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THE HADAMARD VARIATIONAL FORMULA FOR THE GROUND STATE VALUE OF $-\Delta u=\lambda|u|^{p-1}u$

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1. Introduction. This article is divided into three parts. In every part we study the Hadamard variational formula for the (non-trivial) ground state value of the semi-linear equation $-\Delta u=\lambda|u|^{p-1}u$.

Let Ω be a bounded domain in R^N ($N\geqq 2$) with smooth boundary $\partial\Omega$. Let ρ be a smooth function on $\partial\Omega$. We denote $\nu(x)$ as the exterior unit normal vector at $x\in\partial\Omega$. If ε is small enough, we have a new domain Ω_ε bounded by

$$\partial\Omega_\varepsilon=\{x+\varepsilon\rho(x)\nu(x);\ x\in\partial\Omega\}.$$

Let p be a fixed number satisfying $1<p<\infty$ for $N=2$, $1<p<(N+2)/(N-2)$ for $N\geqq 3$.

We consider the minimizing problem

$$\lambda_\varepsilon=\inf_{X_\varepsilon}\int_{\Omega_\varepsilon}|\nabla\varphi|^2dx\,, \tag{1.1}$$

where

$$X_\varepsilon=\{\varphi\in H_0^1(\Omega_\varepsilon),\ \varphi\geqq 0,\ \|\varphi\|_{L^{p+1}(\Omega_\varepsilon)}=1\}.$$

For the sake of simplicity we write $\|\ \|_{L^{p+1}(\Omega_\varepsilon)}$ as $\|\ \|_{p+1,\varepsilon}$. It is well known that there exists at least one solution $u_\varepsilon\in C^{3,\alpha}(\bar{\Omega}_\varepsilon)$ satisfying $\|u_\varepsilon\|_{p+1,\varepsilon}=1$, and

$$\begin{aligned} -\Delta u_\varepsilon(x)&=\lambda_\varepsilon u_\varepsilon^p(x) && x\in\Omega_\varepsilon \\ u_\varepsilon(x)&=0 && x\in\partial\Omega_\varepsilon\,, \end{aligned}$$

and $u_\varepsilon>0$ in Ω_ε.

The author calls λ_ε as the Dirichlet ground state value on Ω_ε and u_ε as the Dirichlet ground state solution.

In this note we would like to consider ε-dependence of λ_ε, u_ε. One of the main result of this paper is the following: Here $\lambda_0=\lambda$, $u_0=u$.

THEOREM 1. *Assume that the number of positive solution u which minimize* $(1.1)_0$ *is unique. Assume that* $\mathrm{Ker}(\Delta+\lambda p u^{p-1})=\{0\}$. *Then, we have the following limit.*

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$$\delta\lambda=\lim \varepsilon^{-1}(\lambda_\varepsilon-\lambda) \tag{1.2}$$
$$=-\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial\nu_x}\right)^2\rho(x)d\sigma_x\,,$$

Here $\partial/\partial\nu_x$ denotes derivative along the exterior normal direction. Under the same assumption as above, we have the limit

$$\delta u=\lim_{\varepsilon\to 0}\varepsilon^{-1}(u_\varepsilon-u)$$

exists and it satisfies

$$(-\Delta-\lambda p u^{p-1})\delta u(x)=\delta\lambda u^p(x) \qquad \text{in } \Omega \tag{1.3}$$
$$\delta u(x)=-\rho(x)\frac{\partial u}{\partial\nu_x}(x) \quad \text{on } \partial\Omega\,.$$

Remark. When Ω is a ball the assumption that the number of positive solution is unique is satisfied. See Gidas-Ni-Nirenberg [8]. See also p. 152 of Dancer [4]. The assumption of uniqueness of u can not be satisfied always. Brezis-Nirenberg [3] shows a counter example for uniqueness for Ω=annulus.

In section 2, we prove the Lipschitz continuity with respect to ε of ground state value. In section 3 we prove (1.2) under some assumption of Lipschitz continuity of solutions. In section 4 we give a condition by which we have Lipschitz continuity of solutions. In section 5 we study δu under the assumption that $\mathrm{Ker}(\Delta+\lambda p u^{p-1})=\{0\}$.

The Robin problem.

We consider the minimizing problem

$$\lambda_\varepsilon=\inf_{X_\varepsilon}\left(\int_{\Omega_\varepsilon}|\nabla\varphi|^2dx+k\int_{\partial\Omega_\varepsilon}\varphi^2d\sigma_x\right), \tag{1.4}$$

where

$$X_\varepsilon=\{\varphi\in H^1(\Omega_\varepsilon),\ \|\varphi\|_{p,\varepsilon}=1,\ \varphi\geqq 0\}$$

Here $k>0$ is a positive constant. We see that there exists at least one solution of $u_\varepsilon\in X_\varepsilon$ such that it satisfies

$$-\Delta u_\varepsilon(x)=\lambda_\varepsilon u_\varepsilon{}^p(x),\ u_\varepsilon(x)>0 \quad x\in\Omega_\varepsilon \tag{1.5}$$
$$\frac{\partial}{\partial\nu_x}u_\varepsilon(x)+k\quad u_\varepsilon(x)=0 \qquad x\in\partial\Omega_\varepsilon\,.$$

We write $\lambda_0=\lambda$, $u_0(x)=u(x)$. We call λ_ε as the ground state value and u_ε as the ground state solution of $(1.4)_\varepsilon$.

THEOREM 2. *Assume that u is unique. And we assume that*

$$\mathrm{Ker}(\Delta+\lambda p u^{p-1})=\{0\}\,.$$

Then,

(1.6) $$\delta\lambda=\lim_{\varepsilon\to 0}\varepsilon^{-1}(\lambda_\varepsilon-\lambda)$$

exists and is equal to

(1.7) $$\int_{\partial\Omega}(|\nabla_t u|^2-(2\lambda/(p+1))u^{p+1}-(k^2-(N-1)kH_1)u^2)\rho d\sigma_x,$$

where ∇_t denotes the gradient on the tangential plane at $x\in\partial\Omega$. Here H_1 denotes the mean curvature at $x\in\partial\Omega$ with respect to the interior normal direction.

Under the same assumption as above we get

(1.8) $$\begin{aligned}\delta u(x)&=\lim\varepsilon^{-1}(u_\varepsilon(x)-u(x)) \qquad in\ \Omega\\ &=-\delta\lambda(\lambda(p-1))^{-1}u\\ &\quad-\int_{\partial\Omega}\{\nabla_t\Gamma(x,y)\nabla_t u(y)-\Gamma(x,y)(\lambda u(y)^p\\ &\quad+(k^2-(N-1)kH_1(y))u(y))\}\rho(y)d\Omega\end{aligned}$$

where $\Gamma=\Gamma(x,y)$ is the Green function of $-\Delta-\lambda pu^{p-1}$ under the Robin condition on the boundary $\partial\Omega$.

Remark. As far as the author concerns, the semilinear problem (1.5) did not discuss in other articles.

In Part II, section 6, we examine the continuity property of λ_ε. In section 7 we prove (1.8) under the assumption of Lipschitz continuity of u_ε.

Neumann condition.

We consider the minimizing problem.

(1.10) $$\lambda_\varepsilon=\inf_{X_\varepsilon}\int_{\Omega_\varepsilon}|\nabla\varphi|^2dx,$$

where

$$X_\varepsilon=\left\{\varphi\in H^1(\Omega_\varepsilon),\ \|\varphi\|_{p+1,\varepsilon}=1,\int_{\Omega_\varepsilon}|\varphi|^{p-1}\varphi dx=0\right\}.$$

If we replace X_ε by $Y_\varepsilon=\{\varphi\in H^1(\Omega_\varepsilon),\ \|\varphi\|_{p+1,\varepsilon}=1\}$. Then, we see that $\tilde{\lambda}_\varepsilon=\inf_{Y_\varepsilon}\int_{\Omega_\varepsilon}|\nabla\varphi|^2dx=0$, when u_ε=constant. It is easy to show that $\lambda_\varepsilon>0$, and there exists at least one solution u_ε of (1.10) which satisfies

(1.11) $$\begin{aligned}-\Delta u_\varepsilon(x)&=\lambda_\varepsilon|u_\varepsilon|^{p-1}u_\varepsilon(x) && x\in\Omega_\varepsilon\\ \frac{\partial}{\partial\nu_x}u_\varepsilon(x)&=0 && x\in\partial\Omega_\varepsilon.\end{aligned}$$

The author would like to call λ_ε as the second state value and u_ε as the second state solution of (1.10). The condition

$$\int_{\Omega_\varepsilon} |\varphi|^{p-1}\varphi dx=0$$

is natural, since

$$-\int_{\Omega_\varepsilon} \Delta u_\varepsilon(x)dx=-\int_{\partial\Omega_\varepsilon} (\partial u/\partial \nu_x)d\sigma_x=\lambda_\varepsilon\int_{\Omega_\varepsilon} |u_\varepsilon|^{p-1}u_\varepsilon(x)dx\,.$$

We write $\lambda_0=\lambda$, $u_0=u$.

