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ON TWO POINT DISTORTION THEOREMS FOR BOUNDED UNIVALENT REGULAR FUNCTIONS

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1. Let $f(z)$ be a bounded univalent regular function mapping the unit disc D into the unit disc E. We define

$$\Delta_1 f(z) = \frac{(1-|z|^2)}{(1-|f(z)|^2)} f'(z).$$

The expression $|\Delta_1 f(z)|$ is invariant under linear transformations of D and of E. For $z_1, z_2 \in D$ distinct let ρ be the hyperbolic distance between z_1 and z_2 and σ the hyperbolic distance between $f(z_1)$ and $f(z_2)$. These are of course invariant under linear transformations of D and E. A two point distortion theorem for f is an inequality between $|\Delta_1 f(z_1)|$, $|\Delta_1 f(z_2)|$, ρ and σ. To prove such a result it is then sufficient to prove it for a suitable normalization for z_1, z_2, $f(z_1)$ and $f(z_2)$.

Many years ago Blatter [1] gave a similar result for univalent functions in D (not satisfying a boundedness condition) namely

$$|f(z_1) - f(z_2)|^2 \geq \frac{\sinh 2\rho}{8 \cosh 4\rho} \sum_{j=1}^{2} (1-|z_j|^2)^2 |f'(z_j)|^2. \tag{1}$$

Kim and Minda [5] pointed out that the first factor on the right is incorrect and extended the result to obtain

$$|f(z_1) - f(z_2)| \geq \frac{\sinh 2\rho}{2(2\cosh 2p\rho)^{1/p}} (|D_1 f(z_1)|^p + |D_1 f(z_2)|^p)^{1/p} \tag{2}$$

where $D_1 f(z) = (1-|z|^2) f'(z)$ valid for $p \geq P$ with some P, $1 < P \leq 3/2$. In each case there was an appropriate equality statement.

Recently the author [4] has proved that [2] is valid for all $p \geq 1$ and also has given an inequality in the opposite direction.

Ma and Minda [6] have given for bounded univalent regular functions upper and lower bounds for σ in terms of $|\Delta_1 f(z_1)|$, $|\Delta_1 f(z_2)|$ and ρ depending on a

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parameter p conditioned by $p \geq 3/2$. Their proof is based on estimates for length in the hyperbolic metric and results in extremely complicated expressions in $|\Delta_1 f(z_1)|$, $|\Delta_1 f(z_2)|$ and ρ.

In this paper we will obtain two point distortion theorems for bounded univalent regular functions more analogous to those in [4] which are more truly distortion theorems.

2. The proof in [4] was carried out by studying the family $\mathscr{F}$ of functions f regular and univalent in D satisfying, for $0 < r < 1$, $f(-r) = -1$, $f(r) = 1$. Of course $\mathscr{F}$ depends on r but this is kept fixed. The treatment consists of proving two theorems stated here as Theorem T and Theorem F.

Theorem T. *If* $f \in \mathscr{F}$ *and* $p \geq 1$

$$(|f'(-r)|^p + |f'(r)|^p)^{1/p} \leq \frac{4(\cosh 2p\rho)^{1/p}}{(1 - r^2)\sinh 2\rho}.$$

Equality occurs only for functions mapping D on the plane slit along a ray on the positive or negative real axis.

Theorem F. *If* $f \in \mathscr{F}$ *and* $p > 0$

$$(|f'(-r)|^p + |f'(r)|^p)^{1/p} \geq \frac{2^{1/p}\cosh(\rho/2)}{\cosh \rho}.$$

Equality occurs only for a function l_0 *mapping D onto the plane slit along the real axis symmetrically through the point at infinity.*

In the proof of Theorem T there is constructed a one-parameter family of functions $f_b \in \mathscr{F}$ which map D onto an admissible domain [2] with respect to the quadratic differential

$$\kappa_1 \frac{(w - b)}{(w + 1)^2 (w - 1)^2}\, dw^2$$

where κ_1 real has the same sign as b or exceptionally for $b = \infty$

$$\kappa_2 \frac{dw^2}{(w + 1)^2 (w - 1)^2}$$

with $\kappa_2 < 0$. The functions are determined explicitly and the proof is carried out by direct calculation. Incidentally the formula on p. 156 *l*.4 in [4] should read

$$\frac{d}{db} \log|f_b'(r)| = -\frac{1}{b - 1}\frac{1}{(b^2 - 1)^{1/2}} \log\left(\frac{1 + r^2}{2r}\frac{(b + 1)^{1/2} - (b - 1)^{1/2}}{(b + 1)^{1/2} + (b - 1)^{1/2}}\right)$$

but the correct formula is used in the remainder of the proof.

