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KODAI MATH. J.
24 (2001), 393–420

RESIDUAL DISCRIMINANT AND RESIDUAL BIFURCATION LOCUS OF A FUNCTION GERM SINGULAR ALONG AN EULER FREE DIVISOR

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We are interested in a complex analytic function germ f defined in $(\boldsymbol{C}^n, 0)$ singular along a complete intersection germ $(H, 0)$ of positive dimension, and with singularities more complicated than those of H only at the origin of $\boldsymbol{C}^n$ (that is, f is of finite relative codimension on the right along H). We know ([Pe1]) that such a function germ f admits a mini-versal unfolding F. We would like to have informations about the objects which can control the analytic types of deformed function germs arising from f with singularities along H.

In the classical setting of isolated complete intersection singularities, it is well known that the discriminant of its mini-versal unfolding is a free divisor (see [Sa1] for hypersurfaces and [Lo] in the general case) and that the logarithmic stratification of this discriminant controls the analytic types of the deformations arising from f (see [Wi], [Sa1] for hypersurfaces and [Ti] in the general case). It is also known that the vector fields tangent to this discriminant are liftable and we know how to produce a basis of such tangent vector fields (see [Sa2], [Bru1] or [Ter1] for hypersurfaces and [Go] for the general case). The same kind of results are also true for the bifurcation set of the mini-versal unfolding (see [Ter2] and [Bru2] and [Go]).

Despite the existence of the right objects for these questions in the non-isolated case, that is, RC_F the residual critical locus, RD_F the residual discriminant and RB_F the residual bifurcation locus introduced by Pellikaan [Pe2], as far as the author knows, there are no such results in the case of hypersurface germ of finite codimension on the right along a variety of positive dimension.

Such theorems for any complete intersection germ H (CI for short) of positive dimension, as given singular set, are not easy to find.

In the first section we recall some notations and basic facts about non-isolated hypersurfaces singularities and state a theorem which enables to compute an equation for the residual discriminant when this last one is a hypersurface. In the second section, the main part of this paper, we suppose that H is a hypersurface and we show that such nonisolated singularities are closely related in

1991 *Mathematics Subject Classification*: Primary 32S05, 32S30, 58K60; Secondary 32B10, 32S60, 58K40.

Received October 2, 2000; revised May 1, 2001.

some way with function germs with isolated singularities on H (see [BR]). We suppose that H is a free divisor and is Euler (see Section 2.3 or [Al1, p. 2]). With a ringed structure over the residual critical locus RC_F different from that given by Pellikaan [Pe2] and under the hypothesis that $\dim RC_G \cap (H \times \boldsymbol{C}^{p-1}) = p-2$ (where G is the truncated mini-versal unfolding associated with F and p is the dimension of the parameter space of the mini-versal unfolding F), we show that it can be endowed with a reduced complete intersection structure of dimension p. By means of Fitting ideals, the residual discriminant RD_F becomes a reduced free divisor. Moreover we show that the vector fields tangent to RD_F are liftable by $\tilde{F}$ (the unfolding mapping associated with F). The same results are also true for the residual bifurcation locus RB_F.

In the last section, thanks to the lists of simple singularities given by Zaharia ([Za1], [Za2]) and when H is a smooth space of codimension 2, we exhibit a family of finite relative codimension on the right along H (which is not a suspension of any function germ singular along a smooth hypersurface) such that the residual discriminant of any function germ of this kind is a complete intersection of codimension 2.

1. Residual critical locus and residual discriminant, first facts

1.1. Introduction

Let $x := (x_1, \dots, x_n)$ denote coordinates at the origin of $\boldsymbol{C}^n$ and y a coordinate at $0 \in \boldsymbol{C}$. Let $\mathcal{O}_x$ denote the local $\boldsymbol{C}$-algebra of holomorphic function germs at the origin of $\boldsymbol{C}^n$. Let I be the reduced ideal of a complete intersection germ $(H, 0) \subset (\boldsymbol{C}^n, 0)$. We recall that $\mathcal{R}_I$ is the subgroup of the automorphisms of $\mathcal{O}_x$ which preserve the ideal I and $\mathcal{R}_{I,e}$ is the subgroup of the biholomorphisms which preserve the ideal I (the target may move), and they act naturally on $\mathcal{O}_x$ and I. We also recall that $\mathcal{K}_{I,e}$ denotes the subgroup of the extended contact group $\mathcal{K}_e$ which preserves the ideal I.

If z denotes any system of coordinates at a point 0 of a smooth space (germ) $(\boldsymbol{C}^l, 0)$ $(l \geq 1)$, we denote by Θ_z the $\mathcal{O}_z$-free module of rank l of smooth vector field germs on $(\boldsymbol{C}^l, 0)$. If X is an analytic set germ in $(\boldsymbol{C}^l, 0)$ defined by a reduced ideal J, then we denote by Θ_X or $\Theta_{J,e}$ the $\mathcal{O}_z$-module of the vector fields tangent to X, see [Pe1].

Let $f \in I^2$, which is equal to the primitive ideal associated with I, i.e., the ideal of the holomorphic function germs vanishing over H and with their first derivatives vanishing over H (see [Pe1]). We assume that f is $\mathcal{R}_I$-finitely determined (or of finite relative codimension on the right along H or just of finite relative codimension along H when the context is clear). This means that the complex vector space $I^2/T_e\mathcal{R}_I(f)$ is finite dimensional, with $T_e\mathcal{R}_I(f) := \{\xi \cdot f : \xi \in \Theta_{I,e}\}$.

Remark 1.1. Let f be of finite relative codimension along H, and let us denote by (df) the Jacobian ideal of f. Then $V(T_e\mathcal{R}_I(f)) = V((df))$.

Let $F(x,u) := f(x) + \sum_{k=1}^{p} u_k e_k(x)$ be a mini-versal unfolding of f, where p is the dimension of the complex vector space $I^2/T_e\mathscr{R}_I(f)$ and $\{e_1,\dots,e_p\}$ is a basis of this vector space. Let us denote by $\tilde{I} := I \otimes_{\mathcal{O}_x} \mathcal{O}_{x,u}$ and by $\tilde{\Theta}_{I,e} := \Theta_{I,e} \otimes_{\mathcal{O}_x} \mathcal{O}_{x,u}$ the $\mathcal{O}_{x,u}$-module of the vertical vector fields tangent to $(H \times \boldsymbol{C}^p, 0)$ (in the following the parameter space $(\boldsymbol{C}^p, 0)$ can be seen as the horizontal space) and by $\tilde{T}_e\mathscr{R}_I(f)$ the ideal in $\mathcal{O}_{x,u}$ generated by the $\tilde{\xi} \cdot F$'s, where $\tilde{\xi} \in \tilde{\Theta}_{I,e}$. Then by the Malgrange-Weierstrass Preparation Theorem, the $\mathcal{O}_{x,u}$-module $\tilde{I}^2/\tilde{T}_e\mathscr{R}_I(f)$ is an $\mathcal{O}_u$-module finitely generated by $e_1,\dots,e_p$. Let $f_u(x) := F(x,u)$. We want to investigate where the f_u's present at the origin some singularities more complicated than the singularities of H. For this purpose we introduce the following definitions.

Definition 1.1 ([Pe2, Section 2] and [Ji, Chapter 2.2]). The support in $(\boldsymbol{C}^n \times \boldsymbol{C}^p, 0)$ of the analytic coherent sheaf generated by $\tilde{I}^2/\tilde{T}_e\mathscr{R}_I(f)$ is denoted by RC_F and is called the residual critical locus of F. We call $\tilde{I}^2/\tilde{T}_e\mathscr{R}_I(f)$ the residual Jacobian module of F and denote it by M_{RC_F}. The image RD_F of RC_F by $F \times id_{(\boldsymbol{C}^p,0)}$ is called the residual discriminant of F.

Our definitions use some ringed structure slightly different from those of Pellikaan and Jiang, who are particulary interested in morsification of a germ f (with one-dimensional singular set, in order to find the expected numbers of the generic transversal singularities), but these differences do not change the geometric properties of these zero loci.

Let $\tilde{F} := F \times id_{(\boldsymbol{C}^p,0)} : (\boldsymbol{C}^n \times \boldsymbol{C}^p, 0) \to (\boldsymbol{C} \times \boldsymbol{C}^p, 0)$ be the unfolding mapping associated with the mini-versal unfolding of f, and let π be the projection on the second factor $(\boldsymbol{C}^p, 0)$.

Now let us recall what the 0-Fitting ideal of a module is. Let M be any finitely generated A-module of finite presentation. Then there is an exact sequence

$$A^p \xrightarrow{\Psi} A^q \to M \to 0.$$

The 0-th Fitting ideal of M, denoted by $\mathscr{F}_0(M)$, is the ideal generated in A by the $q \times q$ minors of the matrix of the linear mapping $\Psi : A^p \to A^q$. This ideal does not depend on the presentation and behaves well under base change (see [To, I.2] and [Tei, Section 1]) and is such that $\mathscr{F}_0(M) \subset Ann_A(M)$ and $\sqrt{\mathscr{F}_0(M)} = \sqrt{Ann_A(M)}$, which is a key point, since it enables to provide some analytic ringed structure (non reduced in general) to images of finite mappings (see [Tei]).

Lemma 1.2. *With the above hypotheses, the following restriction germs are finite maps*

$$\tilde{F}|_{RC_F} : RC_F \to (\boldsymbol{C}^p \times \boldsymbol{C}, 0) \quad \textit{and} \quad \pi|_{RC_F} : RC_F \to (\boldsymbol{C}^p, 0).$$

Proof. When RC_F is endowed with the following ringed structure $\mathcal{O}_{x,u}/\mathscr{F}_0(M_{RC_F})$ (since $\mathcal{O}_x/\mathscr{F}_0(I^2/T_e\mathscr{R}_I(f))$ is a finite dimensional complex vector space, it is a finitely generated $\mathcal{O}_u$-module), so [Gu, Theorem 5] tells us that we are in presence of a finite map. $\square$

Since $\tilde{F}|_{RC_F} : RC_F \to (\boldsymbol{C}^p \times \boldsymbol{C}, 0)$ is a finite map, RD_F is the germ of an analytic space in $(\boldsymbol{C}^p \times \boldsymbol{C}, 0)$ defined (for instance) by the ideal $\mathscr{F}_0(\tilde{F}_*(M_{RC_F}))$ in $\mathcal{O}_{y,u}$, where $\mathscr{F}_0$ is the 0-th Fitting ideal of the given module. Note that the dimension of RC_F is always smaller than or equal to p.

Remark 1.2. The modules sheaves generated by M_{RC_F}, denoted by $\mathscr{M}_{RC_F}$, and respectively by $\mathcal{O}_{x,u}/(d_xF)$, denoted by $\tilde{\mathcal{O}}_{\Sigma(F)}$, coincide on the open set $(\boldsymbol{C}^n \times \boldsymbol{C}^p, 0) \backslash (H \times \boldsymbol{C}^p, 0)$, where (d_xF) is the ideal generated by $\partial F/\partial x_1, \dots, \partial F/\partial x_n$.

Definition 1.3. We denote by RB_F the discriminant locus of the finite analytic mapping $\pi|_{RD_F} : (RD_F, 0) \to (\boldsymbol{C}^p, 0)$ and RB_F is called the residual bifurcation locus of F, where π is the projection $(y, u) \to u$.

1.2. Providing an equation for the discriminant

The next result is a reformulation of [dPGW, Theorem 3.1] in more general terms, which will be very useful when we want to find some equations of the residual discriminant and to lift the vector fields tangent to the residual discriminant.

Let $M = N/L$ be a given $\mathcal{O}_{x,u}$-module of finite type where N and L are some finitely generated submodules (or ideals) of $\mathcal{O}_{x,u}^l$ for a positive integer l. Let $\mathscr{S}_M$ denote its support in $(\boldsymbol{C}^n \times \boldsymbol{C}^p, 0)$. Let $\tilde{G} = (G(x,u), u)$ be a holomorphic mapping with $G \in \mathcal{O}_{x,u}$. Let π be the projection on $(\boldsymbol{C}^p, 0)$. Note that if M is a free $\mathcal{O}_u$-module then $\tilde{G}|_{\mathscr{S}_M}$ and $\pi|_{\mathscr{S}_M}$ are finite mappings.

