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AN EXTREMAL PROBLEM ON THE CLASSICAL CARTAN DOMAINS, III

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1. Let $D_1, \cdots, D_N$ be the classical Cartan domains. We define the numbers n_{D_v} and λ_{D_v} as follows:

$$n_{D_v}=\begin{cases} rs \quad , & \text{if } D_v=R_{\mathrm{I}}(r, s), \\ \dfrac{p(p+1)}{2}, & \text{if } D_v=\hat{R}_{\mathrm{II}}(p), \\ \dfrac{q(q-1)}{2}, & \text{if } D_v=R_{\mathrm{III}}(q), \\ m \quad , & \text{if } D_v=R_{\mathrm{IV}}(m), \end{cases}$$

and

$$\lambda_{D_v}=\begin{cases} \sqrt{s} \quad , & \text{if } D_v=R_{\mathrm{I}}(r, s), \\ \sqrt{\dfrac{p+1}{2}}, & \text{if } D_v=\hat{R}_{\mathrm{II}}(p), \\ \sqrt{q-1} \quad , & \text{if } D_v=R_{\mathrm{III}}(q) \text{ and } q \text{ is even}, \\ \sqrt{q} \quad , & \text{if } D_v=R_{\mathrm{III}}(q) \text{ and } q \text{ is odd}, \\ \sqrt{m} \quad , & \text{if } D_v=R_{\mathrm{IV}}(m), \end{cases}$$

where

$$R_{\mathrm{I}}(r, s)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0, \text{ where } Z \text{ is an } r\times s \text{ matrix}\}, \quad (r\leqq s),$$

$$R_{\mathrm{II}}(p)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0, \text{ where } Z \text{ is a symmetric matrix of order } p\},$$

$$\hat{R}_{\mathrm{II}}(p)=\{Z=(z_{ij}):\ z_{ij}=\sqrt{2}\,x_{ij}\ (i\neq j),\ z_{ii}=x_{ii}, \text{ where } X=(x_{ij})\in R_{\mathrm{II}}(p)\},$$

$$R_{\mathrm{III}}(q)=\{Z=(z_{ij}):\ I-Z\bar{Z}'>0, \text{ where } Z \text{ is a skew-symmetric matrix of order } q\},$$

$$R_{\mathrm{IV}}(m)=\{z=(z_1, \cdots, z_m):\ 1+|zz'|^2-2z\bar{z}'>0,\ 1-|zz'|>0\}.$$

We set

$$D=\lambda_{D_1}D_1\times\cdots\times\lambda_{D_N}D_N, \qquad n=n_{D_1}+\cdots+n_{D_N},$$

and denote the family of holomorphic mappings from D into the unit hyperball

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B_n in $\boldsymbol{C}^n$ by $\mathcal{F}(D)$. In [5] we proved that

$$\sup_{f\in\mathcal{F}(D)}\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=n^{-n/2}, \tag{1}$$

where $(\partial f/\partial z)$ is the Jacobian matrix of f:

$$\left(\frac{\partial f}{\partial z}\right)=\begin{pmatrix}\dfrac{\partial f_1}{\partial z_1}\cdots\dfrac{\partial f_1}{\partial z_n}\\ \cdots\cdots\cdots\cdots\\ \dfrac{\partial f_n}{\partial z_1}\cdots\dfrac{\partial f_n}{\partial z_n}\end{pmatrix},\qquad f=(f_1,\cdots,f_n).$$

In this paper we shall prove that $f_0(z)=z/\sqrt{n}$ is the unique extremal mapping, up to unitary transformations:

THEOREM. *If f is a mapping in $\mathcal{F}(D)$ such that*

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=n^{-n/2},$$

then $\sqrt{n}\,f$ is a unitary transformation of $\boldsymbol{C}^n$.

2. We shall prove the theorem for the case that $N=2$, $D_1=\hat{R}_{\mathrm{II}}(p)$ (p is odd) and $D_2=R_{\mathrm{IV}}(m)$ ($m\geqq 3$). The same argument as in [5] gives the proof of the general case.