To treat the case of bounded univalent regular functions we construct functions analogous to the f_b. We fix r, $0 < r < 1$, and a, $0 < a < r$, and denote by $\mathscr{G}$ the family of functions regular and univalent in D with values in E such that $g(-r) = -a$, $g(r) = a$.

There are three types of special functions in $\mathscr{G}$. In the first instance for ψ, $0 < \psi < \pi$ the quadratic differential

$$Q^*(w, \psi)\, dw^2 = -\frac{(w - e^{-i\psi})^2 (w - e^{i\psi})^2}{(w - a)^2 (w + a)^2 (w - a^{-1})^2 (w + a^{-1})^2}\, dw^2 \tag{3}$$

is positive in E and has the following trajectory structure. The open arcs on $|w| = 1$ joining $e^{i\psi}$ and $e^{-i\psi}$ are trajectories and there is a further trajectory γ in E joining $e^{i\psi}$ and $e^{-i\psi}$ which divides E into two circle domains for $Q^*(w, \psi)\, dw^2$ containing respectively the double poles $-a, a$. If we make two appropriate symmetric incisions from $e^{-i\psi}$, $e^{i\psi}$ along γ into E we obtain a domain E_ψ which can be mapped conformally onto D so that $-a, a$ go to $-r, r$. This induces on D a quadratic differential

$$Q(z, \phi)\, dz^2 = -K^{-1} \frac{(z - e^{-i\phi})^2 (z - e^{i\phi})^2}{(z - r)^2 (z + r)^2 (z - r^{-1})^2 (z + r^{-1})^2}\, dz^2 \tag{4}$$

with a positive constant K. We denote the corresponding mapping from D to E_ψ by h_0. The values of ϕ fill an open arc $0 < \phi_0 < \phi < \pi - \phi_0$, $0 < \phi_0 < \pi$.

A second type of special function is obtained by mapping E into the ζ-plane by the function

$$l(w) = \frac{(1 - a^2)^2}{a(1 + a^2)} \frac{w}{(1 + w)^2} + \frac{2a}{1 + a^2} \tag{5}$$

$l(E)$ is the plane slit on the positive real axis from $(1 + 6a + 4a^4)/(4a(1 + a^2))$ to ∞ and $l(-a) = -1$, $l(a) = 1$. Thus for $g \in \mathscr{G}$, $lg \in \mathscr{F}$. For

$$b_+ = \frac{1 + 6r^2 + r^4}{4r(1 + r^2)} \le b \le \frac{1 + 6a^2 + a^4}{4a(1 + a^2)} = \hat{b}$$

the function $l^{-1} f_b$ is in $\mathscr{G}$ and the quadratic differential

$$\lambda \frac{(\zeta - b)}{(\zeta + 1)^2 (\zeta - 1)^2}\, d\zeta^2$$

for a suitable $\lambda > 0$ induces on E a quadratic differential

$$-\frac{(w - c)(w - c^{-1})(w - 1)^2}{(w - a)^2 (w + a)^2 (w - a^{-1})^2 (w + a^{-1})^2}\, dw^2 \tag{6}$$

where $c = l^{-1}(b)$.

The third type of special function can be obtained by a similar construction with the function

$$\tilde{l}(w) = \frac{(1-a^2)^2}{a(1+a^2)} \frac{w}{(1-w)^2} - \frac{2a}{1+a^2}$$

to obtain a function $\tilde{l}^{-1} f_b$ and a quadratic differential

$$-\frac{(w-\hat{c})(w-\hat{c}^{-1})(w+1)^2}{(w-a)^2(w+a)^2(w-a^{-1})^2(w+a^{-1})^2}\, dw^2 \tag{7}$$

with $\hat{c} = \tilde{l}^{-1} b$.