Theorem 1.4. *Suppose that M is a free $\mathcal{O}_u$-module of rank s. If $\alpha_1, \dots, \alpha_s \in \mathcal{O}_{x,u}$ project to a $\boldsymbol{C}$-basis of $M/(\pi \circ \tilde{G})^* m_u M$, then for each $j = 1, \dots, s$, $G\alpha_j$ is uniquely written as a $(\pi \circ \tilde{G})^* \mathcal{O}_u$-linear combination in $\alpha_1, \dots, \alpha_s$ say*

$$G\alpha_j = \sum_{i=1}^{s} (\pi \circ \tilde{G})^* a_{i,j} \cdot \alpha_i. \tag{r_j}$$

Let $A := [a_{i,j}]$ and let $\delta := I_s - A$. Then the following sequence is exact

$$0 \to \mathcal{O}_{y,u}^s \xrightarrow{\delta} \mathcal{O}_{y,u}^s \xrightarrow{\pi} M \to 0$$

where $\pi(e_i)$ is the projection of the i-th basis vector of $\mathcal{O}_{x,u}^s$. Moreover the ideal $\langle \det \delta \rangle$ of $\mathcal{O}_{y,u}$ defines the image $\tilde{G}|_{\mathscr{S}_M}(\mathscr{S}_M)$.

Proof. We claim that the module of the $\mathcal{O}_{y,u}$-relations amongst the α_i's is generated by the (r_j)'s. Let $Rel := \{(\phi_1, \dots, \phi_s) \in \mathcal{O}^s_{y,u} : \sum_{j=1}^s (\phi_j \circ \tilde{G})\alpha_j = 0\}$. Let $R_j = ye_j - \sum_{i=1}^s a_{i,j}e_i$ then $Rel = (R_1, \dots R_s)\mathcal{O}_{y,u}$. To see this, let $\Phi = (\phi_1, \dots, \phi_s) \in Rel$. For $j = 1, \dots, s$, we can decompose $\phi_j = \beta_j + y\gamma_j$, where $\beta_j \in \mathcal{O}_u$ and $\gamma_j \in \mathcal{O}_{y,u}$. Thus for each j

$$\tilde{G}^*\gamma_j \cdot \alpha_j = \sum_{i=1}^s (\pi \circ \tilde{G})^* b_{i,j} \cdot \alpha_i$$

with $b_{i,j} \in \mathcal{O}_u$. Let $S_j = \gamma_j e_j - \sum_{i=1}^s b_{i,j}e_i$, then we have $S_j \in Rel$.

$$\begin{aligned}
\Phi &= \sum_{j=1}^s (\beta_j + y\gamma_j)e_j = \sum_{j=1}^s \beta_j e_j + y\sum_{j=1}^s \left(S_j + \sum_{i=1}^s b_{i,j}e_i\right) \\
&= \sum_{j=1}^s \beta_j e_j + y\sum_{j=1}^s S_j + \sum_{j=1}^s\sum_{i=1}^s b_{i,j}\left(R_i + \sum_{k=1}^s a_{i,k}e_k\right) \\
&= \sum_{k=1}^s \left(\beta_k + \sum_{j=1}^s\sum_{i=1}^s b_{i,j}a_{i,k}\right)e_k + y\sum_{j=1}^s S_j + \sum_{j=1}^s\sum_{i=1}^s b_{i,j}R_i.
\end{aligned}$$

Since R_j, S_j, and Φ are in Rel we obtain that

$$\sum_{k=1}^s (\pi \circ \tilde{G})^*\theta_k \cdot \alpha_k = 0$$

where $\theta_k = \beta_k + \sum_{j=1}^s \sum_{i=1}^s b_{i,j}a_{i,k} \in \mathcal{O}_u$. Since the α_k's form an $\mathcal{O}_u$-free basis of M, we deduce that $\theta_k = 0$ for any k. Then $\Phi \in m_{y,u}\, Rel + (R_1, \dots, R_s)\mathcal{O}_{y,u}$ and thus

$$Rel \subset m_{y,u}\, Rel + \langle R_1, \dots, R_s\rangle\mathcal{O}_{y,u},$$

and then by Nakayama's Lemma Rel is a $\mathcal{O}_{y,u}$-module (as module sheaf, we see that Rel is coherent), finitely generated by $R_1, \dots, R_s$. Then the sequence of $\mathcal{O}_{y,u}$-modules

$$\mathcal{O}^s_{y,u} \xrightarrow{\delta} \mathcal{O}^s_{y,u} \xrightarrow{\pi} M_{RC_F} \to 0$$

is exact with $\delta b_j = A_j$.

If δ is not injective, there exists a vector $\theta \neq 0$ in $\ker\delta$. Thus there is a Zariski (analytic) open set on which θ never vanishes. Then $\det\delta$ vanishes over an open set, but this is a contradiction, since $\det\delta$ is a Weierstrass polynomial in y of degree s with coefficients in $\mathcal{O}_u$. □

Remark 1.3. The previous theorem applies when the depth of M_{RC_F} is equal to p and with $\tilde{G} = \tilde{F}$, in which case it is a free $\mathcal{O}_u$-module of rank p. So RD_F is a hypersurface.

2. The hypersurface case

When the ideal I is principal ($I = (h)$), i.e., when we consider functions singular along a (reduced) hypersurface $H := \{h = 0\}$, we can say more in this case than in the general case above.

2.1. Hypersurfaces singular along a given hypersurface H and hypersurfaces with isolated singularities along H

Arnol'd, in [Ar], studied a function having isolated singularities along a smooth hypersurface (the boundary). This was generalized by [Ly1] when the boundary is an isolated hypersurface singularity. They have given classifications of such isolated singularities and connected them to some groups generated by reflections. They give also some lists of such simple singularities. The general work dealing with functions having isolated singularities on a variety has been done by Bruce and Roberts in [BR]. They study a function f_0 with an isolated singularity on an analytic set germ H, defined by a reduced ideal I, and they define a notion of right finite determinacy for these function germs. They give necessary and sufficient (numerical) conditions to be of finite determinacy (on the right) in that case, which is equivalent to the finiteness of $\dim_{\mathbf{C}}(\mathcal{O}_x/(\xi_1 \cdot f_0, \dots, \xi_t \cdot f_0))$, where $\xi_1, \dots, \xi_t$ $(t \geq n)$ denote a minimal generating system of $\Theta_{I,e}$. In that case we say that f_0 has an isolated singularity along H or that f_0 is of finite codimension on the right along H. This is stronger than just having an isolated singularity at the origin, since it means that outside the origin the hypersurface $f_0^{-1}(0)$ is tranverse to the leaves of the foliation given by the vector fields tangent to H (the logarithmic strata of H). In this context, note that Dimca, in [Di], has given some conditions on any isolated hypersurface singularity H to have simple functions (for functions with isolated singularities on H). In the same way Tibăr has also given conditions for functions with isolated singularities on an analytic germ H to be simple when H is an ICIS or, in some other cases, when H has nonisolated singularities [Ti].

In the sequel we will show that under some additional hypotheses on H, every function of finite relative codimension on the right along an hypersurface H comes, in fact, from a function having an isolated singularity on H.

Let f be any function in I^2 of finite relative codimension and let h be a generator of I. Then the following composition of $\mathcal{O}_x$-modules homomorphisms is onto:

$$\frac{I^2}{T_e\mathcal{R}_I(f)} \xrightarrow{p_0} \frac{I^2}{T_e\mathcal{K}_I(f)} \xrightarrow{\delta_h} \frac{\mathcal{O}_x}{T_e\mathcal{K}_I(f_0)} \xrightarrow{p_1} \frac{\mathcal{O}_x}{T_e\mathcal{K}(f_0)} \xrightarrow{p_2} 0,$$

where f_0 is the function germ such that $f = h^2 f_0$, p_0, p_1 and p_2 are the obvious projection maps and δ_h is the homomorphism which maps $h^2\alpha$ to α. This means that f_0 has an isolated singularity at the origin and that a basis of the finite dimensional complex vector space $I^2/T_e\mathcal{R}_I(f)$, say $\{h^2 e_1, \dots, h^2 e_p\}$, is mapped onto a generating family of the finite dimensional complex vector space $\mathcal{O}_x/T_e\mathcal{K}(f_0)$. Then the mini-versal unfolding F of f is of the form $F = h^2 F_0$, where $F_0(x,u) := f_0(x) + \sum_{k=1}^{p} u_k e_k(x)$ is a $\mathscr{K}_e$-versal unfolding of f_0. We can suppose that $e_p(x) = -1$. We have $\xi_j \cdot h = h\alpha_j$ and $\xi_j \cdot F = h^2 A_j$ for $j = 1, \dots, t$. We verify that $A_j(x,u) = 2\alpha_j(x)F_0(x,u) + \xi_j(x) \cdot F_0(x,u)$. Note that the $\mathscr{O}_{x,u}$-modules $(h^2)/\tilde{T}_e\mathscr{R}_I(f)$ and $\mathscr{O}_{x,u}/(A_1, \dots, A_t)$ are canonically isomorphic. The following proposition tells us more about f_0 than being just an isolated hypersurface singularity.

Proposition 2.1. *Let H be a hypersurface defined by a reduced equation $\{h = 0\}$ and with finite logarithmic stratification. If $f = h^2 f_0$ is of finite relative codimension along H, then the restriction of f_0 to any logarithmic strata is a submersion except at the origin, that is, $\dim_{\boldsymbol{C}}(\mathscr{O}_x/(\xi_1 \cdot f_0, \dots, \xi_t \cdot f_0)) < \infty$.*

Proof. The conclusion means that for every $x_0 \in H \backslash \{0\}$, there exists a vector field ξ tangent to H such that $(\xi \cdot f_0)(x_0) \neq 0$. Since $\dim_{\boldsymbol{C}}(\mathscr{O}_x/(2\alpha_1 f_0 + \xi_1 \cdot f_0, \dots, 2\alpha_t f_0 + \xi_t \cdot f_0)) < \infty$ (by hypothesis), thus we obtain that $\dim_{\boldsymbol{C}}(\mathscr{O}_x/(f_0, \xi_1 \cdot f_0, \dots, \xi_t \cdot f_0,)) < \infty$. If $\mathscr{O}_x/(\xi_1 \cdot f_0, \dots, \xi_t \cdot f_0)$ is not a finite dimensional vector space, then by the Curve Selection Lemma there is an analytic path Γ on $H \backslash 0$ (since outside H the ξ_i's form a generating family of the vector fields of the ambient space and f_0 has only isolated singularities), such that $0 \in \overline{\Gamma}$, which is contained in the support of the coherent $\mathscr{O}_x$-module sheaf induced from $\mathscr{O}_x/(\xi_1 \cdot f_0, \dots, \xi_t \cdot f_0)$. We can suppose that Γ is contained in a single logarithmic stratum, say $\mathscr{S}$. Hence $d(f_{0|\mathscr{S}})$ vanishes along Γ. Since $f_0(0) = 0$, $f_{0|\mathscr{S}} = 0$. But this is a contradiction to $\dim_{\boldsymbol{C}}(\mathscr{O}_x/(f_0, \xi_1 \cdot f_0, \dots, \xi_t \cdot f_0)) < \infty$. □

Bruce and Roberts were just interested in the finite determinacy on the right on varieties. But it is easy to obtain from their work the notion of contact equivalence along a given variety for function germs having isolated singularities on the variety.

If we were interested in the contact finite determinacy of nonisolated hypersurface (that is $\dim_{\boldsymbol{C}}(I^2/T_e\mathscr{K}_I(f)) < \infty$), then as a consequence of the above discussion and of the previous proof, we have

Corollary 2.2. *Let $f = h^2 f_0 \in I^2$. Then f is of contact finite determinacy relative to H if and only if f_0 is of finite contact determinacy along H. Moreover if the logarithmic stratification of $H = \{h = 0\}$ is finite then f_0 is finitely determined on the right relative to H if and only if f_0 is of finite contact determinacy along H, i.e.,*

$$\dim_{\boldsymbol{C}} \frac{\mathscr{O}_x}{T_e\mathscr{R}_I(f_0)} < \infty \quad \textit{if and only if} \quad \dim_{\boldsymbol{C}} \frac{\mathscr{O}_x}{T_e\mathscr{K}_I(f_0)} < \infty.$$

2.2. The smooth case

In this section we deal with functions singular along a smooth hypersurface through the origin, i.e., $I := (x_n)$. That is the first (and simplest) case to work with, and it really behaves as does any function with an isolated singularity along $H := \{x_n = 0\}$. The $\mathcal{O}_x$-module $\Theta_{I,e}$ is freely generated by $\partial/\partial x_1, \dots, \partial/\partial x_{n-1}$ and $x_n(\partial/\partial x_n)$. We have a converse to Proposition 2.1.

