem 4 follows from Lemma 4.1. Thus, in what follows, we assume that $n\leqq 3$ and

$$\sum_{j\in J}\exp(nq_j)\neq 0 \qquad \text{for any } J\subset\{-1, 0, 1, 2\}, \qquad J\neq\emptyset. \tag{4.3}$$

Since g is not a constant, $q_j-q_k\neq$const. for some $j, k\in\{-1, 0, 1, 2\}$ with $j\neq k$. We have the following two cases.

Case 1): q_j-q_k=const. for some $j, k\in\{-1, 0, 1, 2\}$ with $j\neq k$. In this case, by Theorem 2, we have $n=2$. We shall consider the following two subcases.

Subcase 1.1): $q_{-1}-$ ft$_\beta$=const. for some $j_0\in\{0, 1, 2\}$. Let ' $_1$, j_2 be integers such that { ' $_1$, j_2} = {0, 1, 2} − {j_0}. From (4.1) we have

$$g_3{}^2=b\exp(2q_{j_0})-\exp(2q_{j_1})-\exp(2q_{j_2})$$

with a constant $b(\neq -1)$. By (4.3) we have $b\neq 0$. Then, by Theorem 2, there are the following three possibilities.

1) $g_3=\sqrt{b}\exp(q_{j_0})+id\exp(q_{j_1})$ $(d\in\{\pm 1\})$. In this case we have $-\exp(2q_{j_2})=2id\sqrt{b}\exp(q_{j_0}+q_{j_1})$. Thus

$$\begin{cases}\sqrt{b}\,g_{j_0}+idg_{j_1}-g_3=0\\ ig_{j_2}{}^2-2d\sqrt{b}\,g_{j_0}g_{j_1}=0,\end{cases} \qquad b\notin\{0, -1\}.$$

2) $g_3=\sqrt{b}\exp(q_{j_0})+id'\exp(q_{j_2})$ $(d'\in\{+1\})$. In this case we have $-\exp(2q_{j_1})=2id'\sqrt{b}\exp(q_{j_0}+q_{j_2})$. Thus

$$\begin{cases} \sqrt{b}\,g_{j_0}+id'g_{j_2}-g_3=0 \\ ig_{j_1}{}^2-2d'\sqrt{b}\,g_{j_0}g_{j_2}=0\,, \end{cases} \qquad b\notin\{0,\ -1\}.$$

3) $g_3=id''\exp(q_{j_1})+id'''\exp(q(d'',\ d'''\in\{+1\})$. In this case we have $b\exp(2q_{j_0})=-2d''d'''\exp(q_{j_1}+q_{j_2})$. Thus

$$\begin{cases} id''g_{j_1}+id'''g_{j_2}-g_3=0 \\ bg_{j_0}{}^2+2d''d'''g_{j_1}g_{j_2}=0\,, \end{cases} \qquad \text{fteMO}, -1\}.$$

Subcase 1.2): $q_{j_0}-q_{j_1}=$const. for some $j_0,\ j_1\in\{0,\ 1,\ 2\}$ with $j_0\neq j_1$. Let j_2 be an integer such that $\{j_2\}=\{0,\ 1,\ 2\}-\{j_0,\ j_1\}$ Then $-\exp(2q_{j_0})-\exp(2q_{j_1})=c\exp(2q_{j_1})$ with a constant $c(\neq-1)$. By (4.3) we have $c\neq0$. Thus $g_{j_0}=$ V$-1-c\,g_{j_1}$. Further from (4.1)

$$g_3{}^2=c\exp(2q_{j_1})-\exp(2q_{j_2})+\exp(q)\,.$$

Thus, by Case 3) in Section 3, there are the following three possibilities:

$$1)\qquad \begin{cases} -\sqrt{c}\,g_{j_1}-\sqrt{-1}\,g_{j_2}+g_3=0 \\ g_{j_0}-\sqrt{-1-c}\,g_{j_1}=0\,, \end{cases} \qquad c\notin\{0,\ -1\},$$

$$2)\qquad \begin{cases} 2\sqrt{c}\,g_{j_1}g_3-2cg_{j_1}{}^2+g_{j_2}{}^2=0 \\ g_{j_0}-\sqrt{-1-c}\,g_{j_1}=0\,, \end{cases} \qquad c\notin\{0,\ -1\},$$

$$3)\qquad \begin{cases} 2\sqrt{-1}\,g_{j_2}g_3-cg_{j_1}{}^2+2g_{j_2}{}^2=0 \\ g_{j_0}-\sqrt{-1-c}\,g_{j_1}=0\,, \end{cases} \qquad c\notin\{0,\ -1\}.$$

Case 2): $q_j-q_k\neq$const. for every $j,\ k\in\{-1, 0,\ 1,\ 2\}$ with $j\neq k$. In this case, by Theorem 2, we have $n=2,\ 3$.