Combining these we have quadratic differentials $\hat{Q}(w,t)\, dw^2$ where

$$\hat{Q}(w,t) = -\frac{(w^2-2tw+1)(w-1)^2}{(w-a)^2(w+a)^2(w-a^{-1})^2(w+a^{-1})^2}, \quad t^* \geq t \geq 1,$$

$$\hat{Q}(w,t) = -\frac{(w-2tw+1)^2}{(w-a)^2(w+a)^2(w-a^{-1})^2(w+a^{-1})^2}, \quad 1 \geq t \geq -1,$$

$$\hat{Q}(w,t) = -\frac{(w^2-2tw+1)^2}{(w-a)^2(w+a)^2(w-a^{-1})^2(w+a^{-1})^2}, \quad -1 \geq t \geq -t^*$$

with $t^* = (1/2)(l^{-1}b_+ + (l^{-1}b_+)^{-1})$. Note that the definitions agree at $t = 1, -1$. We denote the corresponding functions in $\mathscr{G}$ by g_t, $t^* \geq t \geq -t^*$.

Lemma 1. *For $g \in \mathscr{G}$, $|g'(-r)|$ is maximized uniquely for g_{t^*}, minimized uniquely for g_{-t^*}, $|g'(r)|$ is maximized uniquely for g_{-t^*}, minimized uniquely for g_{t^*}.*

$g_{t^*}(D)$ is an admissible domain for the quadratic differential

$$-\frac{-(w-1)^2\, dw^2}{(w+a)^2(w+a^{-1})^2(w-a)(w-a^{-1})},$$

$gg_{t^*}^{-1}$ is an admissible function for it. Applying the General Coefficient Theorem [2, 3] with $-a$ as P_1, we have the coefficients

$$\alpha^{(1)} = -\frac{a^2}{2(1-a)^2(1+a^2)}, \quad a^{(1)} = (g'(-r))^{-1} g'_{t^*}(-r).$$

The fundamental inequality gives

$$-\frac{a^2}{2(1-a)^2(1+a^2)} \log\left|\frac{g'_{t^*}(-r)}{g'(-r)}\right| \leq 0$$

or

$$|g'(-r)| \leq |g'_{t^*}(-r)|.$$

The equality statement follows from that in the General Coefficient Theorem.

$g_{-t^*}(D)$ is an admissible domain for the quadratic differential

$$\frac{(w+1)^2\,dw^2}{(w+a)^2(w+a^{-1})^2(w-a)(w-a^{-1})},$$

$gg_{-t^*}^{-1}$ is an admissible function for it. Applying the General Coefficient Theorem with $-a$ as P_1 we have the coefficients

$$\alpha^{(1)} = \frac{a^2}{2(1+a)^2(1+a^2)}, \quad a^{(1)} = (g'(-r))^{-1}g'_{-t^*}(-r).$$

The fundamental inequality gives

$$\frac{a^2}{2(1+a)^2(1+a^2)}\,\log\left|\frac{g'_{t^*}(-r)}{g'(-r)}\right| \le 0$$

or

$$g'(-r) \ge |g'_{-t^*}(-r)|.$$

The equality statement follows from that in the General Coefficient Theorem.
The remaining statements follow by symmetry.

Lemma 2. *The quantity* $|g'(-r)|^p + |g'(r)|^p$, $p > 0$, *is maximized for a function* g_t, *uniquely up to translation along trajectories.*