Proposition 2.3. *Let f_0 be a function with an isolated singularity at the origin along the hypersurface $\{x_n = 0\}$. Then the function $f = x_n^2 f_0$ is of finite relative codimension along $\{x_n = 0\}$.*

Proof. Let Γ be the zeros set of $\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1}$. It is a one-dimensional locus whose intersection with H is reduced to the origin. We have to show that $\{2f_0 + x_n(\partial f_0/\partial x_n) = 0\} \cap \Gamma$ is just the origin. Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$, where each Γ_i, for $i = 1, \dots, k$, is an irreducible curve germ at the origin with holomorphic parametrization $\gamma^i(z)$. We can suppose that $k = 1$. By an easy calculus we find that

$$\frac{d}{dz}[(x_n^2 f_0) \circ \gamma](z) = \gamma_n(z)\frac{d\gamma_n}{dz}(z)\left[\left(2f_0 + x_n\frac{\partial f_0}{\partial x_n}\right) \circ \gamma(z)\right].$$

Since Γ_i goes through H only at the origin, if $z \neq 0$ then $\gamma_n(z)(d\gamma_n/dz)(z) \neq 0$. Thus

$$\left\{2f_0 + x_n\frac{\partial f_0}{\partial x_n} = 0\right\} \cap \Gamma = \left\{\frac{d}{dz}((x_n^2 f_0) \circ \gamma) = 0\right\}$$
$$= \{(x_n^2 f_0) \circ \gamma = 0\} = \{f_0(\gamma) = 0\}$$

which implies that $\{2f_0 + x_n(\partial f_0/\partial x_n) = 0\} \cap \Gamma = \{(x_n(\partial f_0/\partial x_n)) \circ \gamma = 0\} = \{0\}$ since $\partial f_0/\partial x_n \neq 0$ when all the other derivatives are vanishing outside H. $\square$

We have to remember that any f_0 which admits an isolated singularity along $\{x_n = 0\}$ can be written as $g_0(x_1, \dots, x_{n-1}) + x_n g_1(x)$, where g_0 admits an isolated singularity at the origin in $(\boldsymbol{C}^{n-1}, 0)$, and g_1 is such that there is a non-negative integer k such that $(\partial_1^k g/\partial x_n^k)(0) \neq 0$, or there exists an $i \in \{1, \dots, n-1\}$ such that $(\partial g_1/\partial x_i)(0) \neq 0$. These are necessary conditions on g_0 and g_1, but we do not think there are sufficient to produce a f_0 with an isolated singularity along $\{x_n = 0\}$.

Proposition 2.4. *With the above assumptions and notations, RC_F is a smooth analytic set germ of dimension p and thus RD_F is a hypersurface.*

Proof. We see that

$$RC_F := \left\{\frac{\partial F_0}{\partial x_1} = \cdots = \frac{\partial F_0}{\partial x_{n-1}} = 2F_0 + x_n\frac{\partial F_0}{\partial x_n} = 0\right\}.$$

It is clear that F_0 is a versal deformation of the isolated hypersurface singularity $f_0^{-1}(0)$. So we can write $F_0(x,u) = f_0(x,u) + \sum_{k=1}^{p-1} u_k e_k(x) - u_p$. By [Tei, Section 5.5], we know that $\{\partial F_0/\partial x_1 = \cdots = \partial F_0/\partial x_{n-1} = 0\}$ is smooth. Then it is clear that RC_F is also smooth, since $\{2F_0 + x_n(\partial F_0/\partial x_n) = 0\}$ is a graph and tranverse to $\{\partial F_0/\partial x_1 = \cdots = \partial F_0/\partial x_{n-1} = 0\}$. $\square$

Remark 2.1. Note that $\mathcal{O}_{x,u}/(\partial F_0/\partial x_1, \dots, \partial F_0/\partial x_{n-1}, 2F_0 + x_n(\partial F_0/\partial x_n))$ is a reduced algebra. From now we denote it by $\mathcal{O}_{RC_F}$.

Let $\mathcal{O}_{RD_F}$ be defined by the 0-th Fitting ideal of the $\mathcal{O}_{y,u}$-module $\tilde{F}_*(\mathcal{O}_{RC_F})$. As usual, to know the structure of the discriminant with this structure sheaf, we need to know what the generic points of the discriminant are. Since there is only one logarithmic stratum contained in H (H itself), $f_0|_{H\backslash 0}$ is a submersion.

Let $\tilde{\mathcal{O}}$ be the sheaf of holomorphic function germs on $\boldsymbol{C}^n \times \boldsymbol{C}^p$.

LEMMA 2.5. *Let $f = x_n^2 f_0$ of finite relative codimension along $\{x_n = 0\}$. Let $F = x_n^2 F_0$ be its mini-versal unfolding. Then there is a Zariski open dense subset of RC_F of points $(x_0, u_0) = (x_0, u_0^{(1)}, \dots, u_0^{(p)})$ such that*

$$\dim_{\boldsymbol{C}} \frac{\tilde{\mathcal{O}}_{(x_0,u_0)}}{(\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1}, 2f_0 + x_n(\partial f_0/\partial x_n), u_1 - u_0^{(1)}, \dots, u_p - u_0^{(p)})} = 1.$$

Proof. We find that

$$RC_F \cap \{x_n = 0\} = \left\{\frac{\partial F_0}{\partial x_1} = \cdots = \frac{\partial F_0}{\partial x_{n-1}} = F_0 = x_n = 0\right\}.$$

Since $f_0 = k_0(x_1, \dots, x_{n-1}) + x_n k_1(x)$ such that k_0 admits an isolated singularity at the origin of $(\boldsymbol{C}^{n-1}, 0)$, then we can write

$$F_0(x,u) = K_0(x_1, \dots, x_{n-1}, u) + x_n K_1(x,u).$$

It is obvious that K_0 is a $\mathscr{K}_e$-versal unfolding of the isolated hypersurface singularity $k_0^{-1}(0)$. Thus

$$RC_F \cap \{x_n = 0\} = \left\{\frac{\partial K_0}{\partial x_1} = \cdots = \frac{\partial K_0}{\partial x_{n-1}} = K_0 = x_n = 0\right\}.$$

This is a smooth germ in $(\boldsymbol{C}^{n-1} \times 0 \times \boldsymbol{C}^p, 0)$ of dimension $p-1$. This exactly means that

$$RC_F \cap \{x_n \neq 0\} = \left\{\frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0 : x_n \neq 0\right\}.$$

Then the set outside $\{x_n \neq 0\}$ where the f_u's have only Morse singular points with p distinct critical values is a Zariski dense open subset of $RC_F \cap \{x_n \neq 0\}$. Since RC_F is irreducible and $\dim RC_F \cap \{x_n = 0\} \leq p-1$, there is a Zariski dense open subset of RC_F where the f_u's have only Morse singular points with p distinct critical values, which ends the proof. $\square$

So, for a function germ f of finite relative codimension along H equal to 1, we obtain that the maximal ideal of $\mathcal{O}_x$, $m_x = (\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1}, 2f_0 +$

$x_n(\partial f_0/\partial x_n))$. If $f_0 \in m_x^2$, then since $2f_0 + x_n(\partial f_0/\partial x_n) \in m_x^2$, we obtain that $m_x \subset (\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1}) + m_x^2$ and so by Nakayama's Lemma $m_x = (\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1})$ which is impossible. Since the previous dimension is positive, we deduce that $f_0 \in m_x \backslash m_x^2$ with $(\partial f_0/\partial x_n)(0) \neq 0$. In expanding f_0 as a power series in x_n and in making explicit the equations of $(\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_{n-1}, 2f_0 + x_n(\partial f_0/\partial x_n)) = m_x$, we find that $f_0(x) = k_0(x_1, \dots, x_{n-1}) + x_n k_1(x)$ where k_0 is a Morse function in the variables $x_1, \dots, x_{n-1}$, and k_1 is invertible.

Proposition 2.6. *The residual discriminant RD_F endowed with the ringed structure $\mathcal{O}_{RD_F} = \mathcal{O}_{y,u}/\mathcal{F}_0(\tilde{F}_*\mathcal{O}_{RC_F})$ is irreducible and reduced.*

Proof. The irreducibility comes from the smoothness of RC_F. To show that $\mathcal{O}_{RD_F}$ is reduced, there is just to see that it is reduced at a smooth point, see [dPGW, Corollary 1.18]. Let $f(x) = x_n^2(x_n + x_1^2 + \dots + x_{n-1}^2)$ be a generic point of the residual discriminant. Then $F(x,u) = x_n^2(x_n + x_1^2 + \dots + x_{n-1}^2 - u)$ is a mini-versal unfolding of f. We see that $\mathcal{O}_{RC_F} = \mathcal{O}_{x,u}/(x_1, \dots, x_{n-1}, 3x_n - 2u)$, and by computations we have $\mathcal{F}_0(\tilde{F}_*\mathcal{O}_{RC_F})$ is generated by $y - (4/27)u^3$, hence reduced. $\square$

Another important question about the geometry of the (residual) discriminant is to exhibit a basis of the vector fields tangent to the (residual) discriminant in order to know if it is a free divisor, and if such vector fields are liftable by the unfolding map associated to the mini-versal unfolding, since this is the case for isolated complete intersection singularities (see [Lo, Corollary 6.13] and [Go, Theorem 3] or [Sa1], [Ter1], [Bru1] for hypersurfaces). As in the isolated singularity case, instead of considering the mini-versal unfolding we consider the truncated mini-versal unfolding:

Definition 2.7. Let $f = h^2 f_0$ be a function of finite relative codimension along $\{x_n = 0\}$ equal to p. Let $h^2 e_1, \dots, h^2 e_p$ be a $\boldsymbol{C}$-basis of $I^2/T_e\mathcal{R}_I(f)$, with $e_p(x) = -1$. Then the unfolding $G(x,v) = h^2(x)[f_0(x) + \sum_{i=1}^{p-1} v_i e_i(x)]$ is called the truncated mini-versal unfolding of f.

Now we return to the case $h(x) = x_n$. To make the distinction between the truncated mini-versal unfolding G and the mini-versal unfolding F of f, we will denotes RC_G, RC_F, RD_G and RD_F their respective residual critical loci and discriminants.

Then RC_G is the support of the coherent $\mathcal{O}_{x,v}$-module sheaf generated by

$$M_{RC_G} := \frac{I^2 \otimes_{\mathcal{O}_x} \mathcal{O}_{x,v}}{\{Y \cdot G : Y \in \Theta_H \otimes_{\mathcal{O}_x} \mathcal{O}_{x,v}\}}$$

and RD_G is the image of RC_G by $\tilde{G} := G \times id_{(\boldsymbol{C}^{p-1},0)}$.

Let $F(x,u) = x_n^2[f_0(x) + \sum_{i=1}^{p-1} u_i e_i(x) + u_p e_p(x)] = x_n^2 F_0(x)$. Such an F_0 is a versal unfolding of the isolated hypersurface singularity $f_0^{-1}(0)$. Let us define the function $G_0(x,v)$ such that $G_0(x,v)+u_pe_p(x) := F_0(x,u)$. Then $G(x,v) = x_n^2G_0(x,v)$. By easy calculus, we obtain

$$RC_G = \left\{(x,v) : \frac{\partial G_0}{\partial x_1} = \cdots = \frac{\partial G_0}{\partial x_{n-1}} = 2G_0 + x_n\frac{\partial G_0}{\partial x_n} = 0\right\}.$$

We define $\mathcal{O}_{RC_G} := \mathcal{O}_{x,v}(\partial G_0/\partial x_1, \dots, \partial G_0/\partial x_{n-1}, 2G_0 + x_n(\partial G_0/\partial x_n))$ and $\mathcal{O}_{RD_G} := \mathcal{O}_{x,v}/\mathscr{F}_0(\tilde{G}_*\mathcal{O}_{RC_G})$. We have to note that $\mathcal{O}_{RC_G}$ defines a complete intersection. The first thing to see is that the codimension of RC_G is $\leq n$. Since

$$RC_G \cap \{v_1 = \cdots = v_{p-1} = 0\} = \operatorname{supp}\frac{m_xI^2}{T_e\mathscr{R}_I(f)} = \{0\}$$

we deduce from this that the dimension of RC_G is $\leq p-1$, and so RC_G becomes a (non-reduced) complete intersection of dimension $p-1$. The consequence of this is that M_{RC_G} is a free $\mathcal{O}_v$-module of rank $p-1$. This also prove that we can define $\mathcal{O}_{RD_G}$ as done above.