We have the following

THEOREM 3. *We assume that* $\mathrm{Ker}(\Delta+\lambda p|u|^{p-1})=\{0\}$. *We also assume that u is unique up to its signature. Then, we have u_ε ($\varepsilon>0$) such that*

$$\|\tilde{u}_\varepsilon-u\|_{C^2(\bar{\Omega})}=O(\varepsilon)\,. \tag{1.12}$$

See $\tilde{u}_\varepsilon$ for the Notation in section 2. Moreover,

$$\delta\lambda=\lim_{\varepsilon\to 0}\varepsilon^{-1}(\lambda_\varepsilon-\lambda) \tag{1.13}$$

exists and is equal to

$$\delta\lambda=\int_{\partial\Omega}\{|\nabla_t u|^2-((2\lambda)/(p+1))|u|^{p+1}\}\rho d\sigma_x\,. \tag{1.14}$$

Under the same assumption as above we have

$$\delta u(x)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(u_\varepsilon(x)-u(x)) \qquad x\in\Omega \tag{1.15}$$

exists and is equal to

$$\begin{aligned}\delta u(x)=&-\delta\lambda u(x)/(\lambda(p-1))\\ &-\int_{\partial\Omega}\{\nabla_t\Gamma(x,y)\cdot\nabla_t u(y)-\lambda\Gamma(x,y)|u(y)|^{p-1}u(y)\}\rho d\sigma_x\end{aligned} \tag{1.16}$$

Here $\Gamma=\Gamma(x,y)$ is the Green (Neumann) function of $-\Delta-\lambda p|u|^{p-1}$ with respect to the Neumann condition.

In section 9 *we prove the Lipschitz continuity of λ_ε.*

Part I

§2. Lipschitz continuity of ground state value.

In this section we prove the following.

PROPOSITION 2.1. *There exists a constant C independent of ε such that*

$$|\lambda_\varepsilon-\lambda|\leq C\varepsilon\,.$$

Remark. From this proposition we can deduce $\|u\|_{C^{3,\alpha}(\Omega_\varepsilon)}\leq C$. See the Appendix.

Proof of Proposition 2.1. First we would like to construct nice C^∞-diffeomorphism between $\bar{\Omega}$ and $\bar{\Omega}_\varepsilon$. Let U_0 be a neighbourhood of $\partial\Omega$ in R^n such that the following holds:

For any $x\in U_0$ there exists unique x' such that $|x-x'|=\mathrm{dist}(x, \partial\Omega)$. We write $x'=P(x)$. Then, $P\in C^\infty(U_0, \partial\Omega)$. Let $\nu(x)$ be an exterior normal vector at x. Then, $\nu\in C^\infty(\partial\Omega, R^N)$.

We construct the following diffeomorphism. Let Ω' be (Ω'' be, respectively) a bounded domain with boundary $\partial\Omega'=\{x-\delta\nu(x);\ x\in\partial\Omega\}$ ($\partial\Omega''=\{x-2\delta\nu(x);\ x\in\partial\Omega\}$. Fix a compact set K in Ω. Then, $K\subset\Omega''\Subset\Omega'\Subset\Omega$ for any sufficiently small $\delta>0$. Fix small $\varepsilon\geqq 0$. Then, take δ such that $\Omega'\Subset\Omega_\varepsilon$. Take $\varphi\in C^\infty(\bar{\Omega}, \boldsymbol{R})$ such that $0\leqq\varphi\leqq 1$, $\varphi=0$ on $\bar{\Omega}''$ $\varphi=1$ on $\bar{\Omega}\setminus\Omega'$. Then, we set

$$\begin{aligned}\Phi_\varepsilon(x)&=x \qquad x\in\Omega''\\ &=x+\varepsilon\varphi(x)\rho(P(x))\nu(P(x)) \qquad x\in\bar{\Omega}\setminus\Omega''.\end{aligned}$$

Then, we can take ε such that Φ_ε is a bijection $\bar{\Omega}\simeq\bar{\Omega}_\varepsilon$. We see that $\Phi_\varepsilon:\bar{\Omega}\simeq\bar{\Omega}_\varepsilon$ is surjective diffeomorphism. It is easy to see that the following properties (2.1), (2.2), (2.3) hold.

(2.1) If we put $\Phi_\varepsilon(x)=x+\varepsilon S_\varepsilon(x)$, $x\in\bar{\Omega}$.
Then, $S_\varepsilon\in C^\infty(\bar{\Omega}, R^n)$, $\|S_\varepsilon\|_{C^m(\bar{\Omega})^n}\leqq C_m$ (independent of ε) for $m\in\boldsymbol{N}\cup\{0\}$. Conversely, there is $t_\varepsilon\in C^\infty(\bar{\Omega}_\varepsilon, R^n)$ such that $\|t_\varepsilon\|_{C^m(\bar{\Omega}_\varepsilon)^n}\leqq C_m$ (independent of ε) for $m\in N\cup\{0\}$ satisfying $\Phi_\varepsilon^{-1}(x)=x+\varepsilon t_\varepsilon(x)$, $x\in\bar{\Omega}_\varepsilon$.

(2.2) For $x\in K$, $s_\varepsilon(x)=t_\varepsilon(x)=0$.

(2.3) If $x\in$(some neighbourhood of $\partial\Omega)\cap\bar{\Omega}$, then $S_\varepsilon(x)=\rho(P(x))\nu(P(x))$.
If $x\in$(some neighbourhood of $\partial\Omega_\varepsilon)\cap\bar{\Omega}_\varepsilon$, then $t_\varepsilon(x)=-\rho(P(x))\nu(P(x))$.

It is an easy exercise that $J\Phi_\varepsilon(x)=1+O(\varepsilon)$, where $J\Phi_\varepsilon(x)$ denotes Jacobian.

By using the above Φ_ε we can make pull back and push foward of functions. We put $(\Phi_\varepsilon{}^*f)(x)=f(\Phi_\varepsilon(x))$ for function f on $\bar{\Omega}_\varepsilon$.

Notation. If $\varphi\in C^0(\bar{\Omega}_\varepsilon)$, then $\tilde{\varphi}=\Phi^*\varphi$

$$\phi\in C^0(\bar{\Omega}), \qquad \text{then} \quad \hat{\phi}=(\Phi_\varepsilon{}^*)^{-1}\phi.$$

Let Δ denote the Laplacian. Then, we denote

$$\tilde{\Delta}=\Phi_\varepsilon{}^*\Delta\Phi_\varepsilon{}^{*-1}$$
$$\hat{\Delta}=\Phi_\varepsilon{}^{*-1}\Delta\Phi_\varepsilon{}^*.$$

We also write $\tilde{\nabla}=\Phi_\varepsilon{}^*\nabla\Phi_\varepsilon^{*-1}$, $\hat{\nabla}=\Phi_\varepsilon{}^{*-1}\nabla\Phi_\varepsilon{}^*$.

Example. If u satisfies $-\Delta u_\varepsilon(x)=\lambda_\varepsilon u_\varepsilon(x)^p$ $x\in\Omega_\varepsilon$ then, $-\tilde{\Delta}\tilde{u}_\varepsilon(x)=\lambda_\varepsilon\tilde{u}_\varepsilon(x)^p$ in $x\in\Omega$.

For Φ_ε the following result hold. We do not give a proof.

LEMMA 2.2. *We have the following properties* (i)$\sim$(viii).

(i) $|J\Phi_\varepsilon(x)|=1+O(\varepsilon)$ *uniformly for* $x\in\bar{\Omega}$.

(ii) $|J\Phi_\varepsilon^{-1}(x)|=1+O(\varepsilon)$ *uniformly for* $x\in\bar{\Omega}_\varepsilon$.

(iii) $\Phi_\varepsilon{}^*: C^{m,\alpha}(\bar{\Omega}_\varepsilon)\to C^{m,\alpha}(\bar{\Omega})$,
$\Phi_\varepsilon^{*-1}: C^{m,\alpha}(\bar{\Omega})\to C^{m,\alpha}(\bar{\Omega}_\varepsilon)$,
is a bounded linear mapping for any $m\in N\cup\{0\}$, $0\leqq\alpha\leqq 1$.