It is readily seen that $|g_t'(-r)|$, $|g_t'(r)|$ vary continuously with t on $[-t^*, t^*]$ either by domain convergence or by the explicit expressions given below. For any $g \in \mathscr{G}$ there exists a $t \in [-t^*, t^*]$ with $|g'(-r)| = |g'_{t^*}(-r)|$. We apply the General Coefficient Theorem with the quadratic differential $\hat{Q}(w,t)\,dw^2$ for this value of t, the admissible domain $g_t(D)$ and the admissible function gg_t^{-1}. Taking $-r$ as P_1, r as P_2 for $t \in [1, t^*]$ the corresponding coefficients are

$$\alpha^{(1)} = -\frac{a^2(a^2+2ta+1)}{4(1-a)^2(1+a^2)^2}, \quad a^{(1)} = (g'(-r^{-1}))^{-1}g_t'(-r),$$

$$\alpha^{(2)} = -\frac{a^2(a^2-2ta+1)}{4(1+a)^2(1+a^2)^2}, \quad a^{(2)} = (g'(r))^{-1}g_t'(r).$$

The fundamental inequality gives

$$-\frac{a^2(a^2+2ta+1)}{4(1-a)^2(1+a^2)^2}\,\log\left|\frac{g_t'(-r)}{g'(-r)}\right| - \frac{a^2(a^2-2ta+1)}{4(1+a)^2(1+a^2)^2}\,\log\left|\frac{g_t'(r)}{g'(r)}\right| \le 0$$

thus $|g'(r)| \le |g_t'(r)|$. Equality can occur only if g is obtained from g_t by translation along trajectories.

The other cases for t are treated similarly.

Lemma 3. *$|g_t'(-r)|$ decreases strictly monotonically as t goes from t^* to $-t^*$. $|g_t'(r)|$ increases strictly monotonically as t goes from t^* to $-t^*$.*

This follows at once by applying the preceding argument for two values of t and Lemma 1.

3. In order to find the maximum of $|g_t'(-r)|^p + |g_t'(r)|^p$ for reasons of symmetry it is sufficient to consider t in the interval $[0, t^*]$.

Lemma 4. *For t in the range of values $[0,1]$, $p > 0$, $|g_t'(-r)|^p + |g_t'(r)|^p$ decreases strictly monotonically as t goes from 1 to 0.*

From (3) and (4) we have for corresponding values z, w, t, ψ, ϕ and $K > 0$

$$\begin{aligned}&\frac{(z-e^{i\phi})^2(z-e^{-i\phi})^2}{(z+r)^2(z-r)^2(z+r^{-1})^2(z-r^{-1})^2}\\ &= K\frac{(w-e^{i\psi})^2(w-e^{-i\psi})^2}{(w+a)^2(w-a)^2(w+a^{-1})^2(w-a^{-1})^2}\left(\frac{dw}{dz}\right)^2.\end{aligned}$$

Letting $z \to -r$, $w \to -a$; $z \to r$, $w \to a$ respectively we have

$$\frac{r^2(r^2+2r\cos\phi+1)^2}{4(1-r^2)^2(1+r^2)^2} = K\frac{a^2(a^2+2a\cos\psi+1)^2}{4(1-a^2)^2(1+a^2)^2}$$
$$\frac{r^2(r^2-2r\cos\phi+1)^2}{4(1-r^2)^2(1+r^2)^2} = K\frac{a^2(a^2-2a\cos\psi+1)^2}{4(1-a^2)^2(1+a^2)^2}$$

and dividing

$$\frac{(r^2+2r\cos\phi+1)}{(r^2-2r\cos\phi+1)} = \frac{(a^2+2a\cos\psi+1)}{(a^2-2a\cos\psi+1)}.$$

Integrating explicitly $\int(-Q(z,\phi))^{1/2}\,dz$ and $\int(-Q^*(w,\psi))^2\,dw$ with suitable normalizations we get

$$\frac{1}{2}\frac{r(1+2r\cos\phi+r^2)}{(1-r^2)(1+r^2)}\log\frac{z+r}{1+rz} - \frac{1}{2}\frac{r(1-2r\cos\phi+r^2)}{(1-r^2)(1+r^2)}\log\frac{r-z}{1-rz},$$

and

$$\frac{1}{2}\frac{a(1+2a\cos\psi+a^2)}{(1-a^2)(1+a^2)}\log\frac{w+a}{1+aw} - \frac{1}{2}\frac{a(1-2a\cos\psi+a^2)}{(1-a^2)(1+a^2)}\log\frac{a-w}{1-aw}.$$