We would like to know more about how to obtain some gemetrically important sets for F from those corresponding to G.

Let χ be the vector field on $(\boldsymbol{C}\times\boldsymbol{C}^p, 0)$ such that $\chi\circ\tilde{F}(x,u) = -x_n^2(\partial/\partial y) + (\partial/\partial u_p) = d\tilde{F}\cdot(\partial/\partial u_p)$. Let us denote by $\Psi(y,u,t)$ the local flow of χ, with the initial condition $\Psi(y,u,0) = (y,v,2u_p)$, then $\Psi(y,u,t) = (\Psi_0(y,u,t), v, t+2u_p)$ and

$$\Psi(\tilde{F}(x,u),u,t) = (F(x,u) - tx_n^2, v, t+2u_p).$$

Let $\Phi(y,u) := \Psi(y,u,-u_p) = (\Psi_0(y,u,-u_p),u) = (\Phi_0(y,u),u)$. Then we have $\operatorname{Im}(\tilde{G})\times\boldsymbol{C} = \Phi(\operatorname{Im}(\tilde{F}))$ and Φ is a well-defined diffeomorphism of $\boldsymbol{C}\times\boldsymbol{C}^p$ preserving the origin. Let C_F be the projection of RC_F on $(\boldsymbol{C}^n\times\boldsymbol{C}^{p-1},0)$, in forgetting the last coordinate u_p. Then we have

$$\Phi(\tilde{F}(RC_F)) = \Phi(RD_F) = \tilde{G}(C_F)\times\boldsymbol{C}.$$

Since $RD_G\times\{0\} = RD_F\cap\{u_p=0\}$ and $\Phi(y,v,0) = (y,v,0)$, from this, we deduce that

$$RD_G\times\{0\} = RD_F\cap\{u_p=0\} = \Phi(RD_F\cap\{u_p=0\}) = \tilde{G}(C_F)\times\{0\}.$$

This is the fundamental point to know whether or not RD_F is a free divisor. Now we state

THEOREM 2.8. *RD_G is a free divisor.*

The next theorem follows from the proof of Theorem 2.8.

THEOREM 2.9. *RD_F is a free divisor.*

Proof of Theorem 2.8. The method to produce a generating family of vector fields tangent to the discriminant of a projection of an isolated complete intersection singularity is now usual (see also [Bru1], [Ter2] and [Go]). Note that

M_{RC_G} is an $\mathcal{O}_v$-free module of rank $p-1$. Then for each $j = 1, \dots, p$, we have

$$Gx_n^2 e_j = \sum_{i=1}^{p} a_{i,j}(v) x_n^2 e_i \quad \text{in } M_{RC_G},$$

which gives

$$\sum_{i=1}^{p} [a_{i,j}(v) - G(x,v)\delta_{i,j}] x_n^2 e_i + \sum_{k=1}^{n-1} b_{k,j}(x,v) \frac{\partial G}{\partial x_k} + b_j(x,v) x_n \frac{\partial G}{\partial x_n} = 0.$$

If A_G is the $p \times p$-matrix with entries the $(a_{i,j}(v) - y\delta_{i,j})$'s, then $\det A_G = 0$ is an equation of the residual discriminant according to Theorem 1.4. Let $\chi_i \in \Theta_{x,v}$ be the following vector field

$$\chi_j(x,v) = \sum_{i=1}^{p-1} [a_{i,j}(v) - G(x,v)\delta_{i,j}] \frac{\partial}{\partial v_i} + \sum_{k=1}^{n-1} b_{k,j}(x,v) \frac{\partial}{\partial x_k} + b_j(x,v) x_n \frac{\partial}{\partial x_n}.$$

Thus we have $\chi_j \cdot G = -(a_{j,p} - G\delta_{j,p})$ and $\chi_j \cdot u_l = \sum_{i=1}^{p-1} [a_{i,j}(v) - G(x,v)\delta_{i,j}]\delta_{i,l} = (a_{l,j} - G\delta_{l,j})$. Let η_i be the vector field of $\Theta_{y,v}$ defined by

$$\eta_j(y,v) = -(a_{p,j}(v) - y\delta_{p,j}) \frac{\partial}{\partial y} + \sum_{i=1}^{p-1} [a_{i,j}(v) - y\delta_{i,j}] \frac{\partial}{\partial v_i}.$$

Then the η_i's are tangent to $RD_F(G)$ and $d\tilde{G}_{\chi_i} = \eta_i \circ \tilde{G}$. To conclude it is sufficient to show that

LEMMA 2.10. *The η_i's are the generators of the $\mathcal{O}_{y,v}$-module Θ_{RD_G}.*

Proof. The proof is inspired by that of [Go, Theorem 3.1]. First we have to note that outside RD_G the η_i's are linearly independant. Let η be any vector field of $\Theta_{y,v}$ tangent to the residual discriminant RD_G. We form the $p \times (p+1)$-matrix whose p first columns are the coefficients of the vector fields η_i's and the last one is given by the coefficients of η. To finish the proof of this lemma we need the following

LEMMA 2.11. *$\mathcal{O}_{RD_G}$ is reduced.*

Proof. Since $RD_G \times \boldsymbol{C}$ is isomorphic to RD_F by Φ then we have the following isomorhism of local $\boldsymbol{C}$-algebras

$$\Phi^* : \mathcal{O}_{RD_F} \to \mathcal{O}_{RD_G \times \boldsymbol{C}} = \mathcal{O}_{RD_G} \otimes_{\mathcal{O}_{y,v}} \mathcal{O}_{y,u}.$$

Let $k \in \mathcal{O}_{RD_G}$ such that $k \neq 0$ and $k^d = 0$ for a positive integer d. This implies by the $(\Phi^*)^{-1}(k^d) = 0$ and so $(\Phi^*)^{-1}(k) = 0$, which ends the proof. $\square$

End of proof of Lemma 2.10. By Lemma 2.11 and Theorem 1.4, $\{\det A_G = 0\}$ gives a reduced equation of RD_G. Each $p \times p$ minor containing the last column is of the form $\phi_j(y,v)\Delta(y,v)$, where $\{\Delta(y,v)=0\}$ is a reduced equation of RD_G. Thus the vector field $\eta - \sum_{j=1}^{p}(-1)^{p-j}\phi_j\eta_j$ vanishes outside RD_G, and since the η_i's are a free basis of the vector fields in $\boldsymbol{C}\times\boldsymbol{C}^{p-1}\backslash RD_G$, by continuity $\eta = \sum_{j=1}^{p}(-1)^{p-j}\phi_j\eta_j$. □

To finish this subsection we give a basis of the vector fields tangent to RD_F. For any $j=1,\dots,p$, we have

$$Fx_n^2 e_j = \sum_{i=1}^{p}\alpha_{i,j}(u)x_n^2 e_i + \sum_{k=1}^{n-1}\beta_{k,j}(x,u)\frac{\partial F}{\partial x_k} + \beta_j(x,u)x_n\frac{\partial F}{\partial x_n}$$

and thus

$$0 = \sum_{i=1}^{p}[\alpha_{i,j}(u) - F(x,u)\delta_{i,j}]x_n^2 e_i + \sum_{k=1}^{n-1}\beta_{k,j}(x,u)\frac{\partial F}{\partial x_k} + \beta_j(x,u)x_n\frac{\partial F}{\partial x_n}.$$

Now we define the vector fields Λ_i in $(\boldsymbol{C}^{n+p},0)$ and Γ_i in $(\boldsymbol{C}\times\boldsymbol{C}^p,0)$ for $i=1,\dots,p$, by

$$\Lambda_j = \sum_{i=1}^{p-1}[\alpha_{i,j}(u) - F(x,u)\delta_{i,j}]\frac{\partial}{\partial u_i} + \sum_{k=1}^{n-1}\beta_{k,j}(x,u)\frac{\partial}{\partial x_k} + \beta_j(x,u)x_n\frac{\partial}{\partial x_n},$$

$$\Gamma_j = -(\alpha_{p,j}(u) - y\delta_{p,j})\frac{\partial}{\partial y} + \sum_{i=1}^{p-1}[\alpha_{i,j}(u) - y\delta_{i,j}]\frac{\partial}{\partial u_i}.$$

We have $d\tilde{F}\cdot\Lambda_i = \Gamma_i\circ\tilde{F}$. When the χ_i's, vector fields in $(\boldsymbol{C}^n\times\boldsymbol{C}^{p-1},0)$, are seen as vector fields in $(\boldsymbol{C}^n\times\boldsymbol{C}^{p-1}\times\boldsymbol{C},0)$, we also have $\Lambda_i(x,v,0)=\chi_i(x,v)$ and that $\Gamma_i\circ(G(x,v),v,0)=\eta_i\circ(G(x,v),v)$, if we also consider the η_i's as vector fields in $(\boldsymbol{C}\times\boldsymbol{C}^{p-1}\times\boldsymbol{C},0)$. Now to finish to find a basis of the $\mathcal{O}_{y,u}$-module of the vector field tangent to RD_F we need to know the Jacobian matrix of Φ denoted by A and its inverse A^{-1}

$$A = \begin{bmatrix} \dfrac{\partial\Phi_0}{\partial y} & \dfrac{\partial\Phi_0}{\partial u_1} & \dfrac{\partial\Phi_0}{\partial u_2} & \cdots & \dfrac{\partial\Phi_0}{\partial u_p} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\partial\psi_0/\partial y}\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

where $\gamma_0 = (\partial\Phi_0/\partial y)^{-1}$ and $\gamma_i = -\gamma_0(\partial\Phi_0/\partial u_i)$ for $1 \le i \le p$. From $\Phi_0(F(x,u),u) = G(x,v)$, we deduce

$$\frac{\partial\Phi_0}{\partial u_p}(F(x,u),u) = x_n^2\frac{\partial\Phi_0}{\partial y}(F(x,u),u,-u_p) \text{ and so we obtain}$$

$$d(\Phi^{-1})\frac{\partial}{\partial u_p}(y,u) = \left(\frac{\partial\Phi_0}{\partial y}\right)^{-1}(y,u)\left[-\frac{\partial\Phi_0}{\partial u_p}(y,u)\frac{\partial}{\partial y}+\frac{\partial}{\partial u_p}\right]$$

$$d(\Phi^{-1})\frac{\partial}{\partial u_p}(G(x,v),u) = \left[\frac{1}{\partial\Phi_0/\partial y}\chi\right]\circ(F(x,u),u).$$

Since χ is tangent to RD_F by the above computations and since the determinant of the matrix formed by the Γ_i's and χ is a reduced equation of the free divisor RD_F, then by Saito's Lemma [Sa1, Lemma 1.9], we have proved the following

Proposition 2.12. *The vector fields $\Gamma_1,\dots,\Gamma_p$ and χ of $\Theta_{y,u}$ generate freely the $\mathcal{O}_{y,u}$-module of the vector fields tangent to RD_F, and are liftable by $\tilde{F}$.*

2.3. The free divisor case

We recall that a free divisor is a hypersurface whose module of tangent vector fields $(\Theta_{I,e})$ is a free $\mathcal{O}_x$-module.

We say that a hypersurface H (or a principal ideal I) is Euler if there is a vector field E tangent to H such that $E\cdot I = I$. This means that we can choose such a vector field such that, given a generator h of I, then $E\cdot h = h$; we denote this vector field by E_h. In that case $\Theta_{I,e} = \Theta_h^0 \oplus \mathcal{O}_x E_h$, where $\Theta_h^0 := \{\xi : \xi\cdot h = 0\}$, see [Al1, p. 2] or [DM, Lemma 3.3]. Then an Euler hypersurface is a free divisor if and only if Θ_h^0 is an $\mathcal{O}_x$-free module.