Subcase 2.1): $n=2$. In this case we have

$$g_3{}^2=e^q-e^{2q_0}-e^{2q_1}-e^{2q_2}\,.$$

By Theorem 2

$$g_3=e^P(e^{2Q}+\sqrt{2}\,\sigma e^Q-\sigma^2)\,,$$

where P, Q are polynomials and σ is a nonzero constant. Thus by Lemma 2.4

$$\{e^q,\ -e^{2q_0},\ -e^{2q_1},\ -e^{2q_2}\}=\{e^{2P+4Q},\ 2\sqrt{2}\,\sigma e^{2P+3Q},\ -2\sqrt{2}\,\sigma^3e^{2P+Q},\ \sigma^4e^{2P}\}\,.$$

We may assume $e^q=e^{2Q+4P}$ or $e^q=2\sqrt{2}\,\sigma e^{2P+3Q}$. Let (j_0, j_1, j_2) be the permutation of (0, 1, 2) which satisfies

$$-\exp(2q_{j_0})=\sigma^4e^{2P}\,,\qquad -\exp(2q_{j_1})=-2\sqrt{2}\,\sigma^3e^{2P+Q}\,.$$

We may assume $g_{j_0}\equiv i$. Then $\sigma^4e^{2P}\equiv1$. Put

$$p=Q-\log \sigma .$$

Then

$$g_3=d_3(e^{2p}+\sqrt{2}e^p-1), \qquad \{e^q, -\exp(2q_{j_2})\}=\{e^{4p}, 2\sqrt{2}e^{3p}\},$$

$$-\exp(2q_{j_1})=-2\sqrt{2}e^p ,$$

where $d_3\in\{\pm1\}$. Now we have the following two cases.

1) $e^q=e^{4p}$. In this case we have $-\exp(2q_{j_2})=2\sqrt{2}e^{3p}$. Thus

$$g_{j_0}=i, \qquad g_{j_1}=d_1(2\sqrt{2})^{1/2}e^{p/2},$$

$$g_{j_2}=id_2(2\sqrt{2})^{1/2}e^{3p/2}, \qquad g_3=d_3(e^{2p}+\sqrt{2}e^p-1),$$

where $d_1, d_2\in\{\pm1\}$. Therefore

$$\begin{cases} 2\sqrt{2}g_{j_0}{}^2+\sqrt{2}g_{j_1}{}^2+i\ 2\sqrt{2}d_3g_{j_0}g_3-id_1d_2g_{j_1}g_{j_2}=0 \\ g_{j_1}{}^3-i\ \sqrt{2}d_1d_2g_{j_0}{}^2g_{j_2}=0 . \end{cases}$$

2) $e^q=2\sqrt{2}e^{3p}$. In this case we have $-\exp(2q_{j_2})=e^{4p}$. Thus

$$g_{j_0}=i, \qquad g_{j_1}=d_1(2\sqrt{2})^{1/2}e^{p/2},$$

$$g_{j_2}=id_2e^{2p}, \qquad g_3=d_3(e^{2p}+\sqrt{2}e^p-1),$$

where $d_1, d_2\in\{\pm1\}$. Therefore

$$\begin{cases} 2g_{j_0}{}^2+g_{j_1}{}^2-id_2g_{j_0}g_{j_2}+i\ 2d_3g_{j_0}g_3=0 \\ g_{j_1}{}^4-8d_2g_{j_0}{}^3g_{j_2}=0 . \end{cases}$$

Subcase 2.2): $n=3$. In this case we have

$$g_3{}^3=e^q-e^{3q_0}-e^{3q_1}-e^{3q_2} .$$

By Theorem 2

$$g_3=e^P(1+e^Q)$$

with polynomials *P, Q*. Thus by Lemma 2.4

$$\{e^q, -e^{3q_0}, -e^{3q_1}, -e^{3q_2}\}=\{e^{3P+3Q}, 3e^{3P+2Q}, 3e^{3P+Q}, e^{3P}\}.$$

We may assume $e^q=e^{3P+3Q}$ or $e^q=3e^{3P+2Q}$. Let (j_0, j_1, j_2) be the permutation of (0, 1, 2) which satisfies

$$-\exp(3q_{j_0})=e^{3P}, \qquad -\exp(3q_{j_1})=3e^{3P+Q} .$$

We may assume $g_{j_0}\equiv-1$. Then $e^{3P}\equiv1$ and

$$g_3=\omega_3(1+e^Q), \quad \{e^q, -\exp(3q_{j_2})\}=\{e^{3Q}, 3e^{2Q}\}, \quad -\exp(3q_{j_1})=3e^Q ,$$

where $\omega_3\in\{1, e^{\pm i2\pi/3}\}$. We have the following two cases.

1) $e^q=e^{3Q}$. In this case $-\exp(3q_{j_2})=3e^{2Q}$. Thus

$$g_{j_0}=-1,\quad g_{j_1}=\sqrt[3]{3}\,\omega_1 e^{Q/3}, g_{j_2}=\sqrt[3]{3}\,\omega_2 e^{2Q/3},\quad g_3=\omega_3(1+e^Q),$$

where $\omega_1, \omega_2\in\{-1, e^{\pm i\pi/3}\}$. Thus

$$\begin{cases}\sqrt[3]{9}\,\omega_1\omega_2 g_{j_0}{}^2+\sqrt[3]{9}\,\omega_1\omega_2\omega_3{}^2 g_{j_0}g_3+g_{j_1}g_{j_2}=0\\ \omega_1\omega_2 g_{j_1}{}^2-\sqrt[3]{3}\,g_{j_0}g_{j_2}=0\,.\end{cases}$$

2) $e^q=3e^{2Q}$. In this case $-\exp(3q_{j_2})=e^{3Q}$. Thus

$$g_{j_0}=-1,\quad g_{j_1}=\sqrt[3]{3}\,\omega_1 e^{Q/3},\quad g_{j_2}=\omega_2 e^Q,\quad g_3=\omega_3(1+e^Q),$$

where $\omega_1, \omega_2\in\{-1, e^{\pm i\pi/3}\}$. Thus

$$\begin{cases}\omega_2 g_{j_0}-g_{j_2}+\omega_2\omega_3{}^2 g_3=0\\ \omega_2 g_{j_1}{}^3+3g_{j_0}{}^2 g_{j_2}=0\,.\end{cases}$$

Theorem 4 is this proved.

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