S. MORI
KODAI MATH. J.
7 (1984), 56—60

AN APPLICATION OF THE LEMMA ON THE LOGARITHMIC DERIVATIVE IN SEVERAL COMPLEX VARIABLES DUE TO A. VITTER

Dedicated to Professor Mitsuru Ozawa on his 60-th birthday

BY SEIKI MORI

1. A. Vitter [6] generalized Nevanlinna's lemma on logarithmic derivative to the case of meromorphic functions $f: \boldsymbol{C}^n \to \boldsymbol{P}^1(\boldsymbol{C}) \cong \boldsymbol{C} \cup \{\infty\}$, and he obtained the defect relation for holomorphic (or meromorphic) mappings $f: \boldsymbol{C}^n \to \boldsymbol{P}^m(\boldsymbol{C})$ and a collectionn of hyperplanes in general position by using the method due to H. Cartan [2]. Furthermore, he obtained the following generalization of a result of H. Milloux (see Hayman [4, p. 55]):

THEOREM A (Vitter [6]). *Let $f: \boldsymbol{C}^n \to \boldsymbol{P}^m(\boldsymbol{C})$ be a meromorphic mapping of rank n $(m \geqq n+1)$ and*

$$F \equiv f \wedge f_{z_1} \wedge f_{z_2} \wedge \cdots \wedge f_{z_n} : \boldsymbol{C}^n \longrightarrow G(n+1, m+1) \hookrightarrow \boldsymbol{P}^{\binom{m+1}{n+1}-1}(\boldsymbol{C})$$

the tangent mapping of f. Then

$$T_F(r) \leqq (n+1) T_f(r) + O(\log r \cdot T_f(r)) \qquad //,$$

where $G(n+1, m+1)$ denotes the Grassmanian manifold, $T_g(r)$ the Nevanlinna's characteristic function of a meromorphic mapping g into a projective space and the notation "//" means that the stated inequality holds outside exceptional intervals.

In this note, we give an application of Theorem A as follows:

Let V be a smooth hypersurface of degree d in the projective space $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. Then if $d > n+2$, any holomorphic mapping $f: \boldsymbol{C}^n \to V$ must be degenerate in the sense that its Jacobian determinant vanishes identically.

Green [3] showed this theorem for $n=1$. (See also Carlson-Griffiths [1] and Kodaira [5].

We give another proof than that of Carlson-Griffiths and Kodaira, but they showed results under more general situations, e.g., V is an n-dimensional projective algebraic manifold of general type.

Received March 9, 1983

2. Notation and Terminology.

Let $z=(z_1, \cdots, z_n)\in \boldsymbol{C}^n$ be the natural coordinate system in $\boldsymbol{C}^n$. We set $\|z\|^2=\sum_{j=1}^{n} z_j\bar{z}_j$, $\phi=dd^c\log\|z\|^2$, $\phi_k=\phi\wedge\cdots\wedge\phi$ (k-times), $\sigma=d^c\log\|z\|^2\wedge\phi_{n-1}$, $B(r)=\{z\in \boldsymbol{C}^n \mid \|z\|<r\}$, $\partial B(r)=\{z\in \boldsymbol{C}^n \mid \|z\|=r\}$ and $d^c=\frac{\sqrt{-1}}{4\pi}(\bar{\partial}-\partial)$.

Let L be a positive line bundle over $\boldsymbol{P}^m(\boldsymbol{C})$ and ω the curvature form for the line bundle L. For a holomorphic mapping $f: \boldsymbol{C}^n\to \boldsymbol{P}^m(\boldsymbol{C})$, we set

$$T_f(L, r)\equiv\int_0^r\frac{dt}{t}\int_{B(t)}f^*\omega\wedge\phi_{n-1},$$

where $f^*\omega$ denotes the pullback of the form ω under f. The function $T_f(L, r)$ is called the Nevanlinna's characteristic function of f relative to L. In particular, for the hyperplane bundle $[H]$ over $\boldsymbol{P}^m(\boldsymbol{C})$ and the curvature form $\omega_0=dd^c\log\|w\|^2$ ($w\in \boldsymbol{C}^{m+1}-\{0\}$) of the canonical hermitian metric in $[H]$, we use the notation $T_f(r)$ instead of $T_f([H], r)$ for simplicity.