(iv) *For* $\varphi\in C^1(\bar{\Omega}_\varepsilon)$, $(\partial\tilde{\varphi}/\partial\nu)(x)=(\partial/\partial\nu)\varphi(x+\varepsilon\rho(x)\nu(x))$ $x\in\partial\Omega$. *Here* $\partial/\partial\nu$ *denotes the normal derivative at* $\partial\Omega$.

(v) *For* $\varphi\in C^{1+m,\alpha}(\bar{\Omega}_\varepsilon)$, *then*
$\|\tilde{\nabla}u-\nabla u\|_{C^{m,\alpha}(\bar{\Omega}_\varepsilon)^n}\leqq C_m\varepsilon\|\varphi\|_{C^{1+m,\alpha}(\bar{\Omega}_\varepsilon)}$
for $m\in N\cup\{0\}$, $0\leqq\alpha\leqq 1$.

(vi) *For* $C^{2+m,\alpha}(\Omega_\varepsilon)$, *then*
$\|\tilde{\Delta}\varphi-\Delta\varphi\|_{C^{m,\alpha}(\bar{\Omega}_\varepsilon)}\leqq C_m\varepsilon\|\varphi\|_{C^{2+m,\alpha}(\bar{\Omega}_\varepsilon)}$
for $m\in N\cup\{0\}$, $0\leqq\alpha\leqq 1$.

(vii) *For* $\varphi\in C^{m,\alpha}(\overline{\Omega\cup\Omega_\varepsilon})$, $\|\tilde{\varphi}-\varphi\|_{C^{m,\alpha}(\bar{\Omega})}\leqq C_{m,\varepsilon,\varphi}\to 0$ *as* $\varepsilon\to 0$. *And the convergence is uniform for* $\|\varphi\|_{C^{m,\alpha}(\overline{\Omega\cup\Omega_\varepsilon})}\leqq C$.

(viii) *For* $\varphi\in C^{1+m,\alpha}(\overline{\Omega\cup\Omega_\varepsilon})$, *then* $\|\tilde{\varphi}-\varphi\|_{C^{m,\alpha}(\bar{\Omega})}\leqq C_m\varepsilon\|\varphi\|_{C^{1+m,\alpha}(\overline{\Omega\cup\Omega_\varepsilon})}$ *for* $m\in N\cup\{0\}$, $0\leqq\alpha\leqq 1$.

We give a proof of (vii), (viii) only for $n=0$, $\alpha=0$. $\tilde{\varphi}(x)-\varphi(x)=\varphi(\Phi_\varepsilon(x))-\varphi(x)$, where $\sup|\Phi_\varepsilon(x)-x|\leqq C\varepsilon$ and the continuity implies (vii). $\tilde{\varphi}(x)-\varphi(x)=\varphi(\Phi_\varepsilon(x))-\varphi(x)\leqq|\Phi_\varepsilon(x)-x|\,\|\nabla\varphi\|_{C^0(\overline{\Omega\cup\Omega_\varepsilon})}$ implies (viii).

Now we are in a time to prove Proposition 2.1. We have

$$\begin{aligned}\int_{\Omega_\varepsilon}|\nabla\hat{u}|^2dx&=\int_\Omega|\tilde{\nabla}u|^2|J\Phi_\varepsilon|dx\\&=\int_\Omega|\nabla u|^2|J\Phi_\varepsilon|dx+\int_\Omega(\tilde{\nabla}u-\nabla u)(\tilde{\nabla}u+\nabla u)|J\Phi_\varepsilon|dx\\&=\int_\Omega|\nabla u|^2dx+O(\varepsilon)\\&=\lambda+O(\varepsilon).\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\Omega_\varepsilon}|\hat{u}|^{p+1}dx&=\int_\Omega|u|^{p+1}|J\Phi_\varepsilon|dx\\&=\int_\Omega|u|^{p+1}dx+O(\varepsilon)\\&=1+O(\varepsilon).\end{aligned}$$

Since $\hat{u}|_{\partial\Omega_\varepsilon}=0$, $\hat{u}\in H_0^1(\Omega_\varepsilon)$. Therefore,

$$\lambda_\varepsilon\leqq\lambda+O(\varepsilon).$$

Conversely we also get $\lambda\leqq\lambda_\varepsilon+O(\varepsilon)$. Thus, we get the desired result.

§ 3. Variational formula for ground State value.

In the present time, we assume that

$$\|\tilde{u}_\varepsilon - u\|_{C^2(\bar{\Omega})} \leqq C\varepsilon \tag{3.1}$$

as $\varepsilon \to 0$. Under this assumption we will prove Theorem 1. The validity of the assumption (3.1) is discussed in sections 4 and 5.

In this section we use an idea of using Whitney's extension by which the Hadamard variational formula for linear problem is proved. See Fujiwara-Ozawa [5].

We can show the following.

LEMMA 3.1. *There exists a C^3 extension $\bar{u}_\varepsilon$ of u_ε to R^n such that*
(i) $\|\bar{u}_\varepsilon\|_{C^3(R^n)} \leqq C < +\infty$
(ii) $\|\bar{u}_\varepsilon - u\|_{C^2(\bar{\Omega})} \leqq C\varepsilon$.

Proof. (i) is trivial. We have

$$\|\bar{u}_\varepsilon - u\|_{C^2(\bar{\Omega})} = \|\bar{u}_\varepsilon - \tilde{u}_\varepsilon\|_{C^2(\bar{\Omega})} + \|\tilde{u}_\varepsilon - u\|_{C^2(\bar{\Omega})}.$$

Then, by (3.1), $\|\tilde{u}_\varepsilon - u\|_{C^2(\Omega)} \leqq C\varepsilon$. We know that $\tilde{u}_\varepsilon = \tilde{\bar{u}}_\varepsilon$ in $\bar{\Omega}$. Then,

$$\|\bar{u}_\varepsilon - \tilde{u}_\varepsilon\|_{C^2(\bar{\Omega})} = \|\bar{u}_\varepsilon - \tilde{\bar{u}}_\varepsilon\|_{C^2(\bar{\Omega})} \leqq C\varepsilon \|\bar{u}_\varepsilon\|_{C^3(\overline{\Omega \cap \Omega_\varepsilon})} \leqq C\varepsilon$$

by (viii) of Lemma 2.2.

For the sake of simplicity we put $f(t) = |t|^{p-1}t$. Then, $f'(t) = p|t|^{p-1}$.

LEMMA 3.2. *The estimates*
(i) $\|f(\bar{u}_\varepsilon) - f(u)\|_{C^0(\bar{\Omega})} = O(\varepsilon)$
(ii) $\|f(\bar{u}_\varepsilon) - f(u) - f'(u)(\bar{u}_\varepsilon - u)\|_{C^0(\bar{\Omega})} = O(\varepsilon)$
(iii) $\|\Delta\bar{u}_\varepsilon + \lambda_\varepsilon f(\bar{u}_\varepsilon)\|_{C^0(\bar{\Omega})} = O(\varepsilon)$
hold.

Proof. (i) is determined by Lemma 3.1. By the mean value theorem, we have

$$f(\bar{u}_\varepsilon) - f(u) = f'(u + \theta_\varepsilon(x)(\bar{u}_\varepsilon - u))(\bar{u}_\varepsilon - \bar{u}).$$

Then, $\|f(\bar{u}_\varepsilon) - f(u) - f'(u)(\bar{u}_\varepsilon - u)\|_{C^0(\bar{\Omega})} = \|f'(u + \theta_\varepsilon(\bar{u}_\varepsilon - u)) - f'(u))(\bar{u}_\varepsilon - u)\|_{C^0(\bar{\Omega})} \leqq O(\varepsilon)O(1) = o(\varepsilon)$.

We want to prove (iii). We have $\tilde{\Delta}\tilde{u}_\varepsilon + \lambda_\varepsilon f(\tilde{u}_\varepsilon) = 0$ in Ω. Then, $\Delta\bar{u}_\varepsilon + \lambda_\varepsilon f(\bar{u}_\varepsilon) = \Delta(\bar{u}_\varepsilon - \tilde{u}_\varepsilon) + \lambda_\varepsilon(f(\bar{u}_\varepsilon) - f(\tilde{u}_\varepsilon)) + (\Delta - \tilde{\Delta})\tilde{u}_\varepsilon$. Since $\|\bar{u}_\varepsilon - \tilde{u}_\varepsilon\|_{C^2(\bar{\Omega})} \leqq C\varepsilon$, we hove $\|\Delta(\bar{u}_\varepsilon - \tilde{u}_\varepsilon)\|_{C^0(\bar{\Omega})} = O(\varepsilon)$ as in the proof of (ii) in Lemma 3.1. Similarly $\|f(\bar{u}_\varepsilon) - f(\tilde{u}_\varepsilon)\|_{C^0(\bar{\Omega})} = O(\varepsilon)$.