Comparing expansions about $-r$, $-a$ we find

$$\log(w+a) = \log(z+r) - \log(1+rz) + \log(1+aw)$$
$$-\frac{(1-2r\cos\phi+r^2)}{(1+2r\cos\phi+r^2)}\log\frac{r-z}{1-rz} + \frac{(1-2a\cos\psi+a^2)}{(1+2a\cos\psi+a^2)}\log\frac{a-w}{1-aw}$$

so

$$\log g_t'(-r) = \frac{(1-2r\cos\phi+r^2)}{(1+2r\cos\phi+r^2)}\left(-\log\frac{2r}{1+r^2} + \log\frac{2a}{1+a^2}\right)$$
$$-\log(1-r^2) + \log(1-a^2).$$

Similarly

$$\log g_t'(r) = \frac{(1+2r\cos\phi+r^2)}{(1-2r\cos\phi+r^2)}\left(-\log\frac{2r}{1+r} + \log\frac{2a}{1+a^2}\right)$$
$$-\log(1-r^2) + \log(1-a^2).$$

Thus

$$(g_t'(-r))^p + (g_t'(r))^p = \left(\frac{1-a^2}{1-r^2}\right)^p \left(\frac{2a}{1+a^2}\,\frac{1+r^2}{2r}\right)^{p\frac{1-2r\cos\phi+r^2}{1+2r\cos\phi+r^2}}$$
$$+\left(\frac{1-a^2}{1-r^2}\right)^p \left(\frac{2a}{1+a^2}\,\frac{1+r^2}{2r}\right)^{p\frac{1+2r\cos\phi+r^2}{1-2r\cos\phi+r^2}}.$$

A straightforward calculation shows that this decreases strictly monotonically as t goes from 1 to 0.

COROLLARY 1. *To find the maximum of* $|g_t'(-r)|^p + |g_t'(r)|^p$, $p > 0$, *it is enough to consider the values* $t \in [1, t^*]$.

For $t \in [1, t^*]$ the function $g_t'(z)$ is given by $g_t'(z) = (d/dz)(l^{-1}f_b(z))$ where t, b are corresponding values. Thus

$$g_t'(r) = \frac{a(1+a)(1+a^2)}{(1-a)^3} f_b'(r)$$

$$g_t'(-r) = \frac{a(1-a)(1+a^2)}{(1+a)^3} f_b'(-r)$$

and to find the maximum of $|g_t'(-r)|^p + |g_t'(r)|^p$ we find the maximum of

$$\left(\frac{a(1-a)(1+a^2)}{(1+a)^3}\right)^p (f_b'(-r))^p + \left(\frac{a(1+a)(1+a^2)}{(1-a)^3}\right)(f_b'(r))^p \tag{8}$$

on the appropriate interval for b. We consider its derivative

$$p\left(\frac{a(1-a)(1+a^2)}{(1+a)^3}\right)^p (f_b'(-r))^{p-1}\frac{df_b'(-r)}{db} + p\left(\frac{a(1+a)(1+a^2)}{(1-a)^3}\right)(f_b'(r))^{p-1}\frac{df_b'(r)}{db}. \tag{9}$$

The first term is negative the second positive.

Lemma 5. *The ratio of the terms in* (9) *decreases as b increases. Thus* (8) *has a unique maximum on $[b_+, \hat{b}]$. If*

$$p \geq \log\frac{1+r}{1-r}\left(\log\left(\frac{1-a}{1+a}\,\frac{1+r}{1-r}\right)\right)^{-1} \tag{10}$$

the maximum occurs for b_+. Otherwise it occurs for $b \in (b_+\hat{b}]$. Thus if (10) *holds the maximum of $|g_t'(-r)|^p + |g_t'(r)|^p$ occurs for $t = t^*$.*

A direct calculation shows that

$$\frac{d}{db}\,\frac{(f_b'(-r))^p(d/db)f_b'(-r)}{(f_b'(r))^{p-1}(d/db)f_b'(r)} = \left(\frac{f_b'(-r)}{f_b'(r)}\right)\left(\frac{2}{(b+1)^2}\right)\left(\frac{p}{(b^2-1)^{1/2}}\right)\log\left(\frac{1+r^2}{2r}\,\frac{(b+1)^{1/2}-(b-1)^{1/2}}{(b+1)^{1/2}+(b-1)^{1/2}}\right)-1$$

which is negative (see [4], p. 156).