We note that for a holomorphic mapping $f: \boldsymbol{C}^n\to V\hookrightarrow \boldsymbol{P}^m(\boldsymbol{C})$ with reduced representation $f=(f_0, \cdots, f_m)$, its characteristic function $T_f(r)$ is written in the form

$$T_f(r)=\int_{\partial B(r)}\log\sum_{j=0}^{m}|f_j|^2\sigma+O(1).$$

3. Applying Vitter's theorem, we show the following.

THEOREM. *Let $V\subset \boldsymbol{P}^{n+1}(\boldsymbol{C})$ be an n-dimensional smooth hypersurface given by a homogeneous polynomial $P(w)$ of degree d and $dP(w)\neq 0$ in $w\in \boldsymbol{C}^{n+2}-\{0\}$, and $f: \boldsymbol{C}^n\to V$ a holomorphic mapping. Then if $d>n+2$, f must be degenerate in sense that its Jacobian determiant J_f vanishes identically.*

Proof. By definition of V, $P(f(z))=0$ on $\boldsymbol{C}^n$. Hence differentiating this equation in z_j ($j=1, \cdots, n$), we have

$$\sum_{i=0}^{n+1}P_i\cdot f_i(z)_{z_j}=0 \qquad (j=1, \cdots, n), \tag{1}$$

where $P_i=\frac{dP}{dw_i}$. On the other hand, we have by Euler's formula

$$\sum_{i=0}^{n+1}w_iP_i=0 \qquad \text{on} \quad V,$$

so that

$$\sum_{i=0}^{n+1}f_i(z)\cdot P_i(f(z))=0. \tag{2}$$

Suppose that $J_f\not\equiv 0$. Then there is at most one j with $\Delta_j\not\equiv 0$, where

$$\Delta_j = \begin{array}{c} \quad\quad (\hat{j}) \\ \left| \begin{array}{c} f_0 \cdots\cdots f_j \cdots\cdots f_{n+1} \\ (f_0)_{z_1} \cdots (f_j)_{z_1} \cdots (f_{n+1})_{z_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (f_0)_{z_n} \cdots (f_j)_{z_n} \cdots (f_{n+1})_{z_n} \end{array} \right| \end{array}.$$

Solving the equations (1) and (2) in $P_i(f(z))$ $(i=0, 1, \cdots, n+1)$, we have the ratio of $P_0 : \cdots : P_{n+1}$;

$$(3) \qquad (P_0 : \cdots : P_{n+1}) = \left(\frac{\Delta_{0j}}{\Delta_j} P_j : \cdots : \frac{\Delta_{n+1\ j}}{\Delta_j} P_j\right)$$

for j with $\Delta_j \not\equiv 0$, where

$$\Delta_{kj} = (-1) \begin{array}{c} \quad\quad (k) \quad\quad (\hat{j}) \\ \left| \begin{array}{c} f_0 \cdots\cdots f_j \cdots\cdots f_j \cdots\cdots f_{n+1} \\ (f_0)_{z_1} \cdots (f_j)_{z_1} \cdots (f_j)_{z_1} \cdots (f_{n+1})_{z_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (f_0)_{z_n} \cdots (f_j)_{z_n} \cdots (f_j)_{z_n} \cdots (f_{n+1})_{z_n} \end{array} \right| \end{array}.$$

Here "$\wedge$" over (j) means that this column vector is to be deleted. We now denote by G the left hand side and by Φ the right hand side of (3). Since the functions P_i $(i=0, \cdots, n+1)$ have not common zero, $h \equiv \sum_{i=0}^{n+1} |P_i|^2$ can be considered as a metric in $[H]^{d-1}|_V$ which is the restriction over V of the $(d-1)$-th symmetric tensor power $[H]^{d-1}$ of the hyperplane bundle $[H]$ over $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. On the other hand, $\tilde{h} \equiv \sum_{i=0}^{n+1} |w_i^{d-1}|^2$ is also a metric in $[H]^{d-1}|_V$. Thus we have

$$\begin{aligned} T_{G(f)}(r) &= \int_{\partial B(r)} \log \sum_{i=0}^{n+1} |P_i(f)|^2 \sigma + O(1) \\ &= T_f([H]^{d-1}|_V, r) + O(1) \\ &= \int_{\partial B(r)} \log \sum_{i=0}^{n+1} |f_i^{d-1}|^2 + O(1) \\ &= \int_{\partial B(r)} \max_i \log |f_i^{d-1}|^2 + O(1) \\ &= (d-1) \cdot T_f(r) + O(1). \end{aligned}$$

By definition we have

$$\Phi \equiv \left(\Delta_{0j} \frac{P_j}{\Delta_j} : \cdots : \Delta_{n+1\ j} \frac{P_j}{\Delta_j}\right) = (\Delta_{0j} : \cdots : \Delta_{n+1\ j})$$

on $\boldsymbol{C}^n - E$, where $E = \{z \in \boldsymbol{C}^n | \Delta_j(z) = 0\}$. Let δ be the common factor of $\Delta_{0j}, \cdots, \Delta_{n+1\ j}$ such that $F = \left(\frac{\Delta_{0j}}{\delta} : \cdots : \frac{\Delta_{n+1\ j}}{\delta}\right)$ is a reduced representation of Φ. Note that F is the tangent mapping

$$F=f\wedge f_{z_1}\wedge\cdots\wedge f_{z_n}: \boldsymbol{C}^n \longrightarrow G(n+1,\ n+2) \hookrightarrow \boldsymbol{P}^{\binom{n+2}{n+1}-1}(\boldsymbol{C})=\boldsymbol{P}^{n+1}(\boldsymbol{C}).$$