We know that $|\lambda_\varepsilon| \leqq C$. Therefore, as in the Appendix $\|\tilde{u}_\varepsilon\|_{C^3(\bar{\Omega})} \leqq C$, which implies $\|(\Delta - \tilde{\Delta})\tilde{u}_\varepsilon\|_{C^0(\bar{\Omega})} = O(\varepsilon)$.

We prove the following.

LEMMA 3.3. *The equality*

$$\left\|\bar{u}_\varepsilon+\varepsilon\rho\frac{\partial u}{\partial\nu}\right\|_{C^0(\partial\Omega)}=o(\varepsilon)$$

holds.

Proof. We put $x\in\partial\Omega$. Then, $0=u_\varepsilon(x+\varepsilon\rho(x)\nu(x))=\bar{u}_\varepsilon(x+\varepsilon\rho(x)\nu(x))$. On the other hand

$$0=\bar{u}_\varepsilon(x+\varepsilon\rho(x)\nu(x))=\bar{u}_\varepsilon(x)+\varepsilon\rho(x)\frac{\partial}{\partial\nu}\bar{u}_\varepsilon(x)+o(\varepsilon).$$

Here $o(\varepsilon)$ is uniform with respect to $x\in\partial\Omega$. Then,

$$\begin{aligned}\left\|u_\varepsilon+\varepsilon\rho\frac{\partial u}{\partial\nu}\right\|_{C^0(\partial\Omega)}&\leqq\varepsilon\left\|\rho\frac{\partial}{\partial\nu}(\bar{u}_\varepsilon-u)\right\|_{C^0(\partial\Omega)}+o(\varepsilon)\\&\leqq C\varepsilon\|\bar{u}_\varepsilon-u\|_{C^1(\partial\Omega)}+o(\varepsilon).\end{aligned}$$

By Lemma 3.1 we get the desired result.

The following Lemma 3.4 is easy to see. Thus, we omit its proof.

LEMMA 3.4. *For given $\varphi\in C^1(\overline{\Omega_\varepsilon\cup\Omega})$. Then,*

$$\int_{\Omega_\varepsilon}\varphi dx-\int_\Omega\varphi dx=\varepsilon\int_{\partial\Omega}\varphi\rho d\sigma+o(\varepsilon)$$

and $o(\varepsilon)$ is uniform with respect to φ satisfying $\|\varphi\|_{C^1(\overline{\Omega_\varepsilon\cup\Omega})}\leqq C$.

The following Lemma is used in the proof of variational formula for the ground state value.

LEMMA 3.5. *The equation*

$$\int_\Omega f(u)(\bar{u}_\varepsilon-u)dx=-(\varepsilon/(p+1))\int_{\partial\Omega}|u|^{p+1}\rho d\sigma+o(\varepsilon)$$

holds.

COROLLARY 3.6. *The equation*

$$\int_\Omega f(u)(\bar{u}_\varepsilon-u)dx=o(\varepsilon)$$

is valid.

Proof of Lemma 3.5. We have

$$(3.2)\qquad (p+1)\int_\Omega f(u)(\bar{u}_\varepsilon-u)dx=\int_\Omega uf'(u)(\bar{u}_\varepsilon-u)dx+\int_\Omega f(u)(\bar{u}_\varepsilon-u)dx.$$

Here we used $uf'(u)=pf(u)$. (3.2) is equal to

$$=\int_\Omega \bar{u}_\varepsilon f'(u)(\bar{u}_\varepsilon-u)dx-\int_\Omega f'(u)(\bar{u}_\varepsilon-u)^2dx$$

$$+\int_\Omega f(u)\bar{u}_\varepsilon dx-\int_\Omega f(u)udx$$

$$=\int_\Omega (f'(u)(\bar{u}_\varepsilon-u)-(f(\bar{u}_\varepsilon)-f(u)))\bar{u}_\varepsilon dx$$

$$-\int_\Omega f'(u)(\bar{u}_\varepsilon-u)^2dx+\int_\Omega f(\bar{u}_\varepsilon)\bar{u}_\varepsilon dx-\int_\Omega f(u)udx\,.$$

The first term in the right hand side of (3.2) is $o(\varepsilon)$ by Lemma 3.2 (ii). The second term in the right hand side of (3.2) is $O(\varepsilon^2)$ by Lemma 3.1 (ii).

We see that

$$\int_{\Omega_\varepsilon} f(\bar{u}_\varepsilon)\bar{u}_\varepsilon dx=\int_\Omega f(u)udx=1\,.$$

Thus, the third and the fourth term in the right hand side of (3.2) is equal to

$$-\int_{\Omega_\varepsilon} f(\bar{u}_\varepsilon)\bar{u}_\varepsilon dx+\int_\Omega f(\bar{u}_\varepsilon)\bar{u}_\varepsilon dx=-\varepsilon\int_{\partial\Omega} f(\bar{u}_\varepsilon)\bar{u}_\varepsilon\rho d\sigma+o(\varepsilon)$$

$$=-\varepsilon\int_{\partial\Omega} f(u)u\rho d\sigma+o(\varepsilon)=-\varepsilon\int_{\partial\Omega}|u|^{p+1}\rho d\sigma+o(\varepsilon)\,.$$

Here we used Corollary 3.6 and Lemma 3.1, (ii), Lemma 3.2 (i).

We are now in a position to prove Theorem 1. By the Green formula and $u|_{\partial\Omega}=0$, we have

$$\int_\Omega (\Delta u\cdot\bar{u}_\varepsilon-u\Delta\bar{u}_\varepsilon)dx=\int_{\partial\Omega}(\partial u/\partial\nu)\bar{u}_\varepsilon d\sigma\,. \tag{3.3}$$

We have

$$\int_\Omega \Delta u\cdot\bar{u}_\varepsilon dx=-\lambda\int_\Omega f(u)\bar{u}_\varepsilon dx\,.$$

On the other hand,

$$\int_\Omega u\Delta\bar{u}_\varepsilon dx=\int_{\Omega\cap\Omega_\varepsilon} u\Delta u_\varepsilon dx+\int_{\Omega\setminus\Omega_\varepsilon} u\Delta\bar{u}_\varepsilon dx$$

$$=-\lambda_\varepsilon\int_{\Omega\cap\Omega_\varepsilon} uf(u_\varepsilon)dx-\lambda_\varepsilon\int_{\Omega\setminus\Omega_\varepsilon} uf(\bar{u}_\varepsilon)dx+\int_{\Omega\setminus\Omega_\varepsilon} u(\Delta\bar{u}_\varepsilon+\lambda_\varepsilon f(\bar{u}_\varepsilon))dx$$

$$=-\lambda_\varepsilon\int_\Omega uf(\bar{u}_\varepsilon)dx+\int_{\Omega\setminus\Omega_\varepsilon} u(\Delta\bar{u}_\varepsilon+\lambda_\varepsilon f(\bar{u}_\varepsilon))dx$$

$$=(3.4)$$

The second term in the right hand side of (3.4) satisfies

$$|\Omega\setminus\Omega_\varepsilon|\,\|u\|_{C^0(\bar\Omega)}\|\Delta\bar u_\varepsilon+\lambda_\varepsilon f(\bar u_\varepsilon)\|_{C^0(\bar\Omega)}=O(\varepsilon^2)$$

by Lemma 3.2 (iii). Thus,

$$\int_\Omega u\Delta\bar u_\varepsilon dx=-\lambda_\varepsilon\int_\Omega uf(\bar u_\varepsilon)dx+o(\varepsilon).$$

Therefore, the left hand side of (3.3) is equal to

$$-\lambda\int_\Omega f(u)\bar u_\varepsilon dx+\lambda_\varepsilon\int_\Omega uf(\bar u_\varepsilon)dx+o(\varepsilon)$$

$$=(\lambda_\varepsilon-\lambda)\int_\Omega uf(\bar u_\varepsilon)dx+\lambda\int_\Omega(f(\bar u_\varepsilon)u-f(u)\bar u_\varepsilon)dx+o(\varepsilon)$$

$$=\lambda_\varepsilon-\lambda+(\lambda_\varepsilon-\lambda)\int_\Omega u(f(\bar u_\varepsilon)-f(u))dx$$

$$+\lambda(p-1)\int_\Omega f(u)(\bar u_\varepsilon-u)dx+\lambda I_2+o(\varepsilon),$$

using $\int_\Omega uf(u)dx=1$, where

$$I_2=\int_\Omega(f(\bar u_\varepsilon)u-f(u)\bar u_\varepsilon-(p-1)f(u)(\bar u_\varepsilon-u))dx.$$

Since $\lambda_\varepsilon-\lambda=O(\varepsilon)$, $\|f(u_\varepsilon)-f(u)\|_{C^0(\bar\Omega)}=O(\varepsilon)$, we have estimated a term in the above formula. The integrand in I_2 satisfies

$$\|f(\bar u_\varepsilon)u-f(u)\bar u_\varepsilon-(p-1)f(u)(\bar u_\varepsilon-u)\|_{C^0(\bar\Omega)}$$

$$=\|f(\bar u_\varepsilon)u-f(u)u_\varepsilon-(uf'(u)-f(u))(\bar u_\varepsilon-u)\|_{C^0(\bar\Omega)}$$

$$=\|f(\bar u_\varepsilon)u-f(u)u-uf'(u)(\bar u_\varepsilon-u)\|_{C^0(\bar\Omega)}$$

$$=\|u(f(\bar u_\varepsilon)-f(u)-f'(u)(\bar u_\varepsilon-u))\|_{C^0(\bar\Omega)}=o(\varepsilon)$$

by Lemma 3.2 (ii).