Firstly we give an improved proof of (1). Instead of $R_{\mathrm{IV}}(m)$ we consider the following domain

$$R^*_{\mathrm{IV}}(m)=\{z=(z_1,\cdots,z_m):\ 1+|2z_1z_2+z_3^2+\cdots+z_m^2|^2-2z\bar{z}'>0,$$
$$1-|2z_1z_2+z_3^2+\cdots+z_m^2|>0\},$$

which is the image of $R_{\mathrm{IV}}(m)$ under the unitary transformation

$$(z_1,\cdots,z_m)\to\left(\frac{1}{\sqrt{2}}(z_1+iz_2),\ \frac{1}{\sqrt{2}}(z_1-iz_2),\ z_3,\cdots,z_m\right).$$

Now we consider the domain

$$D=\sqrt{\frac{p+1}{2}}\,\hat{R}_{\mathrm{II}}(p)\times\sqrt{m}\,R^*_{\mathrm{IV}}(m),\qquad n=\frac{p(p+1)}{2}+m.$$

We represent the points z in D in the form of vectors in $\boldsymbol{C}^n$

$$z=(x,y),\quad x=(x_{11},\cdots,x_{1p},x_{22},\cdots,x_{2p},\cdots,x_{pp}),\quad y=(y_1,\cdots,y_m).$$

Let f be a mapping in $\mathcal{F}(D)$. We may assume that $f(0)=0$ (see [4]). We set

$$f=(f_1,\cdots,f_n).$$

We denote by σ a one-to-one mapping from $\{1,\cdots,p\}$ onto itself such that

$\sigma(i_0)=\imath_0$ for a certain $\imath_0$ and $\sigma(i)\neq\imath$, $\sigma\circ\sigma(i)=\imath$ for $i\neq i_0$, and denote by τ a mapping from $\{1\}$ into $\{3, \cdots, m\}$. We take a point $w=(u, v)$, where $u=(u_{11}, \cdots, u_{1p}, u_{22}, \cdots, u_{2p}, \cdots, u_{pp})$ is a point such that

$$u_{\imath\jmath}=\begin{cases}\sqrt{p+1}\,\zeta_k & (i=\imath_k,\ j=\sigma(\imath_k)),\\ \sqrt{\dfrac{p+1}{2}}\zeta_{t+1} & (\imath=j=\imath_0),\\ 0 & (\text{otherwise}),\end{cases}$$

$$(p=2t+1,\quad 1\leqq i_1<\cdots<i_t<p,\quad i_k<\sigma(i_k)),$$

or

$$u_{\imath\jmath}=\begin{cases}\sqrt{\dfrac{p+1}{2}}\zeta_\imath & (i=j),\\ 0 & (\text{otherwise}).\end{cases}$$

and $v=(v_1, \cdots, v_m)$ is a point such that

$$v_i=\begin{cases}\sqrt{m}\,\xi_1 & (i=\tau(1)),\\ 0 & (i\neq\tau(1)),\end{cases}$$

or

$$v_i=\begin{cases}\sqrt{\dfrac{m}{2}}\xi_\imath & (i=1, 2),\\ 0 & (i\geqq 3).\end{cases}$$

If $\zeta_\imath$ and $\xi_\imath$ are complex numbers with $|\zeta_\imath|<1$ and $|\xi_\imath|<1$, the point w belongs to D. Hence $f_l(w)$ has an expansion

$$f_l(w)=\sum c^{(l)}_{v_1\cdots v_\alpha\mu_1\mu_\beta}\zeta_1^{\nu_1}\cdots\zeta_\alpha^{\nu_\alpha}\xi_1^{\mu_1}\xi_\beta^{\mu_\beta},\quad c^{(l)}_{0\cdots 0}=0$$

which converges uniformly on every compact subset of the polydisc $\varDelta=\{(\zeta_1, \cdots, \zeta_\alpha, \xi_1, \xi_\beta):|\zeta_\imath|<1, |\xi_\imath|<1\}$, where $\alpha=t+1$ or p, and $\beta=1$ or 2. We set

$$\zeta_\jmath=\rho e^{i\theta_\jmath},\quad \xi_\jmath=\rho e^{i\theta'_\jmath}\qquad (0<\rho<1,\ 0\leqq\theta_\jmath\leqq 2\pi,\ 0\leqq\theta'_\jmath\leqq 2\pi),$$

then we have

$$1\geqq\frac{1}{(2\pi)^{\alpha+\beta}}\int_0^{2\pi}\cdots\int_0^{2\pi}\left[\sum_{l=1}^n|f_l(w)|^2\right]d\theta_1\cdots d\theta_\alpha d\theta'_1 d\theta'_\beta$$

$$=\sum_{l=1}^n\sum_{v_i,\mu_\jmath}|c^{(l)}_{v_1\cdots v_\alpha\mu_1\mu_\beta}|^2\rho^{2(v_1+\cdots+v_\alpha+\mu_1+\mu_\beta)}.$$