Now we suppose that H is a free divisor and Euler. Let h be a given reduced equation of H. Let $\xi_1,\dots,\xi_{n-1}$ be a system of generators of Θ_h^0. The next proposition is a kind of converse to Proposition 2.1. We need some notations to state it.

Let $f_0 \in \mathcal{O}_x$ with $\dim_{\mathbf{C}}(\mathcal{O}_x/(\xi_1\cdot f_0,\dots,\xi_{n-1}\cdot f_0, E_h\cdot f_0))$ finite. Let us denote by Γ the 1-dimensional complete intersection $\{\xi_1\cdot f_0 = \cdots = \xi_{n-1}\cdot f_0 = 0\}$. Then Γ is a finite union of irreducible analytic curves $\Gamma_1,\dots,\Gamma_s$. For $i = 1,\dots,s$, let $\gamma^{(i)}$ be a complex analytic parametrization of the curve Γ_i.

Proposition 2.13. *Let H be a free divisor and Euler with reduced equation h. Let $f_0 \in \mathcal{O}_x$ with $\dim_{\mathbf{C}}(\mathcal{O}_x/(\xi_1\cdot f_0,\dots,\xi_{n-1}\cdot f_0, E_h\cdot f_0))$ finite. Let us consider the following conditions*:

(i) *$\xi_1\cdot f_0,\dots,\xi_{n-1}\cdot f_0$, h is a regular sequence,*

(ii) *$\Gamma\cap H = \Gamma_1\cup\cdots\cup\Gamma_t$ and for each $i = 1,\dots,t$*

$$\frac{d\gamma^{(i)}}{dz}(z) \in \operatorname{span}(\xi_1\circ\gamma^{(i)}(z),\dots,\xi_{n-1}\circ\gamma^{(i)}(z), E_h\circ\gamma(z)),$$

(iii) *There exist a finite number of logarithmic strata of H which meet Γ.*

If f_0 satisfies one of the above conditions, then $f := h^2 f_0$ is of finite relative codimension along H.

Proof. (i) Γ is one-dimensional and we can suppose it is irreducible and parametrized by the analytic arc germ γ, with $\gamma(0) = 0$. By hypothesis $\Gamma \cap H = \{0\}$. Then $h(\gamma(z)) = 0$ if and only if $z = 0$.

Let $M(x)$ be the $n \times n$-matrix whose rows are the coefficients of the vector fields $\xi_1, \ldots, \xi_{n-1}$ and $\xi_n := E_h$. Let us denote by $m_{i,j}$ the $(n-1) \times (n-1)$ minor obtained from M by deleting the i-th row and the j-th column. Remember that $\det M(x) = u(x)h(x)$ where u is a unit. Along $\Gamma \backslash \{0\}$, the vector fields $\xi_1, \ldots, \xi_{n-1}$ and E_h are linearly independent. Thus for any $k = 1, \ldots, n$ we have

$$u(\gamma(z))h(\gamma(z)) \frac{\partial}{\partial x_k} \circ \gamma(z) = \sum_{l=1}^{n} (-1)^{l+k} m_{l,k}(\gamma(z)) \xi_l \circ \gamma(z).$$

Now we have

$$\frac{d}{dz}((h^2 f_0) \circ \gamma(z)) = h \circ \gamma(z)[2f_0\, d_{\gamma(z)} h \cdot \gamma'(z) + h(\gamma(z))\, d_{\gamma(z)} f_0 \cdot \gamma'(z)],$$

and using the above expression of the $(\partial/\partial x_k)$'s we find

$$\frac{d}{dz}((h^2 f_0) \circ \gamma) = \left[\frac{h}{u} \circ \gamma(z)\right] \left[\sum_{k=1}^{n} (-1)^{k+n} m_{k,n}(\gamma(z)) \gamma_k'(z)\right] [(2f_0 + E_h \cdot f_0) \circ \gamma(z)].$$

We see that $[\sum_{k=1}^{n} (-1)^{k+n} m_{n,k}(\gamma(z)) \gamma_k'(z)]$ is the determinant of the $n \times n$-matrix whose $(n-1)$-th first rows are the coefficients of $\xi_1, \ldots, \xi_{n-1}$ and the last one is made with the coefficients of $\gamma'(z)$. Since the vector fields $\xi_1, \ldots, \xi_{n-1}$ are a basis of the vector fields tangent to the levels of h and since $\Gamma \cap H = \{0\}$ the curve γ is transverse to the levels $\{h = w_0\ (\neq 0)\}$ when w_0 is ranging a small open punctured neighbourhood of the origin in $\boldsymbol{C}$. Thus this determinant is non-zero. Thus $(2f_0 + E_h \cdot f_0) \circ \gamma(z) = 0$ if and only if $(\partial/\partial z)((h^2 f_0) \circ \gamma(z)) = 0$ if and only if $f_0 \circ \gamma(z) = 0$. Then $(2f_0 + E_h \cdot f_0) \circ \gamma(z) = 0$ if and only if $(E_h \cdot f_0) \circ \gamma(z) = 0$, which is by hypothesis $z = 0$.

(ii) Let us suppose that $t = 1$ and $(2f_0 + E_h \cdot f_0) \circ \gamma(z) \equiv 0$, with $\gamma^{(1)} = \gamma$. By hypothesis there are analytic functions in z $\alpha_1, \ldots, \alpha_{n-1}, \alpha$ such that

$$\frac{d(f_0 \circ \gamma)}{dz}(z) = \sum_{k=1}^{n-1} \alpha_k(z)(\xi_k \cdot f_0) \circ \gamma(z) + \alpha(z)(E_h \cdot f_0) \circ \gamma(z).$$

Thus we obtain the following differental equation $2\alpha(z) f_0 \circ \gamma(z) + (d(f_0 \circ \gamma)/dz)(z) = 0$. Since $f_0 \circ \gamma(0) = 0$, this implies that $f_0 \circ \gamma \equiv 0$ and so $E_h \cdot f_0 \circ \gamma \equiv 0$, which a contradiction to the hypothesis we have on f_0.

(iii) We can suppose that there is only one logarithmic stratum which meets Γ and so we are in the case (ii). $\square$

We have the following

Proposition 2.14. *When H is a free divisor, then residual Jacobian module M_{RC_F} is a Cohen-Macaulay $\mathcal{O}_{x,u}$-module of dimension p.*

Proof. Since $\Theta_{I,e}$ is free as $\mathcal{O}_x$-module (then $t = n$), M_{RC_F} is an $\mathcal{O}_{x,u}$-module canonically isomorphic to $\mathcal{O}_{x,u}/(A_1, \dots, A_n)$. Since RC_F is of dimension $\leq p$, $\mathcal{O}_{x,u}/(A_1, \dots, A_n)$ is necessarily of dimension p. This means that $A_1, \dots, A_n$ is an $\mathcal{O}_{x,u}$-regular sequence and then the depth and dimension of $\mathcal{O}_{x,u}/(A_1, \dots, A_n)$ as $\mathcal{O}_{x,u}$-module are equal to p. $\square$

As it was done in the smooth case, given a mini-versal unfolding F of a finite relative codimensional germ f along H, we introduce the truncated mini-versal unfolding G and then we define, as in the smooth case, the $\mathcal{O}_{x,v}$-module M_{RC_G}, the analytic sets RC_G and RD_G and the local analytic algebras $\mathcal{O}_{RC_G}$ and $\mathcal{O}_{RD_G}$. As in the smooth case we prove that $RD_F \cong RD_G \times \boldsymbol{C}$ by an automorphism of $\mathcal{O}_{y,u}$ built by means of the vector field χ defined as $\chi \circ \tilde{F} = \mathrm{d}\tilde{F}(\partial/\partial u_p)$.

Damon in his trilogy [Da-I], [Da-II], [Da-III] computed discriminant and bifurcation locus of a versal unfolding of a mapping $\boldsymbol{C}^k \to \boldsymbol{C}^l$ under various equivalences preserving a variety (a free divisor or a free complete intersection) at the source or at the target. He noticed that there are such varieties on which there is no mapping whose discriminant is a free divisor. The explanation of this is that the variety has to present what he called generic Morse-type singularities, which is equivalent (when some conditions on k or l and numbers related to the geometry of the logarithmic stratification of the variety are satisfied) to the existence of functions of (extended) codimension one for the equivalences he considered. This condition is sufficient to provide a reduced equation to the discriminant by means of Fitting ideals, which here means the freeness. In our context the same kind of phenomenon appears. Let us consider the following

Example I. Let $h(x, y) = x^3 + y^3$. It defines an Euler free divisor in $\boldsymbol{C}^2$. The tangent vector fields are generated by

$$E = \frac{1}{3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \quad \text{and} \quad \xi = y^2\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y}.$$

Let $f_0(x, y) = x$ and $f(x, y) = h^2 f_0$. Then, f is of codimension 2 along the Euler free divisor $H := \{h = 0\}$. $T_e\mathscr{R}_I(f)$ is the ideal generated by xh^2 and y^2h^2. Let $F_0(x, y, u) = f_0(x, y) + u_1 y + u_2$. Then $F = h^2 F_0$ is a mini-versal unfolding of f. Let G be the truncated mini-versal unfolding of f. We find that $RC_G := \{x = y = 0\} \subset \boldsymbol{C}^3$. So RD_G is just the u_1-axis in $\boldsymbol{C}^2$. When we compute a resolution of M_{RC_G} as an $\mathcal{O}_{t,u_1}$-module, where T is coordinate at the origin of the target space $(\boldsymbol{C}, 0)$. Then the matrix whose determinant is the 0-th Fitting ideal of the resolution is t^2 Id (this is the matrix of the $\mathcal{O}_{u_1}$-endomorphism of M_{RC_G} of multiplication by G). RD_G is a free divisor, but the Fitting ideal does not provide a reduced ideal. Note that $RC_G \subset H \times \boldsymbol{C}$.

Example II. This is true for any Euler free divisor $x^k + y^l$, with $k > l > 2$ and $f(x, y) = x$. The smallest relative codimension along H is $l - 1$. The truncated residual discriminant is always the hyperplane of the truncated parameter space. But the 0-th Fitting ideal is generated by t^{l-1}, not reduced. In that case $RC_G \subset H \times \boldsymbol{C}$.

Example III. As above it is easy to see that when the Euler free divisor is of the form $x^2 + y^k$, $k > 1$, then this time everything works as in the smooth case.

The following proposition is a first step to understand what could be the relevant phenomena seen above.

Proposition 2.15. *Let H be an Euler free divisor. Let f be a function of finite relative codimension p along H and let G be a truncated mini-versal unfolding. Then the following sequence of $\mathcal{O}_{y,v}$-modules is exact*:

$$0 \to \mathscr{L} \to \mathcal{O}_{y,v}^p \xrightarrow{\alpha} M_{RC_G} \to 0.$$

Then

(i) *$\mathscr{L}$ is the free $\mathcal{O}_{y,v}$-module of rank p of the vector fields tangent to RD_G which are liftable along $H \times \boldsymbol{C}^{p-1}$. Moreover $\mathscr{L}$ is a Lie algebra.*

(ii) *Let F be a mini-versal unfolding giving the previous G. If the dimension of $RC_F \cap (H \times \boldsymbol{C}^p)$ is $\leq p - 1$, then any vector field tangent to RD_F is liftable by $\tilde{F}$.*

Proof. (i) The freeness of $\mathscr{L}$ comes from the fact that M_{RC_G} is Cohen-Macaulay of dimension $p - 1$ and is a free $\mathcal{O}_v$-module of rank p. If we denote by v_p the coordinate y, the map α sends $\partial/\partial v_i$ on e_i. For a vector field $\zeta \in \Theta_{y,v}$, $\alpha(\zeta) = 0$ if and only if there exists a vector field $\eta \in \Theta_{x,v}$ tangent to $H \times \boldsymbol{C}^{p-1}$ such that $d\tilde{G}\eta = \zeta \circ \tilde{G}$. Note that such an $\mathscr{L}$ is a Lie algebra because of the last equality, which goes through the Lie bracket (see [Da-I, 3.3 (ii)]).