Therefore if (8) is decreasing at b_+ the maximal value occurs there. This requires the condition

$$p > \log\frac{1+r}{1-r}\left(\log\left(\frac{1-a}{1+a}\,\frac{1+r}{1-r}\right)\right)^{-1}$$

and the equality follows by a passage to the limit.

Using the invariance properties $|\Delta_1 f(z)|$, ρ and σ and the expression for ρ and σ in terms of r and a

$$\rho = \log\frac{1+r}{1-r}, \quad \sigma = \log\frac{1+a}{1-a}$$

we have the following theorem.

Theorem 1. *If f is regular and univalent in D and maps D into E and if z_1, z_2 are distinct points in D and*

$$p \geq \frac{\rho}{\rho-\sigma}$$

where ρ, σ are the hyperbolic distances between z_1 and z_2, $f(z_1)$ and $f(z_2)$ respectively then

$$|\Delta_1 f(z_1)|^p + |\Delta_1 f(z_2)|^p \le \left[\tanh\frac{\sigma}{2}\,\frac{e^{2\sigma+1}}{(e^{\sigma+1})^2}\left(\tanh\frac{\rho}{2}\,\frac{e^{2\rho}+1}{(e^\rho+1)^2}\right)^{-1}\right]^p \times\left[\left(\frac{e^\rho(e^\sigma+1)}{e^\sigma(e^\rho+1)}\right)^{4p}+\left(\frac{e^\sigma+1}{e^\rho+1}\right)^{4p}\right]$$

Equality occurs if and only if f maps D onto E slit along a ray on the hyperbolic line determined by $f(z_1)$ and $f(z_2)$.

If $p < \rho/(\rho-\sigma)$ this result does not obtain.

4. A bound in the opposite sense can be obtained immediately from Theorem F. Let m_0 be the function in $\mathcal{G}$ mapping D onto E with equal rectilinear slits proceeding from ± 1. l_0 is the function $((1+r^2)/r)(z/(1+z^2))$ and if $\tau(w) = ((1+a^2)/a)(w/(1+w^2))$ m_0 is $\tau^{-1} l_0$. For any $g \in \mathcal{G}$, $\tau g \in \mathcal{F}$. Thus by Theorem F for $p > 0$

$$|(\tau g)'(-r)|^p + |(\tau g)'(r)|^p \ge (l_0'(-r))^p + (l_0'(r))^p = 2\left(\frac{1-r^2}{r(1+r^2)}\right)^p$$

and

$$|g'(-r)|^p + |g'(r)|^p \ge 2\left(\frac{a(1+a^2)}{1-a^2}\cdot\frac{1-r^2}{r(1+r^2)}\right)^p.$$

Moreover

$$|\Delta_1 g(-r)|^p + |\Delta_1 g(r)|^p \ge 2\left(\frac{a(1+a^2)}{(1-a^2)^2}\,\frac{(1-r^2)^2}{r(1+r^2)}\right)^p.$$

Equality occurs only for m_0.

Using the invariance properties we have proved the following theorem.

Theorem 2. *If f is regular and univalent in D mapping D into E and z_1, z_2 are distinct points of D, we have for $p > 0$*

$$(|\Delta_1 f(z_1)|^p + |\Delta_1 f(z_2)|^p)^{1/p} \ge \frac{e^{2\sigma}+1}{e^{2\rho}+1}\left(\frac{e^\rho+1}{e^\sigma+1}\right)^3\left(\frac{\cosh(\sigma/2)}{\cosh(\rho/2)}\right)^4$$

where ρ is the hyperbolic distance between z_1 and z_2, σ is the hyperbolic distance between $f(z_1)$ and $f(z_2)$. Equality occurs only for the function mapping D onto E slit along symmetric rays on the hyperbolic line determined by $f(z_1)$ and $f(z_2)$.

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