Summing up these facts we get by Lemma 3.5

$$(3.3)=\lambda_\varepsilon-\lambda-\varepsilon(p-1)/(p+1)\lambda\int_{\partial\Omega}|u|^{p+1}\rho d\sigma+o(\varepsilon)$$

$$=\lambda_\varepsilon-\lambda+o(\varepsilon).$$

By Lemma 3.3, the right hand side of (3.3) is equal to

$$-\varepsilon\int_{\partial\Omega}(\partial u/\partial\nu)^2\rho d\sigma+o(\varepsilon).$$

Therefore, $\lambda_\varepsilon-\lambda=-\varepsilon\int_{\partial\Omega}(\partial u/\partial\nu)^2\rho d\sigma+o(\varepsilon)$, which implies Theorem 1.

§ 4. Variational formula for ground state solution.

In this section we assume the following.

$$\operatorname{Ker}(\Delta+\lambda p u^{p-1})=\{0\}. \tag{4.1}$$

And we will show the following important result.

PROPOSITION 4.1. *Assume that the minimizer of* (1.1) *is unique. Assume that* (4.1) *holds. Then, we have* (3.1).

Remark. The condition (4.1) will be closely related to bifurcation phenomena.

Proof of Proposition 4.1. By the regularity theorem (in the Appendix) $\|u_\varepsilon\|_{C^{3,\alpha}(\bar\Omega_\varepsilon)}\leqq C$. Thus, $\|\tilde u_\varepsilon\|_{C^{3,\alpha}(\bar\Omega)}\leqq C$. We take $0<\alpha'<\alpha$. Then, $C^{3,\alpha}(\bar\Omega)\hookrightarrow C^{3,\alpha'}(\bar\Omega)$ is a compact embedding. Thus, K given by $K=\{\tilde u_\varepsilon;\, 0<\varepsilon\ll 1\}$ is compact in $C^{3,\alpha'}(\bar\Omega)$. As a corollary of this compactness result, we get the following:

Assume that the ground state solution on Ω is unique, then for any ground state solution on Ω_ε $(\varepsilon>0)$, u_ε, we have $\tilde u_\varepsilon\to u$ strongly in $C^{3,\alpha'}(\bar\Omega)$. Thus, $\|\tilde u_\varepsilon-u\|_{C^3(\bar\Omega)}\to 0$.

We have $(\Delta+\lambda p|u|^{p-1})(\tilde u_\varepsilon-u)=(\Delta+\lambda f'(u))(\tilde u_\varepsilon-u)=(\Delta-\tilde\Delta)\tilde u_\varepsilon-(\lambda_\varepsilon-\lambda)f(\tilde u_\varepsilon)-\lambda(f(\tilde u_\varepsilon)-f(u)-f'(u)(\tilde u_\varepsilon-u))=g_\varepsilon$. Here we used $\tilde\Delta\tilde u_\varepsilon+\lambda_\varepsilon f(\tilde u_\varepsilon)=0$. Also $\tilde u_\varepsilon=u=0$ on $\partial\Omega$.

Thus, by the assumption and (4.1)

$$\|\tilde{\tilde u}_\varepsilon-u\|_{C^{2,\alpha'}(\bar\Omega)}\leqq C\|g_\varepsilon\|_{C^{\alpha'}(\bar\Omega)}. \tag{4.2}$$

Here

$$\|(\Delta-\tilde\Delta)\tilde u_\varepsilon\|_{C^{\alpha'}(\bar\Omega)}\leqq C\varepsilon\|\tilde u\|_{C^{2,\alpha'}(\bar\Omega)} \qquad \text{(Lemma 2.2 (vi))}$$

$$|\lambda_\varepsilon-\lambda|\leqq C\varepsilon.$$

Thus,

$$\|f(\tilde u_\varepsilon)-f(u)-f'(u)(\tilde u_\varepsilon-u)\|_{C^{\alpha'}(\bar\Omega)}=o(\|\tilde u_\varepsilon-u\|_{C^{\alpha'}(\bar\Omega)})$$

without any use of $\|\tilde u_\varepsilon-u\|_{C^2(\bar\Omega)}\leqq C\varepsilon$. We have

$$\|\tilde u_\varepsilon\|_{C^{2,\alpha'}(\Omega)}\leqq C, \qquad \|f(\tilde u_\varepsilon)\|_{C^{\alpha'}(\bar\Omega)}\leqq C.$$

Then,

$$\|f_\varepsilon\|_{C^{\alpha'}(\bar\Omega)}\leqq C\varepsilon+o(\|\tilde u_\varepsilon-u\|_{C^{\alpha'}(\bar\Omega)})$$

Then, by (4.2)

$$\|\tilde u_\varepsilon-u\|_{C^{2,\alpha'}(\bar\Omega)}\leqq C\varepsilon+o(\|\tilde u_\varepsilon-u\|_{C^{2,\alpha'}(\bar\Omega)}).$$

Therefore, we get the desired result.

5. Explicit representation of δu.

Assume that the ground state solution in Ω is unique. Assume that $\mathrm{Ker}(\Delta+\lambda p u^{p-1})=\{0\}$. Then, we want to show that

$$\delta u(x)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(u_{\varepsilon}(x)-u(x))$$

exists and which is equal to

$$-((\delta\lambda)/(\lambda(p-1)))u(x)+\int_{\partial\Omega}\frac{\partial\Gamma(x, y)}{\partial\nu_y}\frac{\partial u(y)}{\partial\nu_y}\rho(y)d\sigma_y .$$

Here $\Gamma(x, y)$ is the Green function of the operator $-(\Delta+\lambda p|u|^{p-1})$ under the Dirichlet condition.

Proof. We put $f(t)=|t|^{p-1}t,\ f'(t)=p|t|^{p-1}$. We use the same notation as before. By the Green formula we have

$$\begin{aligned}
\bar{u}_{\varepsilon}(x)-u(x)&=-\langle(\Delta_y+\lambda f'(u(y)))\Gamma(x, y),\ \bar{u}_{\varepsilon}(y)-u(y)\rangle_y\\
&=-\int_{\Omega}\Gamma(x, y)(\Delta_y+\lambda f'(u(y)))(\bar{u}_{\varepsilon}(y)-u(y))dy\\
&\quad-\int_{\partial\Omega}\frac{\partial\Gamma(x, y)}{\partial\nu_y}(\bar{u}_{\varepsilon}(y)-u(y))d\sigma_y\\
&\quad+\int_{\partial\Omega}\Gamma(x, y)\frac{\partial}{\partial\nu_y}(\bar{u}_{\varepsilon}(y)-u(y))d\sigma_y\\
&=-\int_{\Omega}\Gamma(\Delta+\lambda f'(u))(\bar{u}_{\varepsilon}-u)dy\\
&\quad-\int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_y}\bar{u}_{\varepsilon}d\sigma_y\\
&=-J_1-J_2 .
\end{aligned}$$

We fix $x\in\Omega$. Then, $\|\partial\Gamma/\partial\nu_y\|_{C_y^0(\partial\Omega)}\leqq C$. Thus, by Lemma 3.3.

$$J_2=-\varepsilon\int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_y}\frac{\partial u}{\partial\nu_y}\rho d\sigma_y+0(\varepsilon).$$

We examine J_1.