(ii) By hypothesis any vector field defined on $\boldsymbol{C} \times \boldsymbol{C}^p \backslash \tilde{F}(RC_F \cap (H \times \boldsymbol{C}^p))$ can be uniquely extended to a vector field in $\Theta_{y,u}$ by the Hartogs Theorem since $\Delta := \tilde{F}(RC_F \cap (H \times \boldsymbol{C}^p))$ is of codimension 2. So any vector field ζ tangent to $RD_F \backslash \Delta$ is liftable under $\tilde{F}$ in the vector field η defined over $\boldsymbol{C}^n \times \boldsymbol{C}^p \backslash \Sigma$, where $\Sigma := \tilde{F}^{-1}(\Delta)$, which is of codimension 2. So η can be uniquely extended to $\boldsymbol{C}^n \times \boldsymbol{C}^p$. □

So the previous proposition tells us that RD_G (or RD_F) is a free divisor if any liftable vector field by $\tilde{G}$ belongs to $\mathscr{L}$.

The following proposition shows us that what could be a sufficient condition to have a free divisor structure on the residual discriminant, with liftable vector fields.

Proposition 2.16. *Let H be an Euler free divisor and let f be a function of finite relative codimension $p > 1$ along H. Let F be a mini-versal unfolding of f,* *and let G be the truncated mini-versal unfolding of f coming from F. If* $\dim RD_G \cap (H \times \boldsymbol{C}^{p-1}) = p-2$. *Then RD_G is a free divisor and the vector fields tangent to RD_G are liftable by $\tilde{G}$ in vector fields tangent to $H \times \boldsymbol{C}^{p-1}$.*

Proof. In fact we show that any vector field tangent to RD_G is liftable in a vector field tangent to $H \times \boldsymbol{C}^{p-1}$. The proof is exactly the same as that of Proposition 2.15 (ii). So $\Theta_{RD_G} = \mathscr{L}$ and then is a free $\mathcal{O}_{y,v}$-module of rank p, that is, RD_G is a free divisor. □

Remark 2.2. 1) If $\dim RC_F \cap (H \times \boldsymbol{C}^p) = p-1$, then $\mathcal{O}_{RC_F}$ is reduced, since at any point which is not on $H \times \boldsymbol{C}^p$, Θ_H is a basis of the vector fields of the ambient space. By a coordinate change h becomes an invertible coordinate. So the argument of Lemma 2.5 provides a Zariski dense open set of points in RC_F at which the above structure is reduced. Thus the genericity criterion for a Cohen-Macauley space of [dPGW, Corollary 1.18] gives the reduced structure everywhere. This will be very important in the next section.

2) Under the hypotheses of the previous proposition, we obtain a free basis of the $\mathcal{O}_{y,v}$-module of the vector fields tangent to the (truncated) residual discriminant RD_G, in that case, in repeating the proof of Lemma 2.10, because, in constructing the vector fields we want to be a basis of Θ_{RD_G}, we are sure that the determinant of the matrix of these vector fields gives a reduced equation of the residual discriminant, since this matrix is the matrix which appears in the resolution of the residual Jacobian module (that obtained for G) as an $\mathcal{O}_{y,v}$-module, and so its determinant is a generator of the 0-th Fitting ideal of this resolution.

In order to produce a basis of vector fields tangent to RD_F, we just repeat what have been done in the smooth case.

The following corollary is an analog of [Sa2, 1.5]. It will be very useful in the next section to prove that the residual bifurcation set is a free divisor.

Corollary 2.17. *If $RC_G \cap (H \times \boldsymbol{C}^{p-1})$ is of dimension $p-2$, then the following sum of $\mathcal{O}_v$-modules is direct*

$$\Theta_{y,v} = \Theta_{RD_G} \oplus \mathcal{O}_v \left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_{p-1}} \right\}.$$

Proof. This is a straightforward computation once we know a basis of the vector fields tangent to RD_G. □

In the case of a function germ f of finite relative codimension along H equal to 1, there is the following

Proposition 2.18. *Let H be an Euler free divisor such that there exists a function f of finite relative codimension* 1 *along H. Let F be a mini-versal unfolding of f. Then RD_F is a free divisor.*

Proof. Note that $F(x,u) = h^2(x)(f_0(x) + u) = h^2(x)F_0(x,u)$. So the truncated mini-versal unfolding is just f itself. Then in using the mutiplication map by f in M_{RC_G} to obtain a generator of the 0-th Fitting ideal as in Theorem 1.4, then the generator obtained is y. So this provides a reduced structure to RD_G, hence to RD_F. □

COROLLARY 2.19. *Under the previous hypothesis on H and f, then*
(i) *f_0 is a submersion.*
(ii) *There is a vector field X in Θ_H such that $X \cdot f = f$.*
(iii) *Any vector field tangent to RD_F is liftable by $\tilde{F}$ in a vector field tangent to $H \times \boldsymbol{C}^p$.*

Proof. (i) We can suppose that H is Euler and all its tangent vector fields are vanishing at the origin except, possibly, the Euler vector field E. Then $\xi_1 \cdot f_0, \dots, \xi_{n-1} \cdot f_0$, $2f_0 + E \cdot f_0$ are generators of the maximal ideal m_x. Since $\xi_i(0) = (2f_0 + E \cdot f_0)(0) = 0$, this implies that $f_0 \in m_x \backslash m_x^2$ (otherwise $m_x = E \cdot f_0 + m_x^2$).
(ii) Since $f_0 \in m_x$, then there is a vector field $X \in \Theta_H$ such that $X \cdot (h^2 f_0) = h^2 f_0$.
(iii) By (ii) there is a vector field $X \in \Theta_H$ such that $X \cdot f = f$. This means that the vector field $y(\partial/\partial y)$ is liftable by $G = f$ the truncated mini-versal unfolding, and so that the ringed structure over $RD_G = \{0\} \subset \boldsymbol{C}$ given by means of the 0-th Fitting ideal is reduced. □

Remark 2.3. The problem we have is not to decide if the (truncated) residual disriminant is a free divisor or not, but to decide if the vector fields tangent at the (truncated) residual discriminant are liftable in vector fields tangent to a suspension of H, in which case the (truncated) residual discriminant is a free divisor.

As suggested by the very simple examples (and not quite representative of the general situation) before, it seems that the key element is the way that the truncated residual critical locus intersects the suspension of H (in fact the singular set of H). We can think that informations about this are contained in the smallest jets at the origin of the vector fields tangent to H which are non identically zero of the vector fields tangent to H (see Damon and his conditions on the genericity of Morse type singularities on the divisor [Da-I] and the genericity of the locally liftable vector fields [Da-II]).

By now we are not able to find necessary conditions on the divisor H to obtain that for any function germ of finite relative codimension along H in order to be able to lift the vector fields tangent to the (truncated) residual discriminant in vector fields tangent to the suspension of H.

2.4. Bifurcation sets and free divisor

Since we have proved that the residual discriminant of any function germ of finite relative codimension along an Euler free divisor, and under the hypothesis $RC_G \cap (H \times \boldsymbol{C}^{p-1})$ is $= p - 2$, is a free divisor, it is natural to ask if the residual bifurcation set is also a free divisor as it is for an isolated hypersurface singularity ([Bru2], [Ter2] and [Go]).

Let $WP_k(Y,a)$ be the Weierstrass polynomial of degree k, that is, $WP_k(Y,a) = Y^k + a_1 Y^{k-1} + \cdots + a_{k-1}Y + a_k$, where the a_i's are in $\boldsymbol{C}$ for $1 \le i \le k$. Let us denote by $\Pi_k : \boldsymbol{C} \times \boldsymbol{C}^k \to \boldsymbol{C}^k$ the projection on the a_i's plane. Let $W\Delta_k$ be the zero set (smooth) of WP_k in $\boldsymbol{C} \times \boldsymbol{C}^k$ and let $W\Sigma_k \subset \boldsymbol{C}^k$ be the discriminant of $W\Delta_k$ which is also the discriminant of $\Pi_k|_{W\Delta_k}$. We recall that $W\Sigma_k$ is a free divisor and that its smooth points are the u's where WP_k admits a double root with $k-2$ simple roots. Moreover the logarithmic stratification of $W\Sigma_k$ is finite and given by the Samuel stratification (see [DR] or [Me]).

Let D be a free divisor in $(\boldsymbol{C}^p, 0)$ and let $A : (\boldsymbol{C} \times \boldsymbol{C}^{p-1}, 0) \to (\boldsymbol{C} \times \boldsymbol{C}^p, 0)$ with $(y,v) \to A(y,v) = (y, a_1(v), \ldots, a_p(v))$ such that $\delta(y,v) = WP_p \circ A(y,v)$ is a reduced equation of D. Let us denote $\mathcal{O}_D := \mathcal{O}_{y,v}/\delta\mathcal{O}_{y,v}$. Since the singular set $\Sigma(D)$ of this free divisor is a determinantal space of codimension 2 in $(\boldsymbol{C} \times \boldsymbol{C}^{p-1}, 0)$ and since $\pi_D := \Pi_{p-1}|_D$ is a ramified covering of degree p over $(\boldsymbol{C}^{p-1}, 0)$, then any smooth point of $\pi_D(\Sigma(D))$ is a smooth point of the discriminant of π_D which is $Bif(D) = \pi_D(\{\delta = \partial\delta/\partial y = 0\})$.

LEMMA 2.20. *If $\{\delta = \partial\delta/\partial y = 0\} = \Sigma(D)$, then any vector field in Θ_u, tangent to $Bif(D)$, is liftable by Π_{p-1} to a vector field tangent to D.*

Proof. The proof follows in applying [Ly2, Theorem 5] of Lyashko which enables to lift a vector tangent to the bifurcation set (the discriminant of a linear projection) to a vector field tangent to a discriminant, since our hypotheses satisfy those of the quoted result, that is

$$\dim\left\{\delta = \frac{\partial\delta}{\partial y} = \frac{\partial^2\delta}{\partial y^2} = 0\right\} < \dim\left\{\delta = \frac{\partial\delta}{\partial y} = 0\right\}. \qquad \square$$

Now we follow the steps of the proof of Terao to show that the bifurcation set of an isolated hypersurface singularity is also a free divisor [Ter2]. For this purpose from now we suppose that $\{\delta = \partial\delta/\partial y = 0\} = \Sigma(D)$. First we define the $\mathcal{O}_v$-module of the Π_{p-1}-lowerable vector fields that is

$$\mathcal{K} = \{\xi \in \Theta_{y,v} : \xi \cdot v_i \in \mathcal{O}_v \;\; i = 1, \ldots, p-1\} = \mathcal{O}_{y,v}\frac{\partial}{\partial y} + \mathcal{O}_v\Theta_v.$$

Let Π_D be the following $\mathcal{O}_v$-module homomorphism $(\pi_{p-1})_* : \mathcal{K} \cap \Theta_D \to \Theta_v$. Note that the kernel of Π_D is the submodule of $\Theta_{y,v}$ generated by $\delta(\partial/\partial y)$. Let us denote by $\mathcal{G}$ the $\mathcal{O}_v$-submodule of $\Theta_{y,v}$ generated by $\partial/\partial y$ and the $(\partial/\partial v_i)$'s. We have $\mathcal{G} \subset \mathcal{K}$. Now we can state and proof the following

LEMMA 2.21. *Suppose that $\Theta_{y,v} = \Theta_D \otimes_{\mathcal{O}_v} \mathcal{G}$ as $\mathcal{O}_v$-module. Then the $\mathcal{O}_v$-module $(\Pi_{p-1})_*(\mathcal{K} \cap \Theta_D) \subset \Theta_{Bif(D)}$ is free of rank $p-1$.*

Proof. As in [Ter2, Lemma 3.5] the key arguments of the proof are the freeness of Θ_D and the $\mathcal{O}_v$ direct sum in hypothesis.