Letting $\rho\nearrow 1$ we have

$$\sum_{l=1}^n\sum_{v_i,\mu_\jmath}|c^{(l)}_{v_1\cdots v_\alpha\mu_1\mu_\beta}|^2\leqq 1. \tag{2}$$

To obtain (1) from (2) we set

$$\frac{\partial f_l}{\partial x_{ij}}(0)=a_{ij}^{(l)}, \qquad \frac{\partial f_l}{\partial y_i}(0)=b_i^{(l)}$$

and define the numbers A and B as follows:

$$A=\frac{p+1}{2}\sum_{l=1}^{n}\sum_{i=1}^{p}|a_{ii}^{(l)}|^2,$$

or

$$A=\frac{p+1}{2}\sum_{l=1}^{n}\left[|a_{i_0i_0}^{(l)}|^2+2\sum_{k=1}^{t}|a_{i_k\sigma(i_k)}^{(l)}|^2\right]$$

where $p=2t+1$, $1\leqq i_1<\cdots<i_t<p$, $i_k<\sigma(i_k)$, and

$$B=\frac{m}{2}\sum_{l=1}^{n}(|b_1^{(l)}|^2+|b_2^{(l)}|^2) \quad \text{or} \quad m\sum_{l=1}^{n}|b_{\tau(1)}^{(l)}|^2.$$

Since

$$A+B\leqq\sum_{l=1}^{n}\sum_{v_i,\mu_j}|c_{v_1\cdots v_\alpha\mu_1\mu_\beta}^{(l)}|^2,$$

we obtain

$$A+B\leqq 1. \tag{3}$$

From this inequality we can prove

$$\sum_{l=1}^{n}\left[\sum_{i\leqq j}|a_{ij}^{(l)}|^2+\sum_{i=1}^{m}|b_i^{(l)}|^2\right]\leqq 1. \tag{4}$$

Indeed, by taking appropriate p mappings σ, we have, from (3),

$$\frac{p+1}{2}\sum_{l=1}^{n}\left[\sum_{i=1}^{p}|a_{ii}^{(l)}|^2+2\sum_{i<j}|a_{ij}^{(l)}|^2\right]+pB\leqq p.$$

Further from (3) we have

$$\frac{p+1}{2}\sum_{l=1}^{n}\sum_{i=1}^{p}|a_{ii}^{(l)}|^2+B\leqq 1.$$

Adding these two inequalities, we have

$$\sum_{l=1}^{n}\sum_{i\leqq j}|a_{ij}^{(l)}|^2+B\leqq 1. \tag{5}$$

Next, by taking the $m-2$ mappings τ, we have from (5)

$$(m-2)\sum_{l=1}^{n}\sum_{i\leqq j}|a_{ij}^{(l)}|^2+m\sum_{l=1}^{n}\sum_{i=3}^{m}|b_i^{(l)}|^2\leqq m-2.$$

Further from (5) we have

$$2\sum_{l=1}^{n}\sum_{i\leqq j}|a_{ij}^{(l)}|^2+m\sum_{l=1}^{n}\sum_{i=1}^{2}|b_i^{(l)}|^2\leqq 2.$$

Adding these two inequalities we obtain (4).

Let $\lambda_1, \cdots, \lambda_n$ be the characteristic values of $C\bar{C}'$, where $C=(\partial f/\partial z)_{z=0}$. Since $\lambda_1, \cdots, \lambda_n$ are non-negative, we have

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|^2=|\det C\bar{C}'|=\lambda_1\cdots\lambda_n\leqq\left(\frac{\lambda_1+\cdots+\lambda_n}{n}\right)^n$$

$$=\left[\frac{1}{n}\sum_{l=1}^{n}\left(\sum_{i\leqq j}|a_{ij}^{(l)}|^2+\sum_{i=1}^{m}|b_i^{(l)}|^2\right)\right]^n\leqq\left(\frac{1}{n}\right)^n.$$