Hence we have by Theorem A

$$(4)\qquad \begin{aligned} T_{G(f)}(r)&=T_F(r)=(d-1)T_f(r)+O(1)\\ &\leqq(n+1)T_f(r)+O(\log r\cdot T_f(r)) \qquad //. \end{aligned}$$

Thus if f is not rational and if $d>n+2$, the inequality (4) yields a contradiction for sufficiently large values of r. Thus we see $\Delta_j\equiv 0$ for all j. Therefore the Jacobian determinant of f vanishes identically.

Suppose that f is a rational mapping. Then f can be written in the form $f=(1,\ Q_1/R_1,\ \cdots,\ Q_{n+1}/R_{n+1})$ where Q_j and R_j are polynomials. We now consider a generic complex line $\boldsymbol{C}_\xi$ in $\boldsymbol{C}^n$, where ξ is a direction vector in $\boldsymbol{C}^n$ and also we write a parameter in $\boldsymbol{C}_\xi$ as ξ for simplicity. Then the restriction $f|_{\boldsymbol{C}_\xi}$ of f to $\boldsymbol{C}_\xi$ can be written in the form $f=(q_0(\xi):\cdots:q_{n+1}(\xi))$, where $q_0,\ \cdots,\ q_{n+1}$ are polynomials in ξ without common zero. Let k_i be the degree of the polynomial q_i and k the maximum number among k_j's. Then we have

$$\begin{aligned} T_{\hat{f}}(r)&=\frac{1}{4\pi}\int_0^{2\pi}\log\sum_{i=0}^{n+1}|q_i(re^{i\theta})|^2d\theta+O(1)\\ &=k\cdot\log r+O(1), \end{aligned}$$

and hence

$$T_{\hat{G}(f)}(r)=(d-1)\cdot T_{\hat{f}}(r)+O(1)=(d-1)k\cdot T_{\hat{f}}(r)+O(1),$$

where $\hat{g}$ denotes the restriction of the function g to $\boldsymbol{C}_\xi$. On the other hand from the right hand side of (3), we obtain

$$\begin{aligned} T_{\hat{F}}(r)&=\frac{1}{4\pi}\int_0^{2\pi}\log\sum_{h=0}^{n+1}|\hat{\Delta}_{hj}(re^{i\theta})|^2d\theta-N((\hat{\delta})_0,\ r)+O(1)\\ &\leqq\frac{1}{4\pi}\int_0^{2\pi}\log\sum_{h=0}^{n+1}|\sum_{(\alpha_i)}\prod_{\substack{i=0\\ i\neq k}}^{n+1}\hat{f}_i(re^{i\theta})_{z_{\alpha_i}}|^2d\theta+O(1)\\ &\leqq\{k(n+1)-n\}\log r+O(1), \end{aligned}$$

for the degree of $\prod_{\substack{i=0\\ i\neq k}}^{n+1}\hat{f}_i(z)_{z_{\alpha_i}}$ is not greater than $k(n+1)-n$. Thus we have

$$(d-1)k\cdot\log r+O(1)\leqq\{k(n+1)-n\}\cdot\log r+O(1),$$

and hence

$$\{k(d-n-2)+n\}\cdot\log r\leqq O(1).$$

Hence this gives a contradiction if $d\geqq n+2$. Thus we have $\Delta_j\equiv 0$ for all j. Therefore f is degenerate in the sense that $J_f\equiv 0$.

References

[1] J. Carlson and P. A. Griffiths, A defect relation for equidimensional holomorphic mappings between algebraic varieties, Ann. of Math., **95** (1972), 557-584.

[2] H. Cartan, Sur les zéros des combinaisons linéares de p fonctions holomorphes données, Mathematica, **7** (1933), 5-31.

[3] M. L. Green, Some Picard theorems for holomorphic maps to algebraic varieties, Amer. J. Math., **97** (1975), 43-75.

[4] W. K. Hayman, Meromorphic Functions, Oxford Math. Monographs, Oxford University Press, London (1964).

[5] K. Kodaira, Holomorphic mappings of polydiscs into compact complex manifolds, J. Differential Geometry, **6** (1971), 33-46.

[6] A. Vitter, The lemma of the logarithmic derivative in several complex variables, Duke Math. J., **40** (1977), 89-104.

Department of Mathematics
Yamagata University
Yamagata, Japan