$$\begin{aligned}
J_1&=\int_{\Omega\setminus\Omega_{\varepsilon}}\Gamma(\Delta+\lambda f'(u))\bar{u}_{\varepsilon}dx+\int_{\Omega\cap\Omega_{\varepsilon}}\Gamma(\Delta+\lambda f'(u))\bar{u}_{\varepsilon}dx\\
&\quad-\int_{\Omega}\Gamma(\Delta+\lambda f'(u))u\,dx
\end{aligned}$$

$$
\begin{aligned}
&=\int_{\Omega\setminus\Omega_\varepsilon}\Gamma(\Delta\bar{u}_\varepsilon+\lambda_\varepsilon f(\bar{u}_\varepsilon)+\lambda f'(u)\bar{u}_\varepsilon)dx\\
&\quad-\int_{\Omega\setminus\Omega_\varepsilon}\Gamma\lambda_\varepsilon f(\bar{u}_\varepsilon)dx\\
&\quad+\int_{\Omega\cap\Omega_\varepsilon}(-\Gamma\lambda_\varepsilon f(\bar{u}_\varepsilon)+\lambda f'(u)\bar{u}_\varepsilon)dx\\
&\quad-\int_\Omega\Gamma(-\lambda f(u)+\lambda f'(u)u)dx\\
&=\int_\Omega\Gamma(-\lambda_\varepsilon f(\bar{u}_\varepsilon)+\lambda f'(u)\bar{u}_\varepsilon)dx\\
&\quad+\int_{\Omega\setminus\Omega_\varepsilon}\Gamma(\Delta\bar{u}_\varepsilon+\lambda_\varepsilon f(\bar{u}_\varepsilon))dx\\
&\quad-\int_\Omega\Gamma(-\lambda f(u)+\lambda f'(u)u)dx\\
&=-(\lambda_\varepsilon-\lambda)\int_\Omega\Gamma f(u)dx\\
&\quad-(\lambda_\varepsilon-\lambda)\int_\Omega\Gamma(f(\bar{u}_\varepsilon)-f(u))dx\\
&\quad-\lambda\int_\Omega\Gamma(f(\bar{u}_\varepsilon)-f(u)-f'(u)(u_\varepsilon-u))dx\\
&\quad+\int_{\Omega\setminus\Omega_\varepsilon}\Gamma(\Delta\bar{u}_\varepsilon+\lambda_\varepsilon f(\bar{u}_\varepsilon))dx\\
&=-(\lambda_\varepsilon-\lambda)J_3-J_4-\lambda J_5-J_6.
\end{aligned}
$$

We have $(\Delta+\lambda pf'(u))u/(\lambda(p-1))=f(u)$ in Ω and $u/(\lambda(p-1))=0$ on $\partial\Omega$. Therefore, $J_3=-u/(\lambda(p-1))$. For $\alpha'>0$, $\|J_4\|_{C^{2,\alpha'}(\bar{\Omega})}\leqq C|\lambda_\varepsilon-\lambda|\,\|f(\bar{u}_\varepsilon)-f(u)\|_{C^{\alpha'}(\bar{\Omega})}=o(\varepsilon)$ by Lemma 3.1 (ii). We see that $J_5=o(\varepsilon)$ by Lemma 3.2 (ii). We have

$$
\begin{aligned}
|J_6|&\leqq\text{meas}(\Omega\setminus\Omega_\varepsilon)\|\Gamma\|_{C^0(\overline{\Omega\setminus\Omega_\varepsilon})}\|\Delta u_\varepsilon+\lambda_\varepsilon f(u_\varepsilon)\|_{C^0(\bar{\Omega})}\\
&=o(\varepsilon)
\end{aligned}
$$

by Lemma 3.2, (iii).

Summing up these facts, we get $J_1=(\varepsilon\delta\lambda/(\lambda(p-1)))u+o(\varepsilon)$, which implies

$$\bar{u}_\varepsilon-u=\varepsilon\Big(-\delta\lambda u/(\lambda(p-1))+\int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_y}\frac{\partial u}{\partial\nu_y}\rho d\sigma_y\Big)+o(\varepsilon).$$

Part II

6. Continuity of Robin ground state value.

PROPOSITION 6.1. *There exists a constant C independent of ε such that $|\lambda_\varepsilon-\lambda|\leqq C\varepsilon$ holds. Moreover, $\|u_\varepsilon\|_{C^{3,\alpha}(\bar{\Omega}_\varepsilon)}\leqq C$.*

Proof. By the same argument as in the proof of Proposition 2.1, we have

$$\int_{\Omega_\varepsilon}|\nabla\hat{u}|^2dx=\int_\Omega|\nabla u|^2dx+O(\varepsilon)$$

$$\int_{\Omega_\varepsilon}|\hat{u}|^{p+1}dx=1+O(\varepsilon)$$

for $\hat{u}\in H^1(\Omega_\varepsilon)$. Also, we have

$$\int_{\partial\Omega_\varepsilon}k\hat{u}^2d\sigma_x=\int_{\partial\Omega}ku^2J(x)d\sigma_x,$$

where $J(x)$ is a Jacobian of $x\to x+\varepsilon\rho(x)\nu(x)$. It is easy to see that $J(x)=1+O(\varepsilon)$. Therefore, $\int_{\partial\Omega_\varepsilon}k\hat{u}^2d\sigma=\int_{\partial\Omega}ku^2d\sigma+O(\varepsilon)$. Thus, $\int_{\Omega_\varepsilon}|\nabla\hat{u}|^2dx+\int_{\partial\Omega_\varepsilon}k\hat{u}^2d\sigma=\lambda+O(\varepsilon)$. Summing up these facts $\lambda_\varepsilon\leqq\lambda+O(\varepsilon)$. We also have the inverse relations

$$\lambda\leqq\lambda_\varepsilon+O(\varepsilon).$$

We get the desired result.

PROPOSITION 6.2. *Assume that the ground state solution of* (1.4) *is unique. Then, $\tilde{u}_\varepsilon\to u$ strongly in $C^{3,\alpha'}(\bar{\Omega})$.*

Proof. This is an easy consequence of regularity of solution and compact embedding $C^{3,\alpha'}(\Omega)\subset C^{3,\alpha}(\Omega)$ for $\alpha<\alpha'$.

7. Variational formula for the ground state value.

In this section we assume that $\|\tilde{u}_\varepsilon-u\|_{C^2(\bar{\Omega})}=O(\varepsilon)$. This is proved in the later section.

PROPOSITION 7.1. *Under the above condition*

$$\delta\lambda=\int_{\partial\Omega}\left\{|\nabla_t u|^2-(2\lambda/(p+1))u^{p+1}-(k^2-(N-1)kH_1)u^2\right\}\rho d\sigma.$$

Here $H_1=H_1(x)$ is the first mean curvature at $x\in\partial\Omega$ with respect to the inner normal vector. Here ∇_t denotes the gradient on the tangent plane.

Proof of Proposition 7.1 goes as similar as stated before. We need some Lemmas which are characteristic to Robin problem.

LEMMA 7.2. *The equality*

$$\left\|\left(\frac{\partial}{\partial\nu}+k\right)\bar{u}_\varepsilon-\varepsilon\left(-k\rho\frac{\partial u}{\partial\nu}-\rho\frac{\partial^2 u}{\partial\nu^2}+\nabla_t u\nabla_t\rho\right)\right\|_{C^0(\partial\Omega)}=o(\varepsilon).$$

holds.

Proof. $\nu_\varepsilon(x+\varepsilon\rho(x)\nu(x))=(\nu(x)-\varepsilon\nabla_t\rho(x))(1+\varepsilon^2|\nabla_t\rho|^2)^{-1/2}$. Thus,

$$\begin{aligned}\frac{\partial}{\partial\nu_\varepsilon}\bar{u}_\varepsilon(x+\varepsilon\rho(x)\nu(x))&=\nabla\bar{u}_\varepsilon(x+\varepsilon\rho(x)\nu(x))\cdot\nu_\varepsilon(x+\varepsilon\rho\nu(x))\\&=(1+\varepsilon^2|\nabla_t\rho|^2)^{-1/2}\nabla\bar{u}_\varepsilon(x+\varepsilon\rho\nu(x))\cdot(\nu-\varepsilon\nabla_t\rho)\\&=(1+O(\varepsilon^2))(\nabla\bar{u}_\varepsilon(x+\varepsilon\rho\nu(x))\cdot(\nu-\varepsilon\nabla_t\rho)\\&=(1+O(\varepsilon^2))\left(\frac{\partial\bar{u}_\varepsilon}{\partial\nu}(x)+\varepsilon\rho\frac{\partial^2\bar{u}_\varepsilon}{\partial\nu^2}(x)+o(\varepsilon)+\nabla\bar{u}_\varepsilon(x+\varepsilon\rho\nu)\cdot\nabla_t\rho\right)\\&=\frac{\partial\bar{u}_\varepsilon}{\partial\nu}+\varepsilon\rho\frac{\partial^2\bar{u}_\varepsilon}{\partial\nu^2}-\varepsilon\nabla_t u\cdot\nabla_t\rho+o(\varepsilon).\end{aligned}$$

On the other hand $\bar{u}_\varepsilon(x+\varepsilon\rho\nu)=\bar{u}_\varepsilon+\varepsilon\rho(\partial\bar{u}_\varepsilon/\partial\nu)+o(\varepsilon)$. Then, $0=((\partial/\partial\nu_\varepsilon)+k)\bar{u}_\varepsilon(x+\varepsilon\rho\nu)=((\partial/\partial\nu)+k)\bar{u}_\varepsilon-\varepsilon(-k\rho(\partial\bar{u}_\varepsilon/\partial\nu)-\rho(\partial^2/\partial\nu^2)\bar{u}_\varepsilon+\nabla_t u\cdot\nabla_t\rho)+o(\varepsilon)$. Thus, $\|\bar{u}_\varepsilon-u\|_{C^2(\bar{\Omega})}\to 0$ implies our Lemma 7.2. Here it should be noticed that $\|u_\varepsilon-u\|_{C^2(\bar{\Omega})}\leqq C\varepsilon$ does not used here.