Thus we conclude that

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|\leqq n^{-n/2}.$$

3. EXTREMAL MAPPINGS. Let f be a mapping in $\mathcal{F}(D)$ such that

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|=n^{-n/2}.$$

Then f must satisfy the conditions: (i) $f(0)=0$ (see [4]), (ii) $\lambda_1=\cdots=\lambda_n=\dfrac{1}{n}$, hence $C=\dfrac{1}{\sqrt{n}}U$, where U is a unitary matrix of order n, (iii) $c_{v_1\cdots v_\alpha\mu_1\mu_\beta}^{(l)}=0$ for $v_1+\cdots+v_\alpha+\mu_1+\mu_\beta\geqq 2$, $l=1, \cdots, n$, where $c_{v_1\cdots v_\alpha\mu_1\mu_\beta}^{(l)}$ is the term in (2).

Let $\tilde{z}=(\tilde{x}, \tilde{y})$ be a point in D. There is an automorphism φ_1 of $\sqrt{\dfrac{p+1}{2}}\hat{R}_{\mathrm{II}}(p)$ having the properties: (a) $\varphi_1(\tilde{x})=\tilde{u}=(u_{11}, \cdots, u_{1p}, u_{22}, \cdots, u_{2p}, \cdots, u_{pp})$, where $u_{ij}=0$ for $i\neq j$, (b) $\|\varphi_1(x)\|=\|x\|$ for $x\in\sqrt{\dfrac{p+1}{2}}\hat{R}_{\mathrm{II}}(p)$, where $\|x\|$ is the Euclidean norm for $x\in\boldsymbol{C}^{n_1}$, $n_1=\dfrac{p(p+1)}{2}$. The property (b) implies that the restriction of φ_1 to ρB_{n_1} is an automorphism of ρB_{n_1} that fixes 0, where ρ is sufficiently small. Hence φ_1 is a unitary transformation of $\boldsymbol{C}^{n_1}$. Further there is an automorphism φ_2 of $\sqrt{m}\,R_{\mathrm{IV}}^*(m)$ having the properties: (a) $\varphi_2(\tilde{y})=\tilde{v}=(\xi_1, \xi_2, 0, \cdots, 0)$, (b) φ_2 is a unitary transformation of $\boldsymbol{C}^m$. We denote by φ the mapping $(x, y)\to(\varphi_1(x), \varphi_2(y))$. Then φ is an automorphism of D with $\varphi(\tilde{z})=\tilde{w}=(\tilde{u}, \tilde{v})$ and a unitary transformation of $\boldsymbol{C}^n$.

Since $h=f\circ\varphi^{-1}$ is also an extremal mapping, by (ii) we have

$$\left(\frac{\partial h}{\partial z}\right)_{z=0}=\frac{1}{\sqrt{n}}V,$$

and by (iii)

$$h(\tilde{w})=\frac{1}{\sqrt{n}}\tilde{w}V',$$

where V is a unitary matrix. Thus we obtain

$$\|f(\tilde{z})\|=\|h(\tilde{w})\|=\frac{1}{\sqrt{n}}\|\tilde{w}\|=\frac{1}{\sqrt{n}}\|\tilde{z}\|.$$

Hence $\|\sqrt{n} f(z)\|=\|z\|$ for all $z\in D$, where $\|z\|$ is the Euclidean norm for $z\in \boldsymbol{C}^n$. This implies that the restriction of $\sqrt{n} f$ to ρB_n is a holomorphic mapping from ρB_n into itself such that $\left(\frac{\partial\sqrt{n} f}{\partial z}\right)_{z=0}$ is unitary, where ρ is sufficiently small. Therefore we conclude that $\sqrt{n} f$ is a unitary transformation of $\boldsymbol{C}^n$ (see Theorem 8.1.3 in [6]).

4. If B_n is replaced by the unit polydisc U^n in our extremal problem, the extremal mappings need not be unique.

Actually, for every holomorphic mapping f from B_2 into U^2, the inequality

$$\left|\det\left(\frac{\partial f}{\partial z}\right)_{z=0}\right|\leqq 1$$

holds [1], and equality holds not only for $f_0(z)=z$ but also for the mapping $(z_1, z_2)\to\left(z_1+\frac{1}{2}z_2^2,\ z_2+\frac{1}{2}z_1^2\right)$.

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