The natural map $\mathscr{K} \cap \Theta_D \to \mathscr{K}/\mathscr{G}$ is an $\mathcal{O}_v$-isomorphism and since $\mathscr{K}/\mathscr{G} \cong \mathcal{O}_{y,v}/\mathcal{O}_v$, we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{y,v}\left(\delta_D \dfrac{\partial}{\partial y}\right) & \longrightarrow & \mathscr{K} \cap \Theta_D & \xrightarrow{\Pi_D} & \mathfrak{I}(\Pi_D) & \longrightarrow & 0 \\ 0 & \longrightarrow & \delta_D \mathcal{O}_{y,v} & \longrightarrow & \dfrac{\mathcal{O}_{y,v}}{\mathcal{O}_v} & \longrightarrow & \dfrac{\mathcal{O}_D}{\mathcal{O}_v} & \longrightarrow & 0. \end{array}$$

The isomorphism between $\mathscr{K} \cap \Theta_D$ and $\mathcal{O}_{y,v}/\mathcal{O}_v$ is given by $\xi \to [\xi \cdot y]$ (the bracket meaning the residue class). So we have an $\mathcal{O}_v$-isomorphism

$$\alpha : \mathrm{Im}(\Pi) \to \frac{\mathcal{O}_D}{\mathcal{O}_v}$$

$$\Pi(\xi) \to \xi \cdot y \bmod \mathcal{O}_v.$$

Since δ is reduced and π_D finite, as $\boldsymbol{C}$ local algebras we have

$$\frac{\mathcal{O}_D}{\mathcal{O}_v} \xrightarrow{\sim} \frac{\boldsymbol{C}\{y\}}{(y^p)}.$$

And by Nakayama's Lemma [1], $[y], \dots, [y^{p-1}]$ generate freely the $\mathcal{O}_u$-module $\mathcal{O}_D$. So $\mathcal{O}_D/\mathcal{O}_v$ is freely generated over $\mathcal{O}_v$ by $[y], \dots, [y^{p-1}]$, which ends the proof. □

THEOREM 2.22. *Bif(D) is a free divisor.*

Proof. This is immediate since any vector fields η which is tangent to the bifurcation set is liftable under Π_{p-1} in a vector field tangent to D. Then by the last lemma we conclude. □

Let $D = RD_G$ be the residual discriminant of a truncated mini-versal unfolding G of a function germ f singular along an Euler free divisor $H \subset (\boldsymbol{C}^n, 0)$ whose reduced ideal is I. Let p be the dimension of $I^2/T_e\mathscr{R}_I(f)$. By now, we suppose that the dimension of $RC_G \cap (H \times \boldsymbol{C}^{p-1})$ is equal to $p-2$. So the dimension of $RC_F \cap (H \times \boldsymbol{C}^p)$ is equal to $p-1$. We have the following

PROPOSITION 2.23. *Let $\delta_F(y,u)$ be the reduced equation of RD_F, provided by the determinant of the matrix whose columns are the coefficients of the previous basis of the vector fields tangent to RD_F. Then $\{\delta_F = \partial\delta_F/\partial y = 0\} = \Sigma(RD_F)$.*

Proof. Let $\tilde{\mathcal{O}}$ be the sheaf of holomorphic function germ on a small neighbourhood Ω of the origin of $\boldsymbol{C}^n \times \boldsymbol{C}^p$. Let $\mathscr{W} = \Omega \backslash H \times \boldsymbol{C}^p$ and let $\mathscr{V} = \pi_p(\Omega)$.

We recall that the $\tilde{\mathcal{O}}$-module sheaves $\mathscr{M} := \mathscr{M}_{RC_F}$ and $\mathscr{N} := \tilde{\mathcal{O}}_{\Sigma(F)}$ are equal on $\mathscr{W}$. Since $M_{RC_F} := \mathscr{N}_{(0,0)}$ is $\mathcal{O}_u$-free with basis $e_1, \dots, e_p$, we can suppose that Ω is small enough to have representatives of the $e_i = (\partial F/\partial u_i)$'s over Ω.

The Malgrange-Weierstrass Preparation Theorem insures us that for any $u_0 \in \mathscr{W}$ the complex vector space $\mathscr{N}/(U_1, \dots, U_p)$ is a finite dimensional $((U_1, \dots,$

$U_p)$ is a system of local coordinates at u_0 in $\boldsymbol{C}^p$), and is generated by $e_1, \dots, e_p$. This means that F is a versal unfolding of the function germ f_{u_0} with an isolated singularity at x_0. The Morse points form a Zariski open dense subset of any neighbourhood of $(x_0, u_0) \in RC_F \cap \mathscr{W}$. Since $RC_F \cap H \times \boldsymbol{C}^p$ is of codimension 1 in RC_F, this shows that the subset of RC_F of the (x_0, u_0) such that f_{u_0} is a Morse function is open and dense in RC_F.

Now we use an argument similar to [Tei, Section 5.5] to control, on the residual dicriminant, the vanishing locus of the partial derivatives of δ_F in the directions of the unfolding parameters by the vanishing of the partial derivative of δ_F along the y-axis.

Note that $\delta_F \circ \tilde{F}|_{RC_F} \equiv 0$. By the reduced structure of RC_F, we obtain that

$$\delta_F \circ \tilde{F}(x,u) = \sum_{i=1}^{n} \frac{a_i(x,u)}{h^2(x)} \xi_i(x) \cdot F(x,u).$$

Thus we obtain that

$$0 = \frac{\partial(\delta_F \circ \tilde{F})}{\partial u_j} = \sum_{i=1}^{n} \frac{a_i}{h^2} \frac{\partial(\xi_i \cdot F)}{\partial u_j} \mod \tilde{T}_e \mathscr{R}_I(f),$$

$$0 = \xi_j \cdot (\delta_F \circ \tilde{F}) = \sum_{i=1}^{n} \frac{a_i}{h^2} \xi_j \cdot (\xi_i \cdot F) \mod \tilde{T}_e \mathscr{R}_I(f).$$

We easily see that the $n \times n$-matrix $[\xi \cdot (\xi_j \cdot F)]$ is conjugated to the Hessian matrix of f_{u_0}, and so, we deduce that a_i is vanishing over $RC_F \cap \mathscr{W}$. Since RC_F is endowed with a reduced Cohen-Macauley ringed structure it is equidimensional and $RC_F \cap \mathscr{W}$ is an open dense subset of RC_F. Then a_i vanishes over RC_F, which means that a_i belongs to $\tilde{T}_e \mathscr{R}_I(f)$. So we obtain

$$0 = \frac{\partial(\delta_F \circ \tilde{F})}{\partial u_j}\bigg|_{RC_F} = \frac{\partial \delta_F}{\partial y} \circ \tilde{F}|_{RC_F} \cdot \frac{\partial F}{\partial u_j}\bigg|_{RC_F} + \frac{\partial \delta_F}{\partial u_j} \circ \tilde{F}|_{RC_F}.$$

Since RD_F is by definition $\tilde{F}(RC_F)$ the above equality proves that

$$\left\{ \delta_F = \frac{\partial \delta_F}{\partial y} = \frac{\partial \delta_F}{\partial u_1} = \cdots = \frac{\partial \delta_F}{\partial u_p} = 0 \right\} = \left\{ \delta_F = \frac{\partial \delta_F}{\partial y} = 0 \right\}$$

which is the desired result. $\square$

Let G be the truncated mini-versal unfolding of f. Since $\Phi(RD_F) = RD_G \times \boldsymbol{C}$ with $\Phi(y, v, u_p) = (\phi_0(y,u), v, u_p)$, we have $\Phi(\Sigma(RD_F)) = \Sigma(RD_G) \times \boldsymbol{C}$. We also have the following commutative diagrams

$$\begin{array}{ccc} \boldsymbol{C}^n \times \boldsymbol{C}^{p-1} \times \boldsymbol{C} & \xrightarrow{\Phi^{-1}} & \boldsymbol{C}^n \times \boldsymbol{C}^p \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ \boldsymbol{C}^{p-1} \times \boldsymbol{C} & \xrightarrow{\mathrm{id}} & \boldsymbol{C}^p \end{array} \qquad \begin{array}{ccc} RD_G \times \boldsymbol{C} & \xrightarrow{\Phi^{-1}} & RD_F \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ \boldsymbol{C}^{p-1} \times \boldsymbol{C} & \xrightarrow{\mathrm{id}} & \boldsymbol{C}^p \end{array}$$

where p_1 and p_2 are the obvious linear projections on $\boldsymbol{C}^p = \boldsymbol{C}^{p-1} \times \boldsymbol{C}$. Note that the form of Φ gives us that $RB_G \times \boldsymbol{C} = RB_F$. From this we deduce that $p_1|_{RD_G \times \boldsymbol{C}}$ is a diffeomorphism at the point $(y, u) = (y, v, u_p)$ if and only if $p_2 \circ \Phi^{-1}|_{RD_F}$ is also a diffeomorphism at the point $\Phi^{-1}(y, v, u_p) = ((\Phi^{-1})_0, v, u_p)$. This means that the critical set of p_1 is exactly $\Sigma(RD_G)$. So by Theorem 2.22 $Bif(RD_G) = RB_G$ is a free divisor. So we have proved the following

Theorem 2.24. *Let f be a function germ of finite codimension p relatively to an Euler free divisor H. Let f be its mini-versal unfolding. Then the residual bifurcation set of F is a free divisor.*

To finish, as Terao in [Ter2] and Bruce in [Bru2], we can describe a basis of the vector fields tangent to $RB_F = Bif(RD_F)$. The proof of that kind of result is now well known (see also [Go, Section 4]). We begin by producing some vector fields tangent to RB_G.

For any $j = 1, \dots, p-1$ we have in $\mathcal{O}_{x,v}$

$$-G^j h^2 = G^j h^2 e_p = \sum_{i=1}^{n-1} \gamma_{i,j} \xi_j \cdot G + \gamma_j E_{(h)} \cdot G + h(x) \left[\sum_{i=1}^{p-1} c_{i,j}(v) e_j + c_j(v) e_p \right].$$

Now we define the vector fields μ_j in $(\boldsymbol{C}^{p-1}, 0)$, $j = 1, \dots, p-1$

$$\mu_j = \sum_{i=1}^{p-1} b_{i,j}(v) \frac{\partial}{\partial v_i}.$$

Theorem 2.25. *The vector fields μ_j, for $j = 1, \dots, p-1$, form a free basis of Θ_{RB_G}.*

Proof. We build the vector fields $\phi_1, \dots, \phi_{p-1} \in \mathscr{K} \cap \Theta_{RD_G}$ such that $\Pi_{RD_G}(\phi_1), \dots, \Pi_{RD_G}(\phi_{p-1})$ form a free basis of Θ_{RD_G} if and only if $[\phi_1 \cdot y], \dots, [\phi_{p-1} \cdot y]$ is an $\mathcal{O}_v$-free basis of $\mathcal{O}_D/\mathcal{O}_v$ (see [Ter2, Theorem C]). Now we use the basis of Θ_{RD_G} given previously to define inductively ϕ_i $(1 \le i \le p-1)$ by

$$\phi_1 = \eta_p \quad \text{and} \quad \phi_i = y\phi_{i-1} + \sum_{j=1}^{p-1} (\phi_{i-1} \cdot v_j)\eta_j.$$

Thus for any $i = 1, \dots, p-1$ ϕ_i is tangent to RD_G. For $k = 1, \dots, p-1$ we obtain

$$\begin{aligned}
\phi_i \cdot v_k &= y\phi_{i-1} \cdot v_k + \sum_{j=1}^{p-1} (\phi_{i-1} \cdot v_j)\eta_j \cdot v_k \\
&= y\phi_{i-1} \cdot v_k + (\phi_{i-1} \cdot v_k)(a_{k,k} - y) + \sum_{j \neq t}^{p} (\phi_{i-1} \cdot v_j) a_{k,j} \\
&= \sum_{j=1}^{p-1} a_{k,j} (\phi_{i-1} \cdot v_j),
\end{aligned}$$

and this proves that $\phi_i \in \mathscr{K} \cap \Theta_{RD_G}$ and that $\Pi_{RD_G}(\phi_i) = \mu_i$. To finish, we have

$$\phi_i \cdot y = y(\phi_{i-1} \cdot y) + \sum_{j=1}^{p-1} (\phi_{i-1} \cdot v_j)(-a_{j,p}) = y(\phi_{i-1} \cdot y) \bmod \mathcal{O}_v.$$

Thus we find that $[\phi_i \cdot y] = [y^{i-2}\phi_1 \cdot y] = [y^{i-2}(y - a_{p,p}(v))] \in \mathcal{O}_{RD_G}/\mathcal{O}_v$ and thus $[\phi_1 \cdot y], \dots, [\phi_{p-1} \cdot y]$ is a free $\mathcal{O}_v$-generating family of Θ_{RB_G}. □

Corollary 2.26. $\det|\mu_1, \dots, \mu_{p-1}|$ *is a reduced equation of* RB_G.