LEMMA 7.3. *The equality*

$$\left\|\left(\frac{\partial}{\partial\nu}+k\right)\tilde{u}_\varepsilon\right\|_{C^{2,\alpha}(\partial\Omega)}=O(\varepsilon)$$

holds.

Proof. For $x\in\partial\Omega$, $(\partial/\partial\nu)\tilde{u}_\varepsilon(x)=(\partial/\partial\nu)u_\varepsilon(x+\varepsilon\rho(x)\nu(x))=(\partial/\partial\nu_\varepsilon)u_\varepsilon(x+\varepsilon\rho\nu)+O(\varepsilon)$. Then,

$$0=\left(\frac{\partial}{\partial\nu_\varepsilon}+k\right)u_\varepsilon(x+\varepsilon\rho\nu)=\frac{\partial}{\partial\nu}\tilde{u}_\varepsilon(x)+O(\varepsilon)+k\tilde{u}_\varepsilon(x).$$

We are now in a position to prove Theorem 2, (1.7). By the Green formula and $((\partial/\partial\nu)+k)u|_{\partial\Omega}=0$,

$$\int_\Omega(\Delta u\cdot\bar{u}_\varepsilon-u\Delta\bar{u}_\varepsilon)dx=-\int_{\partial\Omega}u\left(\frac{\partial}{\partial\nu}+k\right)\bar{u}_\varepsilon d\sigma. \tag{7.1}$$

As in the proof of theorem for the ground state value of Dirichlet condition we have

$$(7.1)=\lambda_\varepsilon-\lambda-\varepsilon((p-1)/(p+1))\lambda\int_{\partial\Omega}u^{p+1}\rho d\sigma+o(\varepsilon),$$

On the other hand the right hand side of (7.1) is equal to

$$=\varepsilon\int_{\partial\Omega}\left(ku\frac{\partial u}{\partial\nu}\rho+u\frac{\partial^2 u}{\partial\nu^2}\rho-u\nabla_t u\cdot\nabla_t\rho\right)d\sigma+o(\varepsilon)$$

$$=\varepsilon\int_{\partial\Omega}\left(ku\frac{\partial u}{\partial\nu}+|\nabla_t u|^2+(N-1)kHu^2-\lambda u^{p+1}\right)\rho d\sigma+o(\varepsilon)$$

$$=\varepsilon\int_{\partial\Omega}(|\nabla_t u|^2-\lambda u^{p+1}-(k^2-(N-1)kH)u^2)\rho d\sigma+o(\varepsilon).$$

Here there is the equation as the background of our calculus.

$$0=\int_{\partial\Omega}\nabla_t(u\nabla_t u\rho)d\sigma$$

$$=\int_{\partial\Omega}(|\nabla_t u|^2\rho+u|\nabla_t u|^2\rho+u\nabla_t u\cdot\nabla_t\rho)d\sigma$$

$$=\int_{\partial\Omega}\left(|\nabla_t u|^2\rho+u\left(\Delta u-\frac{\partial^2 u}{\partial\nu^2}-(N-1)H\frac{\partial u}{\partial\nu}\right)\rho+u\nabla_t u\cdot\nabla_t\rho\right)d\sigma$$

$$=\int_{\partial\Omega}(|\nabla_t u|^2+(N-1)kHu^2-\lambda u^{p+1})\rho d\sigma$$

$$-\int_{\partial\Omega}u\left(\frac{\partial^2 u}{\partial\nu^2}\rho-\nabla_t u\cdot\nabla_t\rho\right)d\sigma.$$

Summing up these facts we get the desired result. It should be remarked that the relation $\Delta u=-\lambda u^p$, $\partial u/\partial\nu=-ku$, and $\Delta\varphi=\partial^2\varphi/\partial\nu^2+(N-1)H(\partial\varphi/\partial\nu)+\nabla_t{}^2\varphi$ on $\partial\Omega$, etc was used.

8. Variational formula for the ground state solution.

We have the following Proposition.

PROPOSITION 8.1. *If the ground state solution u is unique and* $\mathrm{Ker}(\Delta+\lambda pu^{p-1})=\{0\}$, *then* $\|\tilde{u}_\varepsilon-u\|_{C^2(\bar{\Omega})}=O(\varepsilon)$ *holds.*

The proof is similar to the Dirichlet case using Lemma 7.3. Thus we omit it.

We prove the following Proposition 8.2.

PROPOSITION 8.2. *Assume that u is unique and* $\mathrm{Ker}(\Delta+\lambda pu^{p-1})=\{0\}$. *Then,*

$$\delta u=-(\delta\lambda/(\lambda(p-1)))u$$

$$+\int_{\partial\Omega}\Gamma\left(-\rho\frac{\partial^2 u}{\partial\nu^2}+\nabla_t u\cdot\nabla_t\rho-k\rho\frac{\partial u}{\partial\nu}\right)d\sigma$$

$$=-(\delta\lambda/(\lambda(p-1)))u$$
$$-\int_{\partial\Omega}(\nabla_t\Gamma\cdot\nabla_t\Gamma-\Gamma(\lambda u^p+(k^2-(N-1)kH)u))d\sigma\,.$$

Proof. By the Green formula with $((\partial/\partial\nu)+k)u|_{\partial\Omega}=0$, $((\partial/\partial\nu)+k)\Gamma|_{\partial\Omega}=0$, we have

$$\bar{u}_\varepsilon-u=-\int_\Omega\Gamma(\Delta+\lambda f'(u))(\bar{u}_\varepsilon-u)dy$$
$$+\int_{\partial\Omega}\Gamma\Big(\frac{\partial}{\partial\nu}+k\Big)\bar{u}_\varepsilon d\sigma_y=-P_1+P_2\,.$$

As in the Dirichlet case $P_1=\varepsilon(\delta\lambda/(\lambda(p-1)))u+o(\varepsilon)$ by $\int_\Omega\Gamma f(u)dx=-u/(\lambda(p-1))$. $P_2=\varepsilon\int_{\partial\Omega}\Big(-k\rho\frac{\partial u}{\partial\nu}-\rho\frac{\partial^2 u}{\partial\nu^2}+\nabla_t u\nabla_t\rho\Big)d\sigma+o(\varepsilon)$ for $x\in\Omega$. Therefore, we get the desired result by Lemma 7.2.

Part III

9. Neumann second state value.

In the section we prove the following Proposition 9.1.

PROPOSITION 9.1. *There exists a constant C independent of ε such that*

$$|\lambda_\varepsilon-\lambda|\leqq C\varepsilon \tag{9.1}$$

holds. Moreover,

$$\|u_\varepsilon\|_{C^{3,\alpha}(\bar{\Omega}_\varepsilon)}\leqq C\,. \tag{9.2}$$

Proof. Proof of (9.2) is in the Appendix. We prove (9.1). Recall that $\hat{u}\in H^1(\Omega_\varepsilon)$. Then,

$$\int_{\Omega_\varepsilon}|\nabla\hat{u}|^2dx=\lambda+O(\varepsilon)$$
$$\int_{\Omega_\varepsilon}|\hat{u}|^{p+1}dx=1+O(\varepsilon)\,.$$

We also have

$$\int_{\Omega_\varepsilon}|\hat{u}|^{p-1}\hat{u}dx=\int_\Omega|u|^{p-1}u|J\Phi_\varepsilon|dx=\int_\Omega|u|^{p-1}udx+O(\varepsilon)=O(\varepsilon)\,.$$

We have the following Claim. The proof of this Claim is rather complicated. So we want to prove Proposition 9.1 using this Claim 9.2.