Proof. Since $\mu_1, \dots, \mu_{p-1}$ form a free basis of RB_G, then the determinant of the matrix of their coefficients is only vanishing over RB_G. By Saito's Lemma ([Sa1, Lemma 1.9]) we conclude. □

Corollary 2.27. *The vector fields* μ_j, *for* $j = 1, \dots, p-1$, *and* $\partial/\partial u_p$ *form a free basis of* Θ_{RB_F}.

2.5. Comments on the non-Euler free divisor case

Let $H \subset (\boldsymbol{C}^n, 0)$ be a free divisor which is not necessarily Euler and given by a reduced equation $h \in \mathcal{O}_x$. To produce an Euler free divisor from H with the same geometry, there is just to apply the trick of the good defining equation (see [DM, Section 3]), that is, we look at H in $\boldsymbol{C}^n \times \boldsymbol{C}$ (as a source space with coordinates (x, w)) as the hypersurface $\tilde{H} = H \times \boldsymbol{C}$ which is now Euler and free since

$$\tilde{H} = \{\tilde{h}(x, w) = e^w h(x) = 0\} \quad \text{and} \quad \Theta_{\tilde{H}} = (\Theta_H \otimes_{\mathcal{O}_x} \mathcal{O}_{x,w}) \oplus \mathcal{O}_{x,w}\frac{\partial}{\partial w}.$$

Such an $\tilde{h} \in \mathcal{O}_{x,w}$ provides a reduced equation of $H \times \boldsymbol{C}$ and is called a good defining equation.

We would like to know if, given a non-Euler free divisor H and a function of finite codimension relatively to H, say f, there is a way to find a function g associated with f which will be of finite codimension relatively to $\tilde{H}$ and which will provide a residual discriminant obtained as a one-dimensional fibration along that of f. In the case of isolated hypersurface singularities such a notion exists, which is the stably equivalence in the terminology of Arnol'd.

Let $f = h^2(x)f_0(x)$ be of finite codimension relatively to H and let $k(x, w) = e^{2w}h(x)^2 k_0(x, w)$ with $k_0(x, w) = f_0(x) + w^2$. From the point of view of the isolated singularities along a variety the functions f_0 and k_0 are stably equivalent. The $\mathcal{O}_{x,w}$-module $\Theta_{\tilde{h}}^{(0)}$ defined as the submodule of the vector fields tangent to the levels of $\tilde{h}$ is freely generated by $\chi_i(x, w) = \xi_i(x) - 2a_i(x)(\partial/\partial w)$ for $i = 1, \dots, n$. So the vector fields $\partial/\partial w$ and the χ_i's form an $\mathcal{O}_{x,w}$-free basis of the vector fields tangent to $\tilde{H}$.

Since

$$\frac{(h)^2}{T_e\mathscr{R}_{(h)}(f)} \cong \frac{\mathcal{O}_{x,w}}{(\xi_i \cdot f_0 + 2a_i f_0, w)},$$
$$\frac{(\tilde{h})^2}{T_e\mathscr{R}_{(\tilde{h})}(k)} \cong \frac{\mathcal{O}_{x,w}}{(\xi_i \cdot k_0 + 2a_i k_0, k_0 + w)}.$$

Note that the following ideals in $\mathcal{O}_{x,w}$, $(T_e\mathscr{K}_{(h)}, w) = (f_0, \xi_i \cdot f_0 + 2a_i f_0, w)$ and $T_e\mathscr{K}_{(\tilde{h})} = (k_0, \xi_i \cdot k_0 + 2a_i k_0, k_0 + w)$ are equal. But we do not know if the above local $\boldsymbol{C}$-algebras are isomorphic or not.

3. Counter-example when H is a smooth space of codimension 2

Let $I = (x_1, x_2)$. Then $\Theta_{I,e}$, the $\mathcal{O}_x$-module of the vector fields tangent to I, is generated by $\partial/\partial x_3, \ldots, \partial/\partial x_n$, and $x_1(\partial/\partial x_1)$, $x_2(\partial/\partial x_1)$, $x_1(\partial/\partial x_2)$, $x_2(\partial/\partial x_2)$. In his papers ([Za1] and [Za2]) Zaharia gives some normal forms of the simple nonisolated singularities along a smooth space of codimension 2. We give just below two of these classes where the respective residual critical loci are rather different. Let $z_i = x_i$ for $i = 1, 2$ and $w_j = x_{j+2}$ for $j = 1, \ldots, n-2$.

The first case is given by the normal form $IIA_s : f(z, w) = w_1 z_1^2 + w_2 z_2^2 + z_1 z_2 w_3^{s+1}$. The family $\{z_1 z_2 w_3^k\}_{\{k=0,\ldots,s-1\}}$ is a $\boldsymbol{C}$-basis of the vector space $I^2/T_e\mathscr{R}_I(f)$, and thus $F(x, u) = w_1 z_1^2 + w_2 z_2^2 + z_1 z_2 (w_3^{s+1} + u_1 w_3^{s-1} + \cdots + u_s)$. Let us denote $(w_3^{s+1} + u_1 w_3^{s-1} + \cdots + u_s)$ by $Q_u(w_3)$. After computations we obtain that

$$\tilde{T}_e\mathscr{R}_I(f) = \{z_1^2, z_2^2, z_1 z_2 Q_u(w_3), z_1 z_2 Q_u'(w_3), z_1 z_2 w_i \;\; i \neq 3\},$$
$$\tilde{T}_e\mathscr{R}_I(f) \cap (z_1 z_2) = (z_1 z_2)(z_1, z_2, Q_u(w_3), 2w_i \;\; i \neq 3).$$

We see that among the generators $z_1 z_2, z_1 z_2 w_3, \ldots, z_1 z_2 w_3^{s-1}$ of M_{RC_F} there is an $\mathcal{O}_u$-relation whereas any subfamily of $(s-1)$ elements is $\mathcal{O}_u$-free:

$$\begin{aligned} z_1 z_2 [(s+1) Q_u - x_3 Q_u'] &= z_1 z_2 [2u_1 w_3^{s-1} + 3u_2 w_3^{s-2} + \cdots + (s+1) u_s] \\ &= 0 \bmod \tilde{T}_e\mathscr{R}_I(f). \end{aligned}$$

Thus $RC_F = \{z_1 = z_2 = Q_u(w_3) = Q_u'(w_3) = w_1 = w_2 = 0\}$ and it is a complete intersection of dimension $s-1$, so the residual discriminant RD_F is necessarily of codimension at least 2 in $(\boldsymbol{C} \times \boldsymbol{C}^s, 0)$. We can verify that the residual discriminant of IIA_s is actually the discriminant of A_s embedded in $0 \times \boldsymbol{C}^s$. We can also verify that in the classes $IIA - D - E$ ([Za2]) there is always an $\mathcal{O}_u$-relation between the $\mathcal{O}_u$ generators of M_{RC_F}, but every subfamily is $\mathcal{O}_u$-free. The residual critical locus is still a complete intersection of dimension $s-1$ contained in H, and so the corresponding residual discriminant is in $0 \times \boldsymbol{C}^s$.

These facts are quite general since we have by staightforward calculus ([Gr]) the following

PROPOSITION 3.1. *Let* $f(z,w) = z_1^2 w_1 + z_2^2 w_2 + z_1 z_2 g(z,w)$, *with* g *such that* $(\partial g/\partial w_1)(0)(\partial g/\partial w_2)(0) \neq 1$. *Then* f *is finitely determined relatively to* $H = \{z_1 = z_2 = 0\}$ *if and only if* g_0 *the restriction of* g *to* $\{z_1 = z_2 = w_1 = w_2 = 0\}$, *has an isolated singularity at the origin of* $(\boldsymbol{C}^{n-4}, 0)$.

PROPOSITION 3.2. *Let* f *be a finitely determined germ relatively to the smooth space* $H = \{z_1 = z_2 = 0\}$ *and* f *is of the form* $z_1^2 w_1 + z_2^2 w_2 + z_1 z_2 g(z,w)$. *When* $(\partial g/\partial w_1)(0)(\partial g/\partial w_2)(0) \neq 1$ *then*

(i) RC_F *is a smooth complete intersection of dimension* $\dim_{\boldsymbol{C}}(I^2/T_e\mathscr{R}_I(f)) - 1 = s - 1$.

(ii) $RD_F = 0 \times \Delta(g_0) \subset 0 \times \boldsymbol{C}^s$, *where* $\Delta(g_0)$ *is the discriminant of the isolated hypersurface singularity* g_0.

It is easy to show that for any function germ singular along $H = \{z_1 = z_2 = 0\}$ and of finite relative codimension which can be written as $f(z,w) = z_1^2 w_1 + z_2^2 w_2 + z_1 z_2 g(z,w)$, then (in passing to jets) the condition $(\partial g/\partial w_1)(0) \cdot (\partial g/\partial w_2)(0) \neq 1$ is a generic condition for such function germs.

The next normal form is $IIB_s : f(z,w) = w_2 z_1 z_2 + w_1 z_1^2 + z_2^2(z_2 + w_1^s)$. Then $z_2^2, z_2^2 w_1, \dots, z_2^2 w_1^{s-1}$ is a $\boldsymbol{C}$-basis of $I^2/T_e\mathscr{R}_I(f)$ and thus $F(x,u) = w_2 z_1 z_2 + w_1 z_1^2 + z_2^2(z_2 + w_1^s + u_1 w_1^{s-1} + \cdots + u_s)$. Let us denote $(w_1^s + u_1 w_1^{s-1} + \cdots + u_s)$ by $Q_u(w_1)$. A system of generators for $\tilde{T}_e\mathscr{R}_I(f)$ are $z_1 z_2, z_1^2 + z_2^2 Q_u'(w_1)$, $z_2^2(3z_2 + 2Q_u(w_1))$, $w_2 z_2^2$ and $z_1^2 w_1$. We find that the residual critical locus is

$$\begin{aligned} RC_F &= \{z_1 = w_2 = z_2 Q_u'(w_1) = 3z_2 + 2Q_u(w_1) = w_1 Q_u'(w_1) 0\} \\ &= \{z_1 = w_2 = Q_u'(w_1) = 3z_2 + 2Q_u(w_1) = 0\} \cup \{z_1 = w_2 = z_2 \\ &= w_1 = u_s = 0\} = C_{R,1} \cup C_{R,2}. \end{aligned}$$

Note that $C_{R,1}$ is a graph of codimension 4 and not contained in H, while $C_{R,2}$ is smooth of codimension 5 and so contained in H but not in $C_{R,1}$ (then M_{RC_F} cannot be a free $\mathcal{O}_u$-module). Note that we have $\Sigma_R(\tilde{F}) = C_{R,1}$. That kind of result is also quite general for function germs of finite relative codimension which can be written as $f(z,w) = w_2 z_1 z_2 + w_1 z_1^2 + z_2^2 g(z,w)$ and with the condition $(\partial g/\partial w_1)(0) + ((\partial g/\partial w_2)(0))^2 = 0$ (otherwise they can be written in the generic form above by means of a change of coordinates which preserves the origin and H). In that case the residual critical locus RC_F is the union of two distinct complete intersections $C_{R,1}$ and $C_{R,2}$ of respective dimensions $\dim_{\boldsymbol{C}}(I^2/T_e\mathscr{R}_I(f))$ and $\dim_{\boldsymbol{C}}(I^2/T_e\mathscr{R}_I(f)) - 1$. In that case with the notations of the above propositions g_0 is still a function with an isolated singularity (which was expected in looking at the above generic function germs), but this is not enough at all to have f of finite relative codimension (for a bit more see [Gr]).

We have some open questions about this last case. For instance we do not know if the hypersurface $\tilde{F}(C_{R,1})$ is a free divisor. What are necessary and sufficient conditions on g for f of the form $w_2 z_1 z_2 + w_1 z_1^2 + z_2^2 g(z,w)$ to be of finite relative codimension ? Do the residual discriminant of any such function f of finite relative codimension control, by its partition into logarithmic strata, the analytic types of the germs obtained by deformation of f? (A positive answer to this last question could explain the impossible adjacencies occuring in the list given by Zaharia [Za2]).

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