Claim 9.2. There exists a constant $C_\varepsilon\in R$ such that $v_\varepsilon=u_\varepsilon+C_\varepsilon$ satisfies

$$\int_{\Omega_\varepsilon} |v_\varepsilon|^{p-1} v_\varepsilon dx = 0$$

and $C_\varepsilon = O(\varepsilon)$.

We use this Claim. Then, $v_\varepsilon \in H^1(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 dx = \lambda + O(\varepsilon)$$

$$\int_{\Omega_\varepsilon} |v_\varepsilon|^{p+1} dx = \int_{\Omega_\varepsilon} |u_\varepsilon + C_\varepsilon|^{p+1} = \int_{\Omega_\varepsilon} |u_\varepsilon|^{p+1} dx + O(\varepsilon) = 1 + O(\varepsilon).$$

Therfore, $\lambda_\varepsilon \leqq \lambda + O(\varepsilon)$.

Conversely we get $\lambda \leqq \lambda_\varepsilon + O(\varepsilon)$ using the diffeomorphism $u_\varepsilon \rightleftarrows \tilde{u}_\varepsilon$. Thus, we have the desired result.

Proof of Claim 9.2. We generalize Claim 9.2 as the following statement. *Fix* $\varphi_\varepsilon \in C^0(\bar{\Omega})$ *such that*

(i) $\|\varphi_\varepsilon\|_{C^0(\bar{\Omega})} \leqq C$

(ii) $\|\varphi_\varepsilon\|_{L^{p+1}(\Omega)} \geqq C' > 0$.

(iii) $\int_\Omega |\varphi_\varepsilon|^{p-1} \varphi_\varepsilon dx = O(\varepsilon)$.

Then, there exists unique constant $C_\varepsilon \in R$ *such that*

$$\int_\Omega |\varphi_\varepsilon + C_\varepsilon|^{p-1} (\varphi_\varepsilon + C_\varepsilon) dx = 0 \quad \text{and} \quad C_\varepsilon = O(\varepsilon).$$

Proof. We put $g(t) = |t|^{p-1} t$, then $g'(t) = p|t|^{p-1}$. Thus, for any $x \in \bar{\Omega}$, the function $t \to g(\varphi_\varepsilon(x) + t)$ is strict monotone increasing function. Since $\varphi_\varepsilon \in C^0(\bar{\Omega})$, we have

$$\lim_{t \to \pm\infty} \int_\Omega g(\varphi_\varepsilon(x) + t) dx = \pm\infty$$

The continuity $t \mapsto \int_\Omega g(\varphi_\varepsilon(x) + t) dx$ is easy to see. Therefore, there exists C_ε such that $\int_\Omega g(\varphi_\varepsilon + C_\varepsilon) dx = 0$. We put $F(t) = \int_\Omega g(\varphi_\varepsilon + tC_\varepsilon) dx$. Then, $F'(t) = \int_\Omega C_\varepsilon g'(\varphi_\varepsilon + tC_\varepsilon) dx$. Then, by the mean value theorem there exists $0 < t_\varepsilon < 1$ such that $F(1) - F(0) = F'(t_\varepsilon)$. We know that $F(1) = 0$, $F(0) = O(\varepsilon)$. therefore

$$C_\varepsilon \int_\Omega g'(\varphi_\varepsilon + t_\varepsilon C_\varepsilon) dx = O(\varepsilon).$$

We want to show

$$\int_\Omega g'(\varphi_\varepsilon + t_\varepsilon C_\varepsilon) \geqq C'' > 0. \tag{9.3}$$

We have

$$\int_\Omega |\varphi_\varepsilon + t_\varepsilon C_\varepsilon|^{p-1} dx \geqq \int_{\varphi_\varepsilon > 0} |\varphi_\varepsilon + t_\varepsilon C_\varepsilon|^{p-1} dx$$

$$\geqq \int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p-1} dx, \qquad \text{if} \quad C_\varepsilon>0.$$

If $C_\varepsilon<0$, then

$$\int_\Omega |\varphi_\varepsilon+t_\varepsilon C_\varepsilon|^{p-1} dx \geqq \int_{\varphi_\varepsilon<0} |\varphi_\varepsilon|^{p-1} dx.$$

Therefore,

$$\int_\Omega |\varphi_\varepsilon+t_\varepsilon C_\varepsilon|^{p-1} dx \geqq \min\left(\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p-1} dx, \int_{\varphi_\varepsilon<0} |\varphi_\varepsilon|^{p-1} dx\right).$$

Now we assume that (10.3) does not hold. Then,

$$\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p-1} dx \longrightarrow 0 \quad \text{or} \quad \int_{\varphi_\varepsilon<0} |\varphi_\varepsilon|^{p-1} dx \longrightarrow 0.$$

Without loss of generalities

$$\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p-1} dx \longrightarrow 0.$$

Then, by (i) we have $\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^p dx \to 0$, $\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p+1} dx \to 0$. Therefore,

$$\int_{\varphi_\varepsilon>0} |\varphi_\varepsilon|^{p-1} \varphi_\varepsilon dx \longrightarrow 0.$$

On the other hand $O(\varepsilon)=\int_\Omega |\varphi_\varepsilon|^{p-1}\varphi_\varepsilon dx=\left(\int_{\varphi_\varepsilon>0} - \int_{\varphi_\varepsilon<0}\right)|\varphi_\varepsilon|^{p-1}\varphi_\varepsilon dx$. Therefore,

$$\int_{\varphi_\varepsilon<0} |\varphi_\varepsilon|^p dx \longrightarrow 0$$

and

$$\int_{\varphi_\varepsilon<0} |\varphi_\varepsilon|^{p+1} dx \longrightarrow 0.$$

We have a contradiction by (ii). Now the assertion holds.

10. Variational formula.

We impose the assumption

$$\|\tilde{u}_\varepsilon - u\|_{C^2(\bar{\Omega})} = O(\varepsilon) \tag{10.1}$$

and

$$\mathrm{Ker}(\Delta+\lambda p |u|^{p-1}) = \{0\} \tag{10.2}$$

and assume that the minimizer u is unique up to its signature. Then, we have Theorem 3.

We do not give Theorem 3, since it is a routine work for the readers who read Dirichlet and Robin cases.

Appendix.

We state the regularity theorem in the following manner. This is a consequence of famous Sobolev embedding theorem (Adams [1]), Schauder estimate (Agmon-Douglis-Nirenberg [2]), L^p-estimate (Agmon-Douglis-Nirenberg [2]) and bootstrap argument.

Let λ_ε, u_ε be the Dirichlet (Robin, Neumann) ground state value, solution, respectively. Then, there is a locally bounded function F such that $u_\varepsilon \in C^{3,\alpha}(\bar{\Omega}_\varepsilon)$ and $\|u_\varepsilon\|_{C^{3,\alpha}(\bar{\Omega}_\varepsilon)} \leqq F(\lambda_\varepsilon)$. (If $p<2$ then, $\alpha=p-1$, If $p \geqq 2$, $\alpha \in (0, 1)$ can be taken arbitrary.)

The reader who is unfamiliar with Hadamard's variation may be referred by Hadamard [9], Garabedian [6], Garabedian-Schiffer [7].

Our theorem combined with Ozawa [10] we get a singular variational formula Osawa-Ozawa [11] for nonlinear eigenvalues.

References

[1] R. Adams, Sobolev spaces, Academic Press (1975).

[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. pure Appl. Math., **12** (1959), 623-727.

[3] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Commun. Pure and Appl. Math., **36** (1983), 437-477.

[4] E.N. Dancer, The effect of domain shape on the number of positive solutions of certain nonlinear equations. J. of Diff. Equations., **74** (1988), 120-156.

[5] D. Fujiwara and S. Ozawa, Hadamard's variational formula for the Green functions of some normal elliptic boundary problems, Proc. Japan Acad. **54A**, (1978), 215-220.

[6] P.R. Garabedian, Partial Differential Equations, J. Wiley and Sons. Inc., New York (1964).

[7] P.R. Garabedian and M.M. Schiffer, Convexity of domain functionals, J. Anal. Math. **2** (1978), 281-368.

[8] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Commun. Math. Phys., **68** (1979), 209-243.

[9] J. Hadamard, Mémoire sur le problème d'anlyse relatif à l'équilibre des plaques élastiques encastrées, Oeuvres, C.N.R.S. tom **2** (1968), 515-631.

[10] S. Ozawa, Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains, —the Neumann conditions— Osaka J. Math., **22** (1985), 639-655.

[11] T. Osawa and S. Ozawa, Nonlinear eigenvalue problems and singular variation of domains (to appear in Kodai Math. J